SOME RESULTS ON HIGHER SUSLIN TREES

R. DAVID

Higher Suslin trees have become a tool in some forcing constructions in set theory (see, for example, [D1] and [D2]). Most of the constructions using $\omega_1$ Suslin trees can be extended to $\kappa^+$ Suslin trees for any regular cardinal $\kappa$. Some of these are given in §1.

In many such constructions, sequences of Suslin trees are used. In §II we show, in various ways, that the generalization to sequences, even $\omega$-sequences, of $\kappa^+$ Suslin trees cannot be done.

In these constructions the Suslin trees are used as forcing poset (the forcing adds a branch in the tree). There is another way to kill a Suslin tree, namely by adding a big antichain. Some results on this forcing are given in §III.

Our notation is standard. If $T$ is a tree and $x \in T$, then $|x|$ is the height of $x$ in $T$. We define $T_x$ (or $T(x)$) = $\{x \in T : |x| = \alpha\}$ and $T|_x = \{x \in T : |x| < \alpha\}$.

If $p$ and $q$ are forcing conditions, $p \leq q$ means that $p$ has more information than $q$.

If $(T_x : x \in I)$ is a sequence of trees, $\prod T_x$ will always mean the set of $(x_\beta : x \in I)$ such that $x_\beta \in T_x$ and $|x_\beta| = |x_\beta|$ for $x, \beta \in I$.

For functions $b, T, \ldots$ we denote by $b|_x, T|_x, \ldots$ their restriction to $x$.

If $x$ is a sequence of ordinals and $x$ is an ordinal, $x^+ x$ is the sequence obtained by concatenating $x$ at the end of $x$.

Acknowledgement. The referee pointed out many mistakes or mischiefs, made very useful remarks, and corrected some proofs which were incorrect. I would like to thank him for all that work.

§1. $\kappa^+$ Suslin trees. The basic construction of an $\omega_1$ Suslin tree (due to Jensen) is easily generalized to $\kappa^+$ Suslin trees for any regular $\kappa$.

Definition 1. Let $\kappa$ be a regular cardinal, $I$ a set of cardinality less than or equal to $\kappa$. The canonical $\kappa^+$ Suslin tree $T$ (resp. the canonical sequence $(T^x)_x \in I$ of $\kappa^+$ Suslin trees) is defined as follows.

The levels are defined by induction:

$T(\mu + 1)$ (resp. $T^x(\mu + 1)$) = $\{x^+ \xi : \xi < \kappa, x \in T(\mu)$ (resp. $T^x(\mu))\}$.

cf $\mu < \kappa$: extend all the branches.

cf $\mu = \kappa$: let $\eta(\mu)$ be the least $\eta$ such that $L_\eta \models ZF^- + \bar{\mu} = \kappa$ and $T|_\mu$ (resp. $(T^x|_\mu : x \in I)) \in L_\eta$.

Received October 24, 1984; revised June 17, 1986, and July 25, 1988.

© 1990, Association for Symbolic Logic
Let $C(\mu)$ be the set of forcing conditions defined by: $f \in C(\mu) \iff f: |f| < \kappa \rightarrow T |\mu$ (resp. $\bigcup_{\kappa \in T^*} |\mu|; f \leq g \text{ iff } |f| \geq |g| \& \forall \xi < \kappa \ f(\xi) \geq g(\xi)$. 

$C(\mu)$ clearly is $< \kappa$-closed; let $G(\mu)$ be the $L$-least $C(\mu)$-generic over $L_{g(\mu)}$, and define $B_k = \bigcup_{f \in G(\mu), \xi \in \text{dom } f} f(\xi)$. Define the branches in $T(\mu)$ (resp. $T^*(\mu)$) as the $ \{ B_k : \xi < \kappa \}$. It is easy to check, using the standard condensation argument, that $T$ is Suslin and that $\prod T^* = \{ p : I \rightarrow \text{dom } p; \text{card}(\text{dom } p) < \kappa; \forall \alpha \in \text{dom } p \ p(\alpha) \in T^* \}$ satisfies the $\kappa^+$-chain condition.

