

A generic condition implying o-minimality for restricted C^∞ -functions

OLIVIER LE GAL⁽¹⁾

ABSTRACT. — We prove that the expansion of the real field by a restricted C^∞ -function is generically o-minimal. Such a result was announced by A. Grigoriev, and proved in a different way. Here, we deduce quasi-analyticity from a transcendence condition on Taylor expansions. This then implies o-minimality. The transcendence condition is shown to be generic. As a corollary, we recover in a simple way that there exist o-minimal structures that doesn't admit analytic cell decomposition, and that there exist incompatible o-minimal structures. We even obtain o-minimal structures that are not compatible with restricted analytic functions.

RÉSUMÉ. — On montre que génériquement, l'expansion du corps des réels par une fonction C^∞ restreinte est o-minimale. Un résultat du même type utilisant d'autres d'arguments a été annoncé par A. Grigoriev. Ici, nous utilisons une condition de transcendance sur les développements de Taylor pour assurer la quasianalyticité de certaines algèbres différentielles, ce qui implique la o-minimalité. On montre que cette condition de transcendance est générique. Comme corollaire de ce résultat, on donne des preuves simples du fait qu'il existe des structures o-minimales n'admettant pas de décomposition cellulaire analytique, et qu'il existe des structures o-minimales incompatibles. On obtient même des structures o-minimales non compatibles avec les fonctions analytiques restreintes.

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(1) Université de savoie, Bâtiment Le Chablais, Campus Scientifique, 73376 Le Bourget-du-Lac Cedex, France.
Olivier.Le-Gal@univ-savoie.fr

1. Introduction

The geometry of the zero set of a C^∞ -function may be very complicated, since Whitney proves that any closed subset of \mathbb{R}^n is such a zero set. René Thom deep interest on genericity was driven by the expectation that the objects coming from generic smooth mappings should have a *tame* geometry. On the other hand, the axiomatic approach of o-minimality guarantee such a tame geometric behavior. In [Gri05], A. Grigoriev announces that in one sense, these two approaches meet together : o-minimality is in a certain meaning generic. This article aims to give another version of this result, based on the notion of quasianalyticity.

Recall that an expansion $\mathbb{R}_{\mathcal{F}} = (\mathbb{R}, 0, 1, +, \cdot, <, \mathcal{F})$ of the real field by a family of maps \mathcal{F} is **o-minimal** if any definable set has finitely many connected component. Geometrically speaking, a set $X \subset \mathbb{R}^k$ is **definable** in $\mathbb{R}_{\mathcal{F}}$ if it belongs to the smallest collection of subsets of \mathbb{R}^n , for various n , which contains semi-algebraic sets and all the graphs of maps in \mathcal{F} , and is closed under basic set theoretic operations : finite boolean combinations, cartesian products and projections. A map is also said to be **definable** if its graph is definable.

O-minimality provides many geometric properties for definable sets or maps, such as Whitney stratifications. Some structures verify an additional tameness property, that imply, for instance, Lojasiewicz inequalities: $\mathbb{R}_{\mathcal{F}}$ is **polynomially bounded** if for any definable function $f : \mathbb{R} \rightarrow \mathbb{R}$, there exists an integer n such that $f(x)/x^n$ is ultimately bounded. For a more precise introduction to o-minimal geometry, we refer to the books [Cos00] of Coste and [vdD98] of van den Dries. Among others, the following are classical examples of o-minimal structures. Using the Gabrielov's theorem of the complement [Gab96], Denef and van den Dries prove that the structure \mathbb{R}_{an} generated by the family **an** of restricted analytic functions is o-minimal and polynomially bounded [DvdD88]. The same holds for certain structures generated by quasi-analytic Denjoy-Carleman classes, according to the result [RSW03] of Rolin, Speissegger and Wilkie. Wilkie also prove that the structure $\mathbb{R}_{\text{Pfaff}}$, generated by the so-called pfaffian functions, is o-minimal [Wil99].

We fix some notations. Let h be a C^∞ -function in an open subset \mathcal{U} of \mathbb{R} . The **multi-jet** $j_n^m h(x)$ of order m of h at the n -tuple $x = (x_1, \dots, x_n) \in \mathcal{U}^n$ is the $n(m+1)$ -tuple of the values of h and its m first derivatives at the points x_1, \dots, x_n : $j_n^m h(x) = (h(x_1), \dots, h(x_n), \dots, h^{(m)}(x_1), \dots, h^{(m)}(x_n))$. We define h to be **strongly transcendental** if for any finite subset $\{x_1, \dots, x_n\}$ of \mathcal{U} , the values of h and all its derivatives at x_1, \dots, x_n verify with x_1, \dots, x_n at most a finite number of integral algebraic relations. Precisely,

if $\Delta_n = \{(x_1, \dots, x_n) \in \mathbb{R}^n; \exists i \neq j, x_i = x_j\}$ denotes the **diagonals**, h is strongly transcendental if

$$\forall n \geq 1, \forall x = (x_1, \dots, x_n) \in \mathcal{U}^n \setminus \Delta_n, \exists C \in \mathbb{N}, \forall m \in \mathbb{N},$$

$$\text{trdeg}(x, j_n^m h(x)) \geq n(m+2) - C, \quad (1.1)$$

where trdeg denotes the transcendence degree over \mathbb{Q} . We say that a function $h : [0, 1] \rightarrow \mathbb{R}$ is a **restricted C^∞ -function** – denoted by $h \in C^\infty([0, 1])$ – if it is the restriction of some C^∞ -function \bar{h} defined in a neighborhood of $[0, 1]$. If \bar{h} can be chosen to be strongly transcendental, we say that h is a **restricted strongly transcendental function**.

