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ŁOJASIEWICZ INEQUALITY ON NON-COMPACT DOMAINS AND SINGULARITIES AT INFINITY

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We give a version of the Łojasiewicz inequality for the real polynomials on non-compact domains. The inequality takes in account not only distance to a fiber, but also distance to a polar set. It improves the recent results of [D. S. Tiep, H. H. Vui and N. T. Thao, Łojasiewicz inequality for polynomial functions on non-compact domains, *Int. J. Math.* **23**(4) (2012), Article ID: 1250033, 28 pp., doi:10.1142/S0129167X12500334], since we consider a distance to a smaller set. Then we use this new version of the inequality to obtain a sufficient condition for the existence of a vanishing component at infinity for real polynomials in several variables.

Keywords: Real polynomial; global Łojasiewicz inequality; vanishing component.

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1. Introduction

The (classical) Łojasiewicz inequality [3, 4] asserts that, if f is a real analytic function in a neighborhood of a compact set $K \subset \mathbb{R}^n$, then there exist some constants $c > 0$, $\alpha > 0$ such that

$$|f(x)| \geq cd(x, V)^{\alpha} \quad \text{for } x \in K,$$

where $V := \{x \in \mathbb{R}^n : f(x) = 0\}$ is the zero set of f, and $d(,)$ is the Euclidean distance in $\mathbb{R}^n$. The smallest value of α such that the inequality holds is called the Łojasiewicz exponent of f on K.

In general, this inequality does not hold if K is not compact. However, recently, the authors of [7, 6] propose in a non-compact case to replace in the Łojasiewicz inequality the distance to the zero set V by the distance to a possibly larger set $\widetilde{V}$ which contains V. In particular, [6, Theorem 4.2], based on the Ekeland variational principle introduced in [2], claims that if $f(x_1, \ldots, x_n)$ is a real polynomial, monic and of positive degree m in x_1, then there exist some positive real numbers δ, c, α such that for all $x \in f^{-1}(D_\delta)$,

$$|f(x)| \geq cd(x, \widetilde{V})^\alpha,$$

where $D_\delta := (-\delta, \delta)$, and $\widetilde{V} := V \cup V_1 \cup \cdots \cup V_{m-1}$ with, for $i = 1, \ldots, m-1$, $V_i := \big(\frac{\partial^i f}{\partial x_1^i}\big)^{-1}(0)$.

We prove in Sec. 2, Theorem 2.1, which is a version of the Łojasiewicz inequality in the same setup, but with $\widetilde{V} = V \cup V_1$. In Sec. 3, we relate this inequality with singularities at infinity. More precisely, we give a sufficient condition for the existence of vanishing components at infinity for a polynomial in several variables (Theorem 3.1). Note that the full characterization of singularities at infinity for polynomials is known only in the case of two variables (see [1, 6]). We also construct a counterexample for [6, Question 5.23] about the link between sequences of the first type and vanishing component at infinity (Example 3.1).

2. Łojasiewicz Inequality on Non-Compact Domains

In this section, we prove a version of the Łojasiewicz inequality for monic polynomials in $\mathbb{R}^n$. Indeed, since our proof follows closely the proof of [6, Theorem 4.2] we simply show Lemma 2.1, which, used instead of [6, Lemma 4.1] in the proof of [6, Theorem 4.2], gives a proof of Theorem 2.1.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a polynomial. Assume that f is monic of positive degree m in x_1. Set

$$V_1 = \left\{x \in \mathbb{R}^n : \frac{\partial f}{\partial x_1}(x) = 0\right\},$$
$$\widehat{V} = V \cup V_1.$$

Since either $f(x_1, 0, \ldots, 0)$ or $\frac{\partial f}{\partial x_1}(x_1, 0, \ldots, 0)$ has odd degree in x_1, $\widehat{V}$ is not empty. We still set $D_\delta := (-\delta, \delta)$, and denote by $d(\cdot, \cdot)$ the Euclidean distance in $\mathbb{R}^n$. If $V = \emptyset$, recall that $d(x, V) = +\infty$. Our version of the Łojasiewicz inequality is the following.