In [JJ], Jensen and Johnsbraten use an $\omega$-sequence of $\omega_1$ Suslin trees to define a $\Pi^2_2$ formula and build a generic real $a$ such that $L \models \exists x \phi(x), L(a) \models ZF + \phi(a) + \text{\textup{ZF}} + a' \models (ZF + \phi(a) + \phi(a') = 2^\aleph_0 \leq \aleph_1)$.

The basic tool is a construction of a Suslin tree $T$ which has the additional property that if in some extension of $L$ there are two distinct branches in $T$, then $2^\aleph_0$ is collapsed. This again can be easily generalized.

**Proposition 2.** Let $\kappa$ be a regular cardinal. There is in $L$ a $\kappa^+$ Suslin tree $T$ such that the following holds: for any $B \neq B'$ in some extension of $L$ if $L(B, B') \models ZF + B, B'$ are branches in $T$, then $(\kappa^+)^{L(B, B')} > (\kappa^+)^L$.

**Proof.** (Sketch.) Let $Q$ be an elementary extension of $Q$, the set of rational numbers, of cardinality $\kappa$ that is saturated (i.e., every set $A(x)$ of formulas which is finitely satisfiable in $Q$ and has cardinality less than $\kappa$ is satisfiable in $Q$). The proof follows exactly the one in [JJ] with $Q$ instead of $Q$.

**Proposition 3.** Let $\kappa$ be a regular cardinal. There is a $\Pi^2_2$ formula $\phi(x)$ such that:

1. $V = L = \mathbb{R} \times \kappa H_{\aleph_1} \models \phi(x)$;
2. $\kappa^+ = (\kappa^+)^L \Rightarrow \exists \in X \in \kappa H_{\aleph_1} \models \phi(X)$; and
3. there is a forcing notion $P$ definable in $L$ with parameter $\kappa$ that preserves cardinals and the cofinality function such that any $P$-generic extension of $L$ satisfies $3X \in \kappa H_{\aleph_1} \models \phi(X)$.

**Proof.** An immediate generalization of [JJ] (note: this kind of result will be used in a forthcoming paper which gives a $\Pi^2_2$ formula such that two distinct solutions collapse $\aleph_{\alpha+1}$).

### §II. $\omega$-sequences of Suslin trees.

**Proposition 1.** Let $(T_n)_{n \in \omega}$ be a sequence of $\omega_1$ Suslin trees such that, for every $m < \omega$, $\prod_{n < m} T_n$ is $\omega_1$-cc. Then $T_f = \text{the product (with finite support) of the } T_n$ is $\omega_1$-cc.

**Proof.** This follows from the fact (see [JE]) that the direct limit of ccc Boolean algebras is ccc.

**Note.** There is a sequence $(T_n)$ which is a slight variation of the canonical $\omega$-sequence of $\omega_1$ Suslin trees such that $T = \prod T_n$ (the product with full support) is $\omega_1$-distributive.

**Proof.** Change $C(\mu)$ in the following way: a condition is a sequence $f = (f_n)_{n \in \omega}$ such that:

1. $f \in L_{g(\mu)}$; and
2. $\forall n<\omega f_n, |f_n| < \omega \rightarrow T_n |\mu$.

Set $f \leq g$ iff $\forall n |f_n| \geq |g_n|$ and $\forall i < |g_n| f_n(i) \geq g_n(i)$.

It is easy to check that $\forall n<\omega \prod_{n < m} T_n$ is $\omega_1$-cc. I now prove that $T = \prod T_n$ is $\omega_1$-distributive. Let $(D_n)_{n < \omega}$ be a sequence of strongly dense subsets of $T, x = (x_n)_n \in T$. Use the traditional condensation argument to find $\mu < \beta < \omega_1$ and $\pi : L_{\beta} \rightarrow L_{\omega_2}$.
such that $\pi(\mu) = \omega_1$ and $\pi$ is elementary and $\pi(D_i \cap L_\mu) = D_i$. Let $f \in G(\mu)$ be such that $f(i) \supseteq \chi$ for a sequence $(i_n)$ in $L_\mu$. It is easy to check that if $y = B_{i_n}$ then $y \in \bigcap_{i < \alpha} D_i$.