We now set our main results. Recall that the weak Whitney topology – induced by the family of seminorms $\sup_{t \in K} |h^{(k)}(t)|$, where K ranges in the compact subsets of \mathcal{U} – turns $C^\infty(\mathcal{U})$ into a Baire space: any **residual set** – meaning that it contains a countable intersection of open dense sets – is dense.

THEOREM 1.1. — *The set st of restricted strongly transcendental functions is residual in $C^\infty([0, 1])$.*

The proof is presented in section 2. The interest of restricted strongly transcendental functions appears in the following.

THEOREM 1.2. — *Let h be a restricted strongly transcendental function. Then, \mathbb{R}_h is o-minimal, polynomially bounded and admits C^∞ -cell decomposition.*

The scheme of the proof of this theorem is merely an adaptation of the methods developed in [LGR08] to construct an o-minimal structure that does not admit C^∞ -cell decomposition, joint with the main theorem of [RSW03], which asserts that o-minimality follows from quasi-analyticity. We have devoted section 3 to this proof.

In the last section, we prove as corollaries that there exist o-minimal structures that does not admit analytic cell decomposition, and that there exists a pair of functions (h_1, h_2) such that \mathbb{R}_{h_1} and \mathbb{R}_{h_2} are o-minimal but \mathbb{R}_{h_1, h_2} is not. Such results were first obtained in [RSW03], with the use of quasi-analytic Denjoy-Carleman classes. Here, we can improve the last, by imposing one of the two functions to be analytic. It gives rise to the following, which seems to be actually unknown :

COROLLARY 1.3. — *There exists an o-minimal expansion \mathbb{R}_h of the real field such that $\mathbb{R}_{h, \text{an}}$ is not o-minimal.*

We conclude with some questions that arise from this work. While they are generic, we don't explicitly know any strongly transcendental function. Is it possible to give the construction of such a function? On another hand, one could easily enhance our result for expansions of $(\mathbb{R}, +, \cdot, <)$ by finitely many functions, meaning that the k -tuples (h_1, \dots, h_k) of restricted C^∞ -functions such that $\mathbb{R}_{h_1, \dots, h_k}$ is o-minimal are generic among k -tuples of restricted C^∞ -functions. Does such a result hold for infinite families? The question becomes complicated even to formulate, since it is not clear what topology should be considered on the set of all families of restricted C^∞ -functions.

Notations. — Throughout this paper, \mathbb{N} denotes the set of natural numbers and \mathbb{R} the real field. The letters i, j, k, ℓ, m, n, p and q denote non negative integers, x the n -tuple (x_1, \dots, x_n) , and t a single real variable. The letters f and g denote maps or their germs, and h a one variable C^∞ -function. We use the notation \mathfrak{st} for the set of restricted strongly transcendental functions, and \mathfrak{an} for restricted analytic functions.

2. Genericity and Transcendence

In this section, we show the genericity of the transcendence condition (1). Recall that the transcendence degree of a k -tuple $(x_1, \dots, x_k) \in \mathbb{R}^k$ is the minimum of the dimension of any integral algebraic subsets of \mathbb{R}^k that contains the point (x_1, \dots, x_k) . Let h be a C^∞ -function on a open bounded subset \mathcal{U} of \mathbb{R} . Saying that h is strongly transcendental then means that for any given $x \in \mathcal{U}^n \setminus \Delta_n$, the codimension of the integral algebraic subsets of $\mathbb{R}^{n(m+2)}$ that contain the point $(x, j_n^m h(x))$ is ultimately bounded when m tends to infinity. Actually, we will show a stronger result: the bound on this codimension is generically n . The bound is in particular uniform in x .

LEMMA 2.1. — *Let \mathcal{U} be an open bounded subset in \mathbb{R} , and X be an algebraic set of codimension $n + 1$ in $\mathbb{R}^{n(m+2)}$. Then, the set $E(X) = \{h \in C^\infty(\mathcal{U}); \forall x \in \mathcal{U}^n \setminus \Delta_n, (x, j_n^m h(x)) \notin X\}$ is residual.*

Proof. — Fix an algebraic set $X \subset \mathbb{R}^{n(m+2)}$ of codimension $n + 1$. To prove the lemma, we introduce the followings:

$$\mathcal{U}_i = \{x_1 \in \mathcal{U}; d(x_1, \text{Fr}(\mathcal{U})) \geq \frac{1}{i}\}, \Delta_{n,i} = \{x \in \mathcal{U}^n; d(x, \Delta_n) < \frac{1}{i}\},$$

$$E_i(X) = \{h \in C^\infty(\mathcal{U}); \forall x \in \mathcal{U}_i^n \setminus \Delta_{n,i}, (x, j_n^m h(x)) \notin X\},$$

where d denote the euclidian distance and $\text{Fr}(\mathcal{U})$ the frontier $\overline{\mathcal{U}} \setminus \mathcal{U}$ of \mathcal{U} .