Theorem 2.1. *There exist some positive real numbers* δ, c, α, *such that*

$$|f(x)| \geq cd(x, \widehat{V})^\alpha \quad \textit{for all } x \in f^{-1}(D_\delta).$$

The proof is deduced from Lemma 2.1. We need a definition to introduce it.

Definition 2.1. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a polynomial. A sequence $x^k \to \infty$ in $\mathbb{R}^n$ is said to be *a sequence of the first type of* f if $f(x^k) \to 0$ and $d(x^k, V) \geq \sigma$, for some constant $\sigma > 0$.

Remark 2.1. A sequence of the first type is a sequence along which the classical Łojasiewicz inequality fails for any positive values of c, α.

Some examples of sequence of the first type will be given at the end of the paper. The following lemma is the key to prove Theorem 2.1.

Lemma 2.1. *Let $x^k = (x_1^k, \ldots, x_n^k)$ be a sequence in $\mathbb{R}^n$ such that $f(x^k) \to 0$. Then $d(x^k, \widehat{V}) \to 0$.*

Proof. We proceed by contradiction. Assume that there exists a subsequence b^k of x^k such that $d(b^k, \widehat{V}) > \sigma$ for some $\sigma > 0$. Without loss of generality, we may assume that $b^k = x^k$ for every k.

Set $f_k(x_1) := f(x_1, x'^k) = f(x_1, x_2^k, \ldots, x_n^k)$. So $f_k \in \mathbb{R}_m[x_1]$, where $\mathbb{R}_m[x_1]$ is the space of polynomials in x_1 with degree bounded by m. Since f is monic in x_1, it follows that f_k is monic in x_1. Hence, without loss of generality, we may suppose that f_k is of the following form

$$f_k = x_1^m + \sum_{i=1}^{m} a_i(x'^k) x_1^{m-i},$$

where a_i are polynomials in $n-1$ variables $x_2, \ldots, x_n$.

Since $d(x^k, V) > \sigma$, the polynomial f_k has no root in the interval $I_k = [x_1^k - \sigma, x_1^k + \sigma]$ for any k. By the same way, the derivative f_k' does not have any root in I_k. Hence f_k is monotone and has constant sign on I_k. Without loss of generality, we may assume that $f_k, f_k' > 0$ on I_k. So f_k is positive and increasing on I_k. We have

$$\max_{x_1 \in [x_1^k - \sigma, x_1^k]} f_k(x_1) = f_k(x_1^k) = f(x^k) \to 0.$$

Set $\tilde{f}_k(x_1) = f_k(x_1 + x_1^k)$. Then

$$\max_{x_1 \in [-\sigma, 0]} \tilde{f}_k(x_1) = f(x^k) \to 0. \tag{2.1}$$

It is clear that $\max_{x_1 \in [-\sigma, 0]}$ is a norm on $\mathbb{R}_m[x_1]$. By the fact that $\mathbb{R}_m[x_1]$ is a space of finite dimension, all norms on $\mathbb{R}_m[x_1]$ are equivalent. Now by (2.1), it follows that each coefficient of $\tilde{f}_k$ must tend to 0. On the other hand, since f_k is monic in x_1, $\tilde{f}_k$ is monic in x_1. Hence $\tilde{f}_k$ contains the term x_1^m, which implies that a coefficient of $\tilde{f}_k$ does not tend to 0. This contradiction ends the proof of the lemma. □

We let the reader check that, by using Lemma 2.1 in place of [6, Lemma 4.1], the proof of Theorem 2.1 is similar to the proof of [6, Theorem 4.2].

3. Vanishing Components at Infinity

In this section, we give a sufficient condition for the existence of vanishing components at infinity for polynomials in at least two variables. This follows up the result of [6] in the case of polynomials in two variables. First of all, we give some definitions.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a polynomial with $n \geq 2$. Thom proves in [5] that there exists a finite set B such that

$$f : \mathbb{R}^n \backslash f^{-1}(B) \to \mathbb{R} \backslash B$$

is a local C^∞-trivial fibration. The smallest of such sets B, denoted by $B(f)$, is called the set of bifurcation values of f. The set $B(f)$ consists of the set Σ_f of critical values of f and the set $B_\infty(f)$ of atypical values at infinity of f. A value $t_0 \in \mathbb{R}$ is said to be a *typical value at infinity* of f if there exist $\delta > 0$, $r > 0$ such that the map $f : f^{-1}(D_\delta) \backslash \mathbb{B}_r \to D_\delta$ is a C^∞-trivial fibration, where $\mathbb{B}_r = \{x \in \mathbb{R}^n : \|x\| \leq r\}$. Otherwise, t_0 is called an *atypical value at infinity* of f.