Note 2. If $(T_n)$ is an $\omega$-sequence of $\omega_1$ trees, $T = \prod T_n$ (the product with full support) never is $\omega_1$-cc; then Note 1 shows that it can however preserve $\omega_1$. I do not know whether such a sequence can be found so that $T$ collapses $\omega_1$.

Mitchell (see [LS]) has built a sequence $(T_n)_{n < \omega}$ of $\omega_2$ Suslin trees such that $\prod_{n < m} T_n$ is $\omega_2$-cc for every $m < \omega$ but $T = \prod T_n$ is not $\omega_2$-cc. This can be improved by

**Proposition 2.** There is an $L$-definable sequence $(T_n)_{n < \omega}$ of $\omega_2$, $\omega$-closed Suslin trees such that

1) $\forall m < \omega \prod_{n < m} T_n$ is $\omega_2$-cc, and
2) $\omega_2^L$ is collapsed in every $T = \prod T_n$ generic extension of $L$.

(The conclusion follows from the weaker assumption: CH + $\diamondsuit(\{\mu < \omega_2: cf \mu = \omega_1\})$)

**Proof (Sketch).** Let $F = \{f: |f| < \omega_1 \rightarrow \omega_2; f$ increasing; $\forall x < |f|: x$ successor $\Rightarrow cf(f(x)) = \omega_1$ and $x$ limit $\Rightarrow f(\alpha) = \sup(f(\beta); \beta < \alpha)\}$. The idea is to define

(c): $f \in F$ such that $c_f \in \prod_\mathbb{N} T_n$ and

- $c_0 = (\text{root of } T_0)\gamma$;
- $\forall f, g \in F$
  - $b_1): |c_f| = sup f$,
  - $b_2): f \leq g$ $\Rightarrow c_f \leq c_g$,
  - $b_3): f$ incompatible with $g$ $\Rightarrow c_f \not\prec c_g$,
  - $b_4): f/|f|$ limit $\Rightarrow c_f = \bigcup_{x < |f|} c_f|x$,
  - $b_5): \{c_f\supseteq \chi = \omega_1\}$ is a maximal antichain in $T$ above $c_f$;
- $\forall \alpha \in \mathbb{N}$ $\{f\in F: c_f \leq \alpha\}$ has a unique maximal element, which will be denoted by $f_{\alpha}$.

The construction of the $(T_n)_n$ is essentially that of the canonical trees. A technical hypothesis has to be maintained so that the construction does not break and $\prod_{n < m} T_n$ is $\omega_2$-cc, for which I need the following definition.

Let $C$ be the set of $c = (p, X)$ where

- $a): \alpha = q \supseteq \omega_1 \rightarrow \bigcup_{n < \omega} |p|$ for some $|p| < \omega_2$, and
- $b): X$ is a countable set of functions $\varphi$ such that $\varphi: \omega \rightarrow \omega_2$ and $\forall n < \omega p(\varphi(n)) \in T_n$ (for $\varphi \in X$ let $p_\varphi = (p \setminus \varphi(n))_n$).

Define an order on $C$ by setting, for $c = (p, X)$ and $d = (q, Y)$, $c \leq d$ if $\alpha = \alpha_\mu$; $|p| \geq |q|; \exists \gamma; X \supseteq Y; \text{ and}$

- $\forall \xi < \alpha_\mu p(\xi) \geq q(\xi)$, $\forall \varphi \in Y f_{p_\varphi} = f_{q_\varphi}$ (i.e. $\forall g \in F c_g \leq p_\varphi$ iff $c_g \leq q_\varphi$).