Let us show that $E_i(X)$ is a dense open subset of $C^\infty(\mathcal{U})$. It is open because of the compactness of $\mathcal{U}_i^n \setminus \Delta_{n,i}$. Indeed, if $(h_i)_{i \in \mathbb{N}}$ is a convergent sequence in the complement of $E_i(X)$ with limit h , we can define a sequence $(x_i)_{i \in \mathbb{N}}$ in $\mathcal{U}_i^n \setminus \Delta_{n,i}$ such that $(x_i, j_n^m h_i(x_i)) \in X$. The sequence (x_i) accumulates to a point x in $\mathcal{U}_i^n \setminus \Delta_{n,i}$. Since X is closed, we have $(x, j_n^m h(x)) \in X$, so that h is in the complement of $E_i(X)$. So, the complement of $E_i(X)$ is closed, and $E_i(X)$ is open.

In order to prove that $E_i(X)$ is dense, we fix a function $h \in C^\infty(\mathcal{U})$, and we construct a small perturbation of h that belongs to $E_i(X)$. We obtain this perturbation by adding a polynomial to h . For $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{n(m+1)}) \in \mathbb{R}^{n(m+1)}$, we set

$$h_\varepsilon(t) = h(t) + \varepsilon_1 t^{n(m+1)-1} + \varepsilon_2 t^{n(m+1)-2} + \dots + \varepsilon_{n(m+1)-1} t + \varepsilon_{n(m+1)}$$

and we consider the map

$$\begin{array}{ccc} \varphi : \mathcal{U}_i^n \setminus \Delta_{i,n} & \times & \mathbb{R}^{n(m+1)} & \rightarrow & \mathcal{U}_i^n \times \mathbb{R}^{n(m+1)} \\ & & \varepsilon & \mapsto & x, j_n^m h_\varepsilon(x) \end{array}$$

With this notation, the function h_ε belongs to $E_i(X)$ if for all $x \in \mathcal{U}_i^n \setminus \Delta_{i,n}$, the point $\varphi(x, \varepsilon)$ does not belong to X . Hence, we require an ε as small as desired such that ε lie in the complement of $\pi(\varphi^{-1}(X))$, where $\pi : \mathbb{R}^{n(m+2)} \rightarrow \mathbb{R}^{n(m+1)}$ denotes the projection $\pi(x, \varepsilon) = \varepsilon$.

Since X is an algebraic set of codimension $n+1$, it is the union of finitely many smooth manifolds of codimension at least $n+1$. Moreover, the map φ is a diffeomorphism; it is bijective and has constant rank $n(m+2)$. The pre-image $\varphi^{-1}(X)$ is then the union of finitely many smooth manifolds of codimension at least $n+1$. The projection of each of these manifolds by π might be singular, but according to the Sard theorem, it has measure zero. Hence $\pi(\varphi^{-1}(X))$ is the union of finitely many sets of measure zero, then has measure zero itself.

Let us return to the perturbation h_ε . We have shown that the set of ε such that h_ε does not belong to $E_i(X)$ has measure zero. This set cannot contain any neighborhood of 0. Hence, there exists ε as small as desired such that h_ε belongs to $E_i(X)$, which shows that $E_i(X)$ is dense.

Now, remark that $E(X) = \bigcap_{i \in \mathbb{N}^*} E_i(X)$. Since for all $i \in \mathbb{N}^*$, $E_i(X)$ is an open dense set, $E(X)$ is residual, which achieves the proof. \square

We can now prove theorem 1.1.

Proof. — [Proof of Theorem 1.1] We first show that strongly transcendental functions are residual in $C^\infty(\mathcal{U})$, where \mathcal{U} denote a bounded open

subset of \mathbb{R} . Let E be the set

$$E = \{h \in C^\infty(\mathcal{U}); \forall (n, m) \in \mathbb{N}^2, \text{ for all integral algebraic } X \subset \mathbb{R}^{n(m+2)}, \\ \text{Codim}(X) \geq n + 1 \Rightarrow \forall x \in \mathcal{U}^n \setminus \Delta_n, (x, j_n^m h(x)) \notin X\}.$$

First remark that any function in E is strongly transcendental. Indeed, if h belongs to E , the transcendence degree of $(x, j_n^m h(x))$ is at least $n(m + 1)$ for any point x in $\mathcal{U}^n \setminus \Delta_n$. The transcendence condition (1) is then satisfied with $C = n$. Moreover, the set E is the intersection of all $E(X)$, when X ranges over all integral algebraic subsets of $\mathbb{R}^{n(m+2)}$ of codimension at least $n + 1$, for all integers n, m . There are countable many such (n, m, X) , and according to lemma 2.1, any $E(X)$ is residual. Then E is residual also. Since it contains E , the set of strongly transcendental functions in \mathcal{U} is residual.