So far, the characterization of atypical values at infinity for polynomials in several variables is not known. Atypical values at infinity appear, in particular, when f has a so-called vanishing component at infinity. We follow Tibar and Zaharia [8] for the precise definition.

Definition 3.1 ([8]). We say that f *has a vanishing component at infinity as* $t \to t_0$ if for every $\varepsilon > 0$ and every $R > 0$ there exists $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$, such that $f^{-1}(t)$ admits a connected component W_t with $W_t \cap \mathbb{B}_R = \emptyset$.

Atypical values at infinity and vanishing components are linked by the fact that, if f has a vanishing component at infinity as t tends to t_0, then $t_0 \in B_\infty(f)$. In the following, we give a sufficient condition for f to admit a vanishing component at infinity when $t \to t_0$. The proof is based on our Łojasiewicz inequality (Theorem 2.1). We keep the notations $V := f^{-1}(0)$, $V_1 := \left(\frac{\partial f}{\partial x_1}\right)^{-1}(0)$, and we set $d(V, V_1 \backslash \mathbb{B}_r) := \inf_{x \in V,\, y \in V_1 \backslash \mathbb{B}_r} d(x, y)$.

Theorem 3.1. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a polynomial which is monic and of positive degree in x_1. Assume that f has a sequence of the first type. Suppose that*

$$d(V, V_1 \backslash \mathbb{B}_r) = \inf_{x \in V, y \in V_1 \backslash \mathbb{B}_r} d(x, y) > 0 \tag{3.1}$$

for $r \gg 1$. Then f admits a vanishing component at infinity as t tends to 0. Hence $0 \in B_\infty(f)$.

Proof. If $A \subset \mathbb{R}^n$ is a subset of $\mathbb{R}^n$, we denote by $T_\rho(A)$ the closed tubular neighborhood of radius ρ of A, this means, $T_\rho(A) := \bigcup_{x \in A} (\mathbb{B}_\rho + x)$.

Fix $r > 0$ such that $d(V, V_1\backslash\mathbb{B}_r) > 0$ and $\sigma > 0$ such that f has a sequence of the first type x^k with $d(x^k, V) > \sigma$ for all k. Let $\rho < \min\left\{\frac{d(V,V_1\backslash\mathbb{B}_r)}{2}, \sigma\right\}$. Replacing, if necessary x^k by a subsequence, we may assume that $x^k \notin \mathbb{B}_{r+\rho}$ for any k and x^k is contained in the same connected component U_δ of $f^{-1}(D_\delta)\backslash\mathbb{B}_{r+\rho}$. Then by the assumption,

$$T_\rho(V) \cap T_\rho(V_1\backslash\mathbb{B}_r) = \emptyset, \tag{3.2}$$

where $T_\rho(V), T_\rho(V_1\backslash\mathbb{B}_r)$ denote respectively the closed tubular neighborhoods of radius ρ of V and $V_1\backslash\mathbb{B}_r$.

Let $\delta < c\rho^\alpha$, where δ, c, α are the constants defined in Theorem 2.1. By Theorem 2.1, for $x \in f^{-1}(D_\delta)$, we have $\delta > |f(x)| \geq cd(x, \widehat{V})^\alpha$, so $d(x, \widehat{V}) < \rho$. Hence $f^{-1}(D_\delta) \subset T_\rho(\widehat{V})$. Thus $U_\delta \subset T_\rho(\widehat{V})\backslash\mathbb{B}_{r+\rho}$. Note that

$$T_\rho(\widehat{V})\backslash\mathbb{B}_{r+\rho} = T_\rho(V) \cup T_\rho(V_1)\backslash\mathbb{B}_{r+\rho} \subset T_\rho(V) \cup T_\rho(V_1\backslash\mathbb{B}_r).$$