The induction hypothesis to be maintained is:

$H(\mu)$: $\forall \nu < \mu \forall (q, X) \in C \forall Y \in \prod_{n < m} (T_n)_n$ if $|q| = \nu$ and $\forall n < \mu m (\mu(n) < \gamma_n$ then there is a $p$ such that $(p, X) \in C$ and $(p, X) \leq (q, X)$ and $\alpha_p = \alpha_q$ and $|p| = \mu$ and $\forall n < \mu p(\mu(n)) = \gamma_n$ (this means that countably many points can be extended in such a way that the $f_{\alpha_p}$ are preserved). Now replace (in the definition of the canonical trees) the forcing $C(\mu)$ by

$C(\mu) = \{(p, X) \in C: |p| < \mu\}$

It is routine to check that everything is as wanted.
Remark. The referee pointed out that this property is true for (almost) any $\omega$-sequence of disjoint subtrees of an $\omega_2$-super-Suslin tree. The following is due to him.

First, let us recall Shelah’s definition of $\omega_2$-super-Suslinity. Suppose $T$ is a normal $\omega_2$-tree. Let

$$\text{lev}_\omega(T) = \{(t_i; i < \omega): \text{ for all } i, j \in \omega, t_i \neq t_j \text{ and } \text{ht}(t_i) = \text{ht}(t_j)\}. $$

Letting $t = (t_i; i < \omega)$ and $t' = (t'_i; i < \omega)$, we define $[\text{lev}_\omega(T)]^2 = \{(t, t') \in \text{Lev}_\omega(T)^2: t_i < t'_i \text{ for all } i < \omega\}$, $T$ is $\omega_2$-super-Suslin if there is an $H: [\text{lev}_\omega(T)]^2 \rightarrow \omega_1$ such that whenever $H(||t, t'||) = H||(t, t')||$, letting $t^4 = (t^4_i; i < \omega)$, there is an $i < \omega$ such that $t^4_i$ is compatible with $t_i^1$.

Two key facts about $\omega_2$-super-Suslin trees are, first, that so long as $\mathbb{N}_2$ is not collapsed and no reals are added to a model of set theory in which $T$ is super-Suslin, if $X < T$ is an antichain such that $t \in X$ and $[t' \in T: t' \geq t]$ is not $\omega_2$ Suslin, then $X$ is finite; second, the existence of an $\omega_2$-super-Suslin tree follows from CH and the existence of an $(\omega_1, 1)$ morass. These facts are established in [S-ST].

If $T$, as witnessed by $H$, is an $\omega_2$-super-Suslin tree in $L$ and $(t_n; n \in \omega) \in \text{lev}_\omega(T)^\omega$, then, letting $T_n = T|t_n = \{t \in T: t \sim t_n\}$, we see that $\prod_{n \in \omega} T_n$ collapses $\omega_2$. This can be seen by supposing $b = (b_n; n \in \omega)$ is a $(\prod_{n \in \omega} T_n)^l$-generic sequence of branches and considering $[(t^4_i; i \in \omega): \delta < \alpha < \omega_2]$, where $\delta = \text{ht}(t_n)(\text{for all } n)$ and $t^4_i$ is chosen so that $\text{ht}(t^4_i) = \alpha + 2$ and $b_1|\alpha + 1 = t^4_i|\alpha + 1$ and $t^4_i$ is incompatible with $b_1|\alpha + 2$. Since $(b_n|\alpha + 2; n \in \omega) \in L$, we may insist that, for each $n, (t^4_i; i \in \omega) \in L$ and, hence, $(t_n; n \in \omega), (t^4_i; i \in \omega) \in \text{dom}(H)$. Then, if, for $\alpha \in \omega_2 \setminus (\delta + 1$) we have $f(\alpha) = H((t_n; n \in \omega), (t^4_i; i \in \omega))$, then $f$ is one-to-one and witnesses that $[\omega_2]^{\aleph_0} \leq \omega_1$.