Now, fix a bounded neighborhood \mathcal{U} of $[0, 1]$, and denote by r the restriction map $r : h \in C^\infty(\mathcal{U}) \mapsto h|_{[0, 1]} \in C^\infty([0, 1])$. The map r is surjective (any restricted C^∞ function can be extended to \mathcal{U}), linear and continuous. So, by the open map Theorem, r is an open map, then r concerves genericity. Since $r(E) \subset \mathfrak{st}$, restricted strongly transcendental functions are generic in restricted C^∞ -functions as well. \square

Actually, the same ideas show a slightly different result. Let R be a positive real number, and denote by \mathcal{A}_R the subset of analytic functions in \mathcal{U} given by:

$$\mathcal{A}_R = \{f \in C^\infty(\mathcal{U}); \exists C \in \mathbb{R}, \forall k \geq 0, \forall x \in \mathcal{U}, |f^{(k)}(x)| \leq CR^k k!\}.$$

The R -norm $\|f\|_R = \sup_{k \in \mathbb{N}, x \in \mathcal{U}} \left| \frac{f^{(k)}(x)}{R^k k!} \right|$ turns \mathcal{A}_R into a Banach space. The following proposition claims that strongly transcendental functions are also generic in any affine subspace of $C^\infty(\mathcal{U})$ of direction \mathcal{A}_R . It will be used in the proof of corollary 1.3.

PROPOSITION 2.2. — *Let \mathcal{U} be an open bounded set, $h \in C^\infty(\mathcal{U})$, and $R > 0$. Then strongly transcendental functions are generic in $h + \mathcal{A}_R$, with respect to the topology induced on $h + \mathcal{A}_R$ by the R -norm on \mathcal{A}_R .*

Proof. — We associate to any algebraic set $X \subset \mathbb{R}^{n(m+2)}$ of codimension at least $n + 1$ the set $E_h(X) = E(X) \cap (\mathcal{A}_R + h) = \{g \in (\mathcal{A}_R + h); \forall x \in \mathcal{U} \setminus \Delta_n, (x, j_n^m g(x)) \notin X\}$. Again, $E_h(X)$ is residual in $\mathcal{A}_R + h$: since the perturbation used in the proof of lemma 2.1 was obtained by adding a polynomial – which belongs to \mathcal{A}_R and can be choosen to have an R -norm as small as desired – the same proof holds. Now, since the set of strongly transcendental functions in $\mathcal{A}_R + h$ contains the intersection of all $E_h(X)$

when X ranges over all integral algebraic subsets of $\mathbb{R}^{n(m+2)}$ of codimension $n+1$, for all n, m , strongly transcendental functions are generic in $\mathcal{A}_R + h$.
 \square

3. Transcendence and o-minimality

This section is devoted to the proof of theorem 1.2. The principle is close to a reasoning made in [LGR08] (see part A.). The o-minimality is obtained as a consequence of the main theorem of [RSW03] : $\mathbb{R}_{\mathcal{F}}$ is o-minimal provided that the algebras *generated* by \mathcal{F} are quasianalytic. Here *generated* means with respect to compositions, implicit functions, monomial divisions as well as algebraic operations. In a first step, we introduce a germified version of those algebras, and a family of operators that act naturally on it – namely, the operators corresponding to take implicit functions, to divide by a monomial, to compose and to make algebraic operations. In a second step, we recall two elementary lemmas, which make the action of these operators on Taylor series more precise. Finally, we use those properties to deduce the quasianalyticity of the algebras from strong transcendence.

3.1. Operators, algebras

Our goal is to introduce theorem 3.3, which is a slightly modified version of theorem 5.2 of [RSW03]. It gives a sufficient condition for o-minimality. We first define similar algebras than in [RSW03], together with similar operators than in [LGR08]. Denote by \mathcal{F}_n the algebra of the germs at 0 of maps in $C^\infty(\mathbb{R}^n)$.

DEFINITION 3.1. — *We call elementary operators the following operators:*

1. *Constant operators (of arity 0), defined for each polynomial $P \in \mathbb{R}[x]$ by:*

$$\mapsto P(x) \in \mathcal{F}_n ;$$

2. *Compositions, defined for $(n, m) \in \mathbb{N}^2$ on $\mathcal{F}_n \times \{(g_1, \dots, g_n) \in \mathcal{F}_m^n ; g_i(0) = 0, i = 1, \dots, n\}$ by :*

$$f, g_1, \dots, g_n \mapsto f(g_1, \dots, g_n) \in \mathcal{F}_m ;$$

3. *Monomial divisions, defined for $n \in \mathbb{N}^*$ on $\{f \in \mathcal{F}_n ; f(x_1, \dots, x_{n-1}, 0) = 0\}$ by:*

$$f \mapsto g \in \mathcal{F}_n, \text{ with } g = \frac{f}{x_n} \text{ if } x_n \neq 0, \text{ and } g = \partial_n f \text{ otherwise ;}$$

4. *Implicit function operators, defined for $n \in \mathbb{N}^*$ on $\{f \in \mathcal{F}_{n+1}; f(0) = 0, \partial_{n+1}f(0) = 1\}$ by:*

$$f \mapsto \varphi \in \mathcal{F}_n, \text{ with } f(x_1, \dots, x_n, \varphi(x_1, \dots, x_n)) = 0.$$

*These operators can be composed ones to another as soon as the image of the ones are in the definition set of the other. We call **operator** such a composition.*

We call **Borel map** – denoted by $\widehat{}$ – the application that maps a C^∞ -germ at 0 to its Taylor expansion. Remark that for any operator \mathcal{L} , the Taylor series of $\mathcal{L}(f)$ only depends on the Taylor series of f . We can then extend the Borel map to operators: any operator \mathcal{L} admits a **formal counterpart** $\widehat{\mathcal{L}}$ that verifies $\widehat{\mathcal{L}}(\widehat{f}_1, \dots, \widehat{f}_\ell) = \widehat{(\mathcal{L}(f_1, \dots, f_\ell))}$. For a given $h \in C^\infty(\mathcal{U})$, we now define the algebras generated by h , to be the collection of the germs obtained by the action of the operators on the germ of h at some points of \mathcal{U} . Precisely :