Thus $U_\delta \subset T_\rho(V) \cup T_\rho(V_1\backslash\mathbb{B}_r)$. Since $d(x^k, V) \geq \sigma > \rho$, it follows that $x^k \notin T_\rho(V)$. So by the facts that $T_\rho(V) \cap T_\rho(V_1\backslash\mathbb{B}_r) = \emptyset$, $x^k \in U_\delta$ and U_δ is connected, we deduce that

$$U_\delta \subset T_\rho(V_1\backslash\mathbb{B}_r).$$

Now let W_k be the connected component of $f^{-1}(f(x^k))$ containing x^k. It is clear that $Y_k := W_k\backslash\mathbb{B}_{r+\rho} \subset U_\delta$.

Lemma 3.1. *Let $r_1 > r + \rho$, there exists $N = N(r_1)$ such that for every $k \geq N$, we have $Y_k \cap \mathbb{B}_{r_1} = \emptyset$.*

Proof. Suppose to the contrary, that there exists a subsequence y^{l_k} of x^k such that $Y_{l_k} \cap \mathbb{B}_{r_1} \neq \emptyset$. Replacing x^k by a subsequence if necessary, we may assume that $y_{l_k} = x^k$ and $x^k > r_1$ for all k. So $Y_k \cap \mathbb{S}_{r_1} \neq \emptyset$, where $\mathbb{S}_{r_1} = \{x \in \mathbb{R}^n : \|x\| = r_1\}$. Let $z^k \in Y_k \cap \mathbb{S}_{r_1}$. Since $\mathbb{S}_{r_1}$ is compact, z^k has a convergent subsequence. Again, up to replace x^k and z^k by some subsequences, we may assume that $z^k \to z_0 \in \mathbb{S}_{r_1}$. So we have $f(z_0) = \lim_{k\to 0} f(z^k) = \lim_{k\to 0} f(x^k) = 0$ and then $z_0 \in V$. It is clear that z_0 is contained in the closure of U_δ. Hence $z_0 \in T_\rho(V_1\backslash\mathbb{B}_r)$. Consequently $z_0 \in V \cap T_\rho(V_1\backslash\mathbb{B}_r)$ which contradicts (3.2). This ends the proof of the lemma. □

By Lemma 3.1, for $k \geq N$, we have $Y_k \cap \mathbb{B}_{r_1} = \emptyset$. Thus Y_k is actually a connected component of $f^{-1}(f(x^k))$ and $Y_k = W_k$. So by Lemma 3.1, W_k is a vanishing component at infinity of $f^{-1}(f(x^k))$. The theorem is proved. □

Remark 3.1. We have seen during the proof that f admits a vanishing component at infinity as t tends to 0, which is contained in U_δ. In other words, the sequence of the first type x^k "accompanies" a vanishing component.

The following example shows that the existence of a sequence of the first type does not necessarily imply the existence of a vanishing component at infinity. This is a counterexample for [6, Question 5.23].

Example 3.1. Let $f(x,y,z) = (y^2+(xy-1)^2)z$. We have $V = \{f = 0\} = \{z = 0\}$. Let $s^k = (k, \frac{1}{k}, 1)$, then $f(s^k) = \frac{1}{k^2} \to 0$ and $d(s^k, V) = 1$. So s^k is a sequence of the first type. On the other hand, let $\epsilon \neq 0$, then the level surface $\{f = \epsilon\}$ has only one connected component which is the graph $z = \frac{\epsilon}{y^2+(xy-1)^2}$ over the plan $\{z = 0\}$. Hence f is a globally trivial C^∞ fibration, so f has no bifurcation values. In particular, 0 is not a bifurcation value of f. Consequently, f does not have a vanishing component at infinity as $t \to 0$.

The following example shows that if we replace the condition (3.1) by the condition $V \cap (V_1 \backslash \mathbb{B}_r) = \emptyset$, there may be no vanishing components at infinity of f in a connected component of $f^{-1}(D_\delta)$ which contains a sequence of the first type.