Now, if we knew for each $n$ that $\prod_{m < \omega} T_m \models "T_n \text{ satisfies the } \omega_2\text{-cc}"$, we could conclude that $\prod_{m \leq n} T_n$ satisfies the $\omega_2$-cc, since forcing with $(\prod_{m < \omega} T_n) \ast T_n$ is equivalent to forcing with $\prod_{m \leq n} T_n$. Of course, if $\prod_{m < \omega} T_m$ satisfies the $\omega_2$-cc (and thus has a dense subset which is an $\omega_2$-Suslin tree) and $(b_0, \ldots, b_{n-1})$ is a $\prod_{m < \omega} T_m$-generic sequence of branches, then no reals are added and no cardinals are collapsed in passing from $L$ to $L[b_0, \ldots, b_{n-1}]$, and, since $T$ is $\omega_2$-super-Suslin, only finitely many of the $\{T_m: m \geq n\}$ are not $\omega_2$ Suslin in $L[b_0, \ldots, b_{n-1}]$. However, since $T_n$ itself may fail to be $\omega_2$ Suslin in $L[b_0, \ldots, b_{n-1}]$, to obtain an alternate proof of Proposition 2, it is necessary to be a bit more careful in our choice of $(t_n; n \in \omega) \in L$.

We may assume $T \in L_{\omega_2}$. Choose $X$ such that $X < L_{\omega_2}$, with $(X; T \cap X, H \cap X) \prec (L_{\omega_2}; T, H)$, $H \in X$, and $X \cap \omega_2 = \alpha$, where $\text{cf}(\alpha) = \omega_1$. Let $\beta$ be such that $L_\beta \approx X$. Then $\beta^{-1}(\alpha) = \alpha$ and $\beta^{-1}(T) = T \cap L_\beta$. Define a sequence of nodes $(t_n; n \in \omega)$ as follows: let $t_0 \in T$, $t_0(\alpha) = \alpha$, be arbitrary. Then $t_0 = T \cap L_\beta$ generic over $L_\beta$ and $L_\beta[t_0] \models "\alpha = \omega_2"$. Let $t_0 < t_0$ be on the first level of $T$ (which we assume to be infinite). Since $L_\beta \models "T \cap L_\beta \text{ is super-Suslin}"$, there is a $t_1$ such that $t_1(\alpha) = 1$ and $L_\beta[t_0, t_1] \models "(T|t_0) \cap L_\beta \text{ is Suslin}"$.

Choose $\alpha_0 < \alpha$ such that $L_\beta \models "t_0|\alpha_0 \forces (T|t_0) \cap L_\beta \text{ is Suslin}\"$.

Choose $t_1 \in T|t_0$ of height $\alpha$. Then $L_\beta[t_0, t_1] \models "\alpha = \omega_2"$. Next choose $t_2$ of height 1 in $T$ such that $L_\beta[t_0, t_1] \models "(T|t_2) \cap L_\beta \text{ is Suslin}\"$ and pick $\alpha_1 < \alpha$, $\alpha_1 \geq \alpha_0$, such that

$$L_\beta \models "(t_0|\alpha_1, t_1|\alpha_2) \forces (T|t_0 \times T|t_1) \cap L_\beta \text{ is Suslin}\"."
Choose $t_2 > L_2$ such that $ht(t_2) = \alpha$. Continue in this manner to define $(t_n; n \in \omega)$ and $(\delta_n; n \in \omega)$. Let $s_\omega = \sup \{s_n; n \in \omega\} < \alpha$. For $n < \omega$, let $s_n = t_n | s_\omega$. Then, for each $n \in \omega$,

$$L_p \models \langle T \mid s_0 \times T \mid s_1 \times \cdots \times T \mid s_n \rangle \cap L_{\delta_n} \models \langle T \mid s_{n+1} \rangle \cap L_{\delta_n} \text{ is Suslin} \rangle.$$

Consequently, as $\pi$ is an elementary embedding, for each $n \in \omega$, $\prod_{m \leq n} T | s_m$ satisfies the $\omega_2$-cc.

Minor modifications of this argument show that CH together with the existence of an $(\omega_1, 1)$ morass, which provide an $\omega_2$ super-Suslin tree, suffice to prove Proposition 2.