DEFINITION 3.2. — *Let h be a C^∞ -function on an open set \mathcal{U} . The **algebra generated by h** is the collection $\mathcal{A}(h) = (\mathcal{A}_n(h); n \in \mathbb{N})$ of the smallest algebras $\mathcal{A}_n(h) \subset \mathcal{F}_n$ such that:*

1. *For all $a \in \mathcal{U}$, the germ at 0 of $t \mapsto h(a + t)$ belongs to $\mathcal{A}_1(h)$;*
2. *If $\mathcal{L} : \mathcal{F}_{n_1} \times \dots \times \mathcal{F}_{n_i} \rightarrow \mathcal{F}_m$ is an operator, and if $(f_1, \dots, f_i) \in \mathcal{A}_{n_1}(h) \times \dots \times \mathcal{A}_{n_i}(h)$ belongs to the definiton set of \mathcal{L} , then $\mathcal{L}(f_1, \dots, f_i)$ belongs to $\mathcal{A}_m(h)$.*

Let us introduce a version of theorem 5.2 of [RSW03] convenient to our propose. Recall that an algebra $\mathcal{G} \subset \mathcal{F}_n$ is said to be **quasi-analytic** if the Borel map restricted to \mathcal{G} is injective.

THEOREM 3.3 (Rolin, Speissegger, Wilkie. See 5.2 in [RSW03]). — *Let $\mathcal{U} \subset \mathbb{R}$ be open and $h \in C^\infty(\mathcal{U})$. Denote by \mathcal{H} the collection of all the restrictions of h to a closed interval in \mathcal{U} . If the algebras $\mathcal{A}_n(h)$ are quasi-analytic, $\mathbb{R}_{\mathcal{H}}$ is o -minimal, polynomially bounded, and admits C^∞ -cell decomposition.*

Proof. — Denote by $\mathcal{C} = \{\mathcal{C}_B; B \text{ compact box of } \mathbb{R}^n, n \in \mathbb{N}\}$ the collection of the subalgebras \mathcal{C}_B of $C^\infty(B)$ obtained by the following way.

1. For all compact box $B \subset \mathcal{U}$, $h|_B$ belongs to \mathcal{C}_B ;

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2. For every compact box $B \subset \mathbb{R}^n$, \mathcal{C}_B contains $\mathbb{R}[x_1, \dots, x_n]$;
3. If $g = (g_1, \dots, g_n) \in \mathcal{C}_B^n$, $f \in \mathcal{C}_{B'}$ and $g(B) \subset B'$, then $f \circ g \in \mathcal{C}_B$;
4. If $B' \subset B$ and $f \in \mathcal{C}_B$, $f|_{B'} \in \mathcal{C}_{B'}$;
5. $\partial f / \partial x_i \in \mathcal{C}_B$ for every $f \in \mathcal{C}_B$;
6. If $n > 1$ and $f \in \mathcal{C}_B$ is such that $f(0) = 0$ and $(\partial f / \partial x_n)(x) \neq 0$ for all $x \in \mathcal{C}_B$, then the map α given on B' by $f(x_1, \dots, x_{n-1}, \alpha(x_1, \dots, x_{n-1})) = 0$ belongs to $\mathcal{C}_{B'}$, where B' is the canonical projection of B on \mathbb{R}^{n-1} ;
7. If $f \in \mathcal{C}_B$ and f/x_i admits a C^∞ extension g on B , then $g \in \mathcal{C}_B$.

Remark that \mathcal{C} verifies the conditions (C1)–(C4)(C6)(C7) of [RSW03] (the extension part of condition (C3) holds because any compact box $B \subset \mathcal{U}$ can be extended to a compact box B' with $B \subset \text{Int}(B') \subset B' \subset \mathcal{U}$). Theorem 3.3 then holds provided that the condition C5 of [RSW03] holds, this means that the quasianalyticity of $\mathcal{A}_n(h)$ implies the quasianalyticity of the algebras \mathcal{C}_n of [RSW03] (\mathcal{C}_n is the algebra of the germs at 0 of the maps in \mathcal{C}_B for all box B with $0 \in \text{Int}(B)$). It then suffices to check that the germ at $0 \in \mathbb{R}^n$ of any function $x \mapsto f(a+x)$ belongs to $\mathcal{A}_n(h)$ for any $f \in \mathcal{C}_B$ and any $a \in B \subset \mathbb{R}^n$.