Example 3.2. Let $f(x,y,z) = (y^4+(xy-1)^2)(y^2-z^2-1)(y-z)$ which is monic with respect to y. Then

$$V = \{y = z\} \cup \{y^2 - z^2 - 1 = 0\}.$$

Let $s_k = (k, \frac{1}{k}, 1)$, then $f(s^k) = \frac{1}{k^4}(\frac{1}{k^2} - 2)(\frac{1}{k} - 1) \to 0$ and $d(x, V) \to \frac{1}{\sqrt{2}}$. Hence s^k is a sequence of the first type.

We have

$$\frac{\partial f}{\partial y} = (y^4+(xy-1)^2)(3y^2 - 2yz - z^2 - 1) \\ + (4y^3 + 2x(xy-1)(y^2-z^2-1)(y-z)).$$

Set $V_1 = \{\frac{\partial f}{\partial y} = 0\}$. Let us check that $V \cap V_1 = \emptyset$. Indeed,

(i) assume that $y = z$, then

$$\frac{\partial f}{\partial y} = (y^4+(xy-1)^2)(3y^2 - 2y^2 - y^2 - 1) = -(y^4+(xy-1)^2) \neq 0$$

for all x, y;

(ii) assume that $y = \pm\sqrt{z^2+1}$, then

$$\frac{\partial f}{\partial y} = (y^4+(xy-1)^2)(3(z^2+1) \pm 2z\sqrt{z^2+1} - z^2 - 1) \\ = (y^4+(xy-1)^2)(2(z^2+1) - 2z\sqrt{z^2+1}) \neq 0$$

for all x, z. So $V \cap V_1 = \emptyset$.

Now let us prove that the connected component U_δ of $f^{-1}(D_\delta)$ containing s^k does not contain a vanishing component at infinity of f. Let $\epsilon < \delta$. Let y_ϵ be the smallest positive solution of the equation $y^4(y^2-2)(y-1) = \epsilon$. Then for $\epsilon \ll 1$, we have $y_\epsilon \sim \epsilon^{\frac{1}{4}}$. Let γ be the curve defined by

$$\begin{cases} x(z) = \dfrac{1 - \sqrt{\frac{\epsilon}{(y_\epsilon^2 - z^2 - 1)(y_\epsilon - z)} - y_\epsilon^4}}{y_\epsilon}, \\ y = y_\epsilon, \\ z \in (y_\epsilon, 1]. \end{cases}$$

It is not difficult to check that γ is well-defined and $\gamma \subset f^{-1}(\epsilon) \cap U_\delta$. Set $z_\epsilon = y_\epsilon + \epsilon$. Then

$$\begin{aligned} x(z_\epsilon) &= \frac{1 - \sqrt{\frac{\epsilon}{(y_\epsilon^2 - (y_\epsilon + \epsilon)^2 - 1)(y_\epsilon - y_\epsilon - \epsilon)} - y_\epsilon^4}}{y_\epsilon} \\ &= \frac{1 - \sqrt{\frac{\epsilon}{(-\epsilon^2 - 2y_\epsilon \epsilon - 1)(-\epsilon)} - y_\epsilon^4}}{y_\epsilon} \\ &= \frac{1 - \sqrt{\frac{1}{\epsilon^2 + 2y_\epsilon \epsilon + 1} - y_\epsilon^4}}{y_\epsilon} \\ &= \frac{1 - \frac{1}{\epsilon^2 + 2y_\epsilon \epsilon + 1} + y_\epsilon^4}{y_\epsilon \left(1 + \sqrt{\frac{1}{\epsilon^2 + 2y_\epsilon \epsilon + 1} - y_\epsilon^4}\right)} \\ &= \frac{\frac{\epsilon^2 + 2y_\epsilon \epsilon}{\epsilon^2 + 2y_\epsilon \epsilon + 1} + y_\epsilon^4}{y_\epsilon \left(1 + \sqrt{\frac{1}{\epsilon^2 + 2y_\epsilon \epsilon + 1} - y_\epsilon^4}\right)} \\ &< \frac{\epsilon^2 + 2y_\epsilon \epsilon + y_\epsilon^4}{y_\epsilon}. \end{aligned}$$

Note that $y_\epsilon \sim \epsilon^{\frac{1}{4}}$, so $x(z_\epsilon) \to 0$. Consequently, the connected component of $f^{-1}(\epsilon)$ contained in U_δ is not vanishing at infinity.

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