The rest of this section will now study how the Suslinity of a tree (or a sequence of trees) is preserved by a generic extension.

**Proposition 3.** Let $T$ be a $\kappa^+$ Suslin tree ($\kappa$ a regular cardinal) in a model $M$ of ZF. Let $N$ be a $P$-generic extension of $M$ for a poset $P \in M$ such that either a) $P$ has the $\kappa^+$ strong chain property (i.e. $\forall X \in P$ card $X = \kappa^+$ $\exists Y \subseteq X$ card $Y = \kappa^+$ $\forall p, q \in Y$ $p$ and $q$ are compatible), or b) $P$ is $\kappa$-closed (i.e. for every decreasing sequence $(p_\alpha; \alpha < \kappa)$ in $P$ there is a condition $p$ such that $p \leq p_\alpha$ $\forall \alpha < \kappa$).

Then $T$ still is Suslin in $N$.

(Note. Some cardinals may be collapsed in the extension. For example, if $\kappa = \omega_1$ and $\omega_2^N = \omega_1^M$, then, in $N$, $T$ becomes an $\omega_1$ Suslin tree.)

**Proof.** This result is well known; see e.g. [DJJ] for $\kappa = \omega$. It is enough to show that any antichain of $T$ in $N$ has cardinality at most $\kappa$. Assume not. Let $A$ be a name such that

$$\emptyset \models_{P} A \text{ is a maximal antichain in } T \text{ of size } \kappa^+.$$

Assume first that $P$ has the strong chain property.

Let $B = \{x \in T; \exists p \in P \models x \in A\}$. Clearly $B \supseteq A$. For $x$ in $B$ choose $p_x$ such that $p_x \models x \in A$. Let $C = \{p_x; x \in B\}$. Then there is a $D \subseteq B$ such that card $D = \kappa^+$ and $\forall x, y \in D p_x$ is compatible with $p_y$ (by the claim if card $C = \kappa^+$; by evidence otherwise). Since if $p_x$ is compatible with $p_y$ then $x \parallel y$ (i.e. $x$ is incompatible with $y$), this contradicts the fact that $T$ is Suslin in $M$.

Assume next that $P$ is closed.

It is easy to build a sequence $((p_\alpha; A_\alpha); \alpha < \kappa^+)$ such that $(p_\alpha; \alpha < \kappa^+)$ is a decreasing, $T | A \supseteq A_\alpha$, $p_\alpha \models A \cap T | A = A_\alpha$ and $y \in A$, for some $y$ compatible with the $\alpha$th element of $T$. Let $A = \bigcup A_\alpha$. Clearly $A$ is a maximal antichain in $M$. This implies $p_\alpha \models A = A_\alpha$ for some $\alpha$, which is a contradiction since $\emptyset \models A$ has size $\kappa^+$.

(Note. This proposition can be straightforwardly generalized to assert that the $\kappa^+$-chain property of posets of cardinality $\kappa^+$ is absolute for $P$-generic extension when $P$ satisfies the hypothesis of the proposition.)

Though in general weaker assumptions are not enough to preserve the Suslinity of a tree (e.g. assuming that $P$ is $\kappa^+$-cc or $\kappa^+$-distributive is not enough, since forcing with the tree itself is both), in many cases when $P$ is not intended to kill the Suslinity of $T$, $T$ will remain Suslin after a $P$-generic extension. Conditions of the following kind frequently occur in coding conditions.

**Proposition 4.** Let $P$ be the following set of conditions: $p \in P$ iff $p \models |p| < \kappa^+ \rightarrow 2$ and $\forall \xi \leq |p|\ p \models \xi \in L_{\eta(\xi)}$, where $\eta(\xi)$ is the least $\eta < \xi$ for which $L_{\eta} \models ZF^+
card $\xi = \kappa$. Then:

1) $P$ is $\kappa^+$ distributive but not $\kappa$ closed.

2) Let $T$ be the canonical $\kappa^+$ Suslin tree. Then $T$ remains Suslin in any $P$-generic extension of $L$.

Proof. 1) See [D1].