Each function f in \mathcal{C}_B is obtained from (1) and (2) by applying finitely many operations from (3)–(7). We proceed by induction on the number of operations needed to construct f . If no operation is needed, the germ of $x \mapsto f(a+x)$ is in $\mathcal{A}(h)$ either by applying a constant operator, or because of the first condition in the definition of $\mathcal{A}(h)$. If f is obtained by (3) the claim holds by applying the composition operator. If $f = \partial g / \partial x_i$ is obtained by (5), the germ at $0 \in \mathbb{R}^n$ of $x \mapsto f(a+x)$ is obtained as $[D_{x_{n+1}}(g_a(x_1, \dots, x_{i-1}, x_i + x_{n+1}, x_{i+1}, \dots, x_n) - g'_a)] \circ \phi$ where g_a is the germ of $x \mapsto g(a+x)$, g'_a is the germ $(x, x_{n+1}) \mapsto g_a(x)$, D_{x_n} is the operator of division by x_{n+1} and ϕ_1 is the germ of $x \mapsto (x_1, \dots, x_n, 0)$. If f is obtained by (6), remark that $\mathcal{A}(h)$ is closed by taking implicit function with last partial derivative not necessary 1, by multiplying by a constant before applying the operator. Finally, the germ of a monomial division $g(x)/x_i$ is obtain at $x_i = 0$ by the corresponding operator, and at $x_i = a_i \neq 0$ by applying the implicit function operator to $(x, x_{n+1}) \mapsto g(a+x) - x_{n+1}(a_i + x_i)$. \square

3.2. Two Lemmas on operators

According to theorem 3.3, a restricted strongly transcendental function h is o-minimal as soon as the algebras $\mathcal{A}_n(h)$ are quasianalytic. In order to

obtain this quasi-analyticity, we need two properties, that were introduced in [LGR08]. The following express the algebraic behavior of the formal operators. We denote by $|\beta| = \beta_1 + \dots + \beta_k$ the norm of the multi-index $\beta = (\beta_1, \dots, \beta_k)$.

LEMMA 3.4. — *Let $\mathcal{L} : \mathcal{F}_{i_1}, \dots, \mathcal{F}_{i_m} \rightarrow \mathcal{F}_n$ be an operator. Then, there exists $(a_1, \dots, a_k) \in \mathbb{R}^k$, and for all $\alpha \in \mathbb{N}^n, \ell \in \{1, \dots, m\}$, there exist $\beta_{\ell, \alpha} \in \mathbb{N}^{i_\ell}$ and $P_\alpha \in \mathbb{Q}[x_1, \dots, x_{k+|\beta_{1, \alpha}|+\dots+|\beta_{m, \alpha}|}]$ such that, for all (f_1, \dots, f_m) in the definition domain of \mathcal{L} , we have*

$$\widehat{\mathcal{L}}(\widehat{f}_1, \dots, \widehat{f}_m) = \sum_{\alpha \in \mathbb{N}^n} P_\alpha(a_1, \dots, a_k, f_{1,0}, \dots, f_{1,\beta_{1,\alpha}}, \dots, f_{m,0}, \dots, f_{m,\beta_{m,\alpha}})x^\alpha$$

where $\widehat{f}_p = \sum_{\beta \in \mathbb{N}^{i_p}} f_{p,\beta}x^\beta$.

Proof. — One can check that the statement of the lemma is preserved by composition: if $\mathcal{L}, \mathcal{M}_1, \dots, \mathcal{M}_j$ verify the conclusion of 3.4 then also does $\mathcal{L}(\mathcal{M}_1, \dots, \mathcal{M}_j)$ (see [LGR08] for more details). Hence, by an induction on the number of operators which appear in the definition of \mathcal{L} , the lemma is true if it holds for elementary operators. Now, the statement is obvious for constant operators and monomial divisions (coefficients a_1, \dots, a_k appear for constants operators. They are the coefficients of the polynomial). An easy computation shows that it holds also for composition operators. At last, if \mathcal{L} is an implicit function operator, the coefficients of $\widehat{\mathcal{L}}(\widehat{f})$ are obtained from those of \widehat{f} by solving a triangular system, with coefficients 1 on the diagonal : recall that we define \mathcal{L} only for maps f with $\partial_{n+1}f(0) = 1$. Hence, the maps P_α of the statement are actually polynomials, and not rational maps. \square

The following is a quasi-analyticity property for operators.

LEMMA 3.5. — *Let \mathcal{L} be an operator. Then, $\widehat{\mathcal{L}} = 0$ imply $\mathcal{L} = 0$.*

Proof. — This is lemma 2.4 in [LGR08]. We recall the main steps.

First remark that the operators are continuous. Precisely, let $\mathcal{L} : \mathcal{F}_{i_1} \times \dots \times \mathcal{F}_{i_m} \rightarrow \mathcal{F}_n$ be an operator, $F = (f_1, \dots, f_m)$ be maps whose germs at 0 belong to the definition set of \mathcal{L} . Then there exist a compact neighborhood \mathcal{U} of $0 \in \mathbb{R}^{i_1+\dots+i_m}$, a compact neighborhood \mathcal{V} of $0 \in \mathbb{R}^n$, a neighborhood \mathcal{W} of F in $C^\infty(\mathcal{U}, \mathbb{R}^k)$ and a well defined operator \mathcal{L}' on \mathcal{W} , continuous for the C^∞ -topology on $C^\infty(\mathcal{U})$ and $C^\infty(\mathcal{V})$, which acts on the elements of \mathcal{W} as \mathcal{L} does on germs. This is obtained by an induction on the number of

elementary operators which are involved in \mathcal{L} . Continuity remains true by induction, and is a well known fact for each elementary operators.

Fix now an operator \mathcal{L} , and $F = (f_1, \dots, f_k)$ in its definition domain. By the continuity of \mathcal{L} , we find a compact neighborhood \mathcal{U} of 0 and a neighborhood \mathcal{V} of 0 such that \mathcal{L} is well defined and continuous on a neighborhood \mathcal{W} of F in $C^\infty(\mathcal{U}, \mathbb{R}^k)$. Remark that, if $G \in \mathcal{W}$ is analytic, $\mathcal{L}(G)$ is analytic also: each elementary operator preserves analyticity. Suppose now that $\widehat{\mathcal{L}} = 0$. Then, for any analytic $G \in \mathcal{W}$, $\widehat{\mathcal{L}}(G) = 0$, that means, since $\mathcal{L}(G)$ is analytic, that $\mathcal{L}(G) = 0$. Remind us that analytic functions are dense in $C^\infty(\mathcal{U})$. Being continuous and vanishing on a dense subset, \mathcal{L} vanishes on all \mathcal{W} . Hence, if $\widehat{\mathcal{L}} = 0$, \mathcal{L} vanishes on a neighborhood of any F in its definition domain, then everywhere. \square

3.3. Quasi-analyticity

The two previous lemmas are the main tools to prove that the algebras $\mathcal{A}_n(h)$ are quasi-analytic provided that h is strongly transcendental.

LEMMA 3.6. — *Let $h \in C^\infty(\mathcal{U})$ be strongly transcendental. Then, for all $n \in \mathbb{N}$, the algebra $\mathcal{A}_n(h)$ is quasi-analytic.*

Proof. — Fix $h \in C^\infty(\mathcal{U})$ a strongly transcendental function, $n \in \mathbb{N}$, and $g \in \mathcal{A}_n(h)$ with $\widehat{g} = 0$. We need to show that $g = 0$.

By the definition of $\mathcal{A}_n(h)$, there exists an operator \mathcal{L} such that g is the image by \mathcal{L} of the germs of h at some points b_1, \dots, b_ℓ of \mathcal{U} :

$$\exists (b_1, \dots, b_\ell) \in \mathcal{U}^\ell \setminus \Delta_\ell, g = \mathcal{L}(h_{b_1}, \dots, h_{b_\ell})$$

where h_{b_p} denote the germ at 0 of $t \mapsto h(b_p + t)$. According to lemma 3.4, there exist $(a_1, \dots, a_k) \in \mathbb{R}^k$, a family P_α of polynomials with integral coefficients, and a family of indexes $\beta_{p,\alpha} \in \mathbb{N}$ such that

$$\forall (f_1, \dots, f_j), \widehat{\mathcal{L}}(\widehat{f}_1, \dots, \widehat{f}_j) = \sum_{\alpha \in \mathbb{N}^n} P_\alpha(a_1, \dots, a_k, j_1^{\beta_{1,\alpha}} f_1(0), \dots, j_1^{\beta_{j,\alpha}} f_j(0)) x^\alpha.$$

Applying this formula to $(h_{b_1}, \dots, h_{b_\ell})$, we obtain

$$0 = \widehat{g} = \sum_{\alpha \in \mathbb{N}^n} P_\alpha(a_1, \dots, a_k, j_\ell^{\beta_\alpha} h(b)) x^\alpha, \quad (3.2)$$

where $\beta_\alpha = \max_{p=1, \dots, \ell} (\beta_{p,\alpha})$ and $b = (b_1, \dots, b_\ell)$.

Now recall that h is strongly transcendental, and let us make a remark on the algebraic independence of $(a_1, \dots, a_k, j_\ell^m h(b))$: there exists an integer

N such that the family $\{h^{(p)}(b_q); p \geq N + 1, q = 1, \dots, j\}$ is algebraically independent over $\mathbb{Q}(a_1, \dots, a_k, j_\ell^N h(b))$. Indeed, denote by d_m the transcendence degree

$$d_m = \text{trdeg}(a_1, \dots, a_k, j_\ell^m h(b)),$$

and recall that there exists C such that,

$$\text{trdeg}(j_\ell^m h(b)) \geq (m + 1)\ell - C.$$

Since $d_m \geq \text{trdeg}(j_\ell^m h(b))$, we get $d_m \geq m\ell - C$. On the other hand, we have $d_m - d_{m-1} \leq \ell$. Hence, there are at most $C + d_0$ indexes m such that $d_m - d_{m-1} < \ell$. The researched integer N is the maximum of these indexes.

Let us now express the equality (2) in another way : we set

$$Q_\alpha(y_1, \dots, y_{(\beta_\alpha - N)\ell}) = P_\alpha(a_1, \dots, a_k, j_\ell^N h(b), y_1, \dots, y_{(\beta_\alpha - N)\ell}),$$

where the coefficients of the polynomials Q_α belong to $\mathbb{Q}[a_1, \dots, a_k, j_\ell^N h(b)]$. The equality (2) becomes:

$$0 = \sum_{\alpha \in \mathbb{N}^n} Q_\alpha((h^{(p)}(\widetilde{x}_q))_\alpha) x^\alpha,$$

where $(h^{(p)}(\widetilde{x}_q))_\alpha$ stands for the $(\beta_\alpha - N)\ell$ -tuple formed by all $h^{(p)}(b_q)$ with $p = N + 1, \dots, \beta_\alpha$ and $q = 1, \dots, j$. Since the families $(h^{(p)}(\widetilde{x}_q))_\alpha$ are algebraically independent over the coefficients of the polynomials Q_α , it shows that for all α , $Q_\alpha = 0$. In other word, $\widehat{\mathcal{L}}(\widehat{f}_1, \dots, \widehat{f}_j)$ vanishes as soon as $j_1^N f_i(0) = j_1^N h(b_i)$, for $i = 1, \dots, \ell$. Hence, if we set \mathcal{N} to be the operator

$$\mathcal{N}(f_1, \dots, f_\ell) = \mathcal{L}(H_{b_1}^N + t^{N+1} f_1, \dots, H_{b_\ell}^N + t^{N+1} f_\ell),$$

where $H_{b_i}^N$ denote the Taylor expansion of h to the order N at b_i , we have $\widehat{\mathcal{N}} = 0$. According to lemma 3.5, it follows that $\mathcal{N} = 0$. Hence we have

$$g = \mathcal{L}(h_{b_1}, \dots, h_{b_\ell}) = \mathcal{N}((h_{b_1} - H_{b_1}^N)/t^N, \dots, (h_{b_\ell} - H_{b_\ell}^N)/t^N) = 0.$$

Since for any $g \in \mathcal{A}_n(h)$, $\widehat{g} = 0$ imply $g = 0$, the algebra $\mathcal{A}_n(h)$ is quasi-analytic. \square

We now prove that \mathbb{R}_h is o-minimal if h is a restricted transcendental function.

Proof. — [Proof of theorem 1.2] Let h be a restricted strongly transcendental function. Then there exists \tilde{h} a strongly transcendental function defined in an open neighborhood \mathcal{U} of $[0, 1]$, such that $\tilde{h}|_{[0, 1]} = h$. The algebras

$\mathcal{A}_n(\tilde{h})$ are quasi-analytic by lemma 3.6, so, according to theorem 3.3, $\mathbb{R}_{\mathcal{H}}$ is o-minimal, polynomially bounded and admits C^∞ -cell decomposition, where \mathcal{H} is the collection of all restrictions of \tilde{h} to closed intervals included in \mathcal{U} . Since \mathbb{R}_h is a reduct of $\mathbb{R}_{\mathcal{H}}$, theorem 1.2 holds. \square

4. Corollaries

In this section, we apply the above theorems to give new proofs of some results on o-minimality. In [LGR08], it is asked if the method developed there allows to construct an o-minimal structure that does admit smooth cell decomposition, but not analytic cell decomposition, the main interest of such a proof would be to circumvent the use of the Mandelbrojt's theorem needed in [RSW03]. In one sense, we answer here by the positive. Actually, we do not construct such a structure, but prove its existence.

COROLLARY 4.1. — *There exists a residual subset \mathfrak{T} of restricted C^∞ -function such that, for all function h in \mathfrak{T} , \mathbb{R}_h is o-minimal, admits C^∞ -cell decomposition, but does not admit analytic cell decomposition.*

Proof. — It suffices to recall that the set of restricted C^∞ -functions that are nowhere analytic is residual. Hence, its intersection \mathfrak{T} with restricted strongly transcendental functions is residual, and has the required properties. \square

We also obtain the existence of non compatible o-minimal structures. The interest of this new proof is again to circumvent the use of the Mandelbrojt's theorem needed in [RSW03].

COROLLARY 4.2. — *There exist h_1 and h_2 such that \mathbb{R}_{h_1} and \mathbb{R}_{h_2} are o-minimal, but \mathbb{R}_{h_1, h_2} is not o-minimal.*

Proof. — Fix h a restricted C^∞ -function. We claim that there exist two restricted strongly transcendental functions h_1, h_2 such that $h = h_1 + h_2$. Indeed, let ψ denotes the map, defined on $C^\infty([0, 1])$ by $\psi(g) = h - g$. This map conserves genericity. Hence, the intersection $\psi(\mathfrak{st}) \cap \mathfrak{st}$ is residual, then non empty. A function h_1 belongs to this intersection if there exists h_2 in \mathfrak{st} such that $\psi(h_2) = h_1$, this means $h = h_1 + h_2$. The corollary then stands by choosing an h such that \mathbb{R}_h is not o-minimal : \mathbb{R}_{h_1} and \mathbb{R}_{h_2} are o-minimal, since h_1 and h_2 are strongly transcendantal, while \mathbb{R}_{h_1, h_2} is not, since it defines h . \square

One can improve the last, asking h_2 to be analytic. It gives the following.

COROLLARY 4.3. — *There exists a function h such that \mathbb{R}_h is o-minimal, while $\mathbb{R}_{h,\text{an}}$ is not o-minimal.*

Proof. — Choose an $R > 0$, and a C^∞ -function f on a neighborhood \mathcal{U} of $[0, 1]$, such that $\mathbb{R}_{f|_{[0,1]}}$ is not o-minimal. According to proposition 2.2, the set of strongly transcendental functions is residual in $f + \mathcal{A}_R$, in particular not empty. So, there exists a strongly transcendental function $\bar{h} \in C^\infty(\mathcal{U})$ such that $f - \bar{h}$ is analytic. Define $h = \bar{h}|_{[0, 1]}$. The structure \mathbb{R}_h is o-minimal, since h is strongly transcendental, but $\mathbb{R}_{h,\text{an}}$ is not, since it defines $f|_{[0, 1]}$. \square

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