2) Let $A \subset \kappa^+$ be generic for $P$. Use the standard condensation argument in $L[A]$. Let $\mu = \pi^{-1}(\kappa^+)$. Since $\pi^{-1}(A) = A \cap \mu \in L[\mu]$, the proof goes through as usual.

Definition 5. Let $M$ be a model of ZF, $(T_n)_{n<\omega}$ a sequence of trees, and let $a \subset \omega$ be Cohen generic over $M$. Let $T'$ (resp. $T_n$) be the product (with infinite support) of the $T_n$ as computed in $M$ (resp. $M[a]$).

Question 6. Assume the $(T_n)$ are $\omega_1$ Suslin in $M$ and $T$ preserves $\omega_1$. Then does $T_n$ also preserve $\omega_1$?

The next proposition shows that this is not the case for $\omega_2$.

Proposition 7. Assume the $(T_n)$ are $\omega_2$ Suslin trees. Then forcing with $T_n$ above $M[a]$ collapses $\omega_2$.

Proof. I first show why $T_n$ never is $\omega_2$-cc, since the proof is much simpler.

In $M$, choose $(x_\alpha; \alpha \in \omega_2)$ to be elements of $T$ such that $x_\alpha$ has height $\alpha$. Denote each $x_\alpha$ as $(x^*_{\alpha,n} : n \in \omega)$. Choose in $M$, for every $\alpha \in \omega_2$ and $n \in \omega$, two distinct immediate successors of $x^*_\alpha$, say $x^*_{\alpha,1}$ and $x^*_{\alpha,2}$. Let $y^*_\alpha$ be defined in $M[a]$ by $y^*_{\alpha} = x^*_{\alpha,a}$ (recall that $a$ is a function from $\omega$ to $2$). The set $\{y_\alpha; \alpha \in \omega_2\}$ is an antichain of $T_n$ in $M[a]$; let $\alpha < \beta$ and $y_\alpha < y_\beta$; then $y_\alpha \leq y_\beta$ and since then $a(n) = \varepsilon$ iff $x^*_{\alpha,n} < x^*_{\beta,n}$, $\alpha$ would be a member of $M$; a contradiction.

The proof of the proposition is due to the referee (my original one was incorrect).

Theorem (CH). Suppose $\kappa$ is a cardinal such that $\kappa^\omega = \kappa$ and $(T^n; n \in \omega)$ is a sequence of normal trees of height $\kappa$ with levels of cardinality $\leq \kappa$. Let $P = \langle \mathcal{I}_{\leq \omega_1}, \leq \rangle$, where $q \geq p$ iff $p \supseteq q$. Then in any $P \ast (\prod_{n \in \omega} T^n)$-generic extension there is a function with domain contained in $\omega_1$ (of the ground model) and range cofinal in $\kappa$.

Proof. The $T^n = \{t \in T^n; \text{ht}(t) = \gamma\}$ (where $\text{ht}(t)$ is the height of the node $t$ in the tree $T^n$). Forcing with $\prod_{n \in \omega} T^n$ is equivalent to forcing with $\bigcup_{\gamma \in \omega} \prod_{n \in \omega} T^n_\gamma$, since as $\text{cf}(\kappa) > \omega_1$, the latter is dense in the former. We will abuse notation and write $\prod_{n \in \omega} T^n$ to mean $\bigcup_{\gamma \in \omega} \prod_{n \in \omega} T^n_\gamma$. Let $B$ be a complete Boolean algebra in which $P \subset B$ is dense. First we must get a handle on $B$-valued terms for elements of $\prod_{n \in \omega} T^n$.

Definition. $(A_n; n \in \omega)$ is a spectrum iff

1) for all $n$, $2^{< \omega} \supseteq A_n$ is a maximal antichain in $P$, and

2) if $m < n$ and $p \in A_n$, then there is a $p \in A_m$ such that $p \geq p$.

Notation. Fix $\gamma < \kappa$, $x \in \Sigma$, iff there is a spectrum $(A_n(x); n \in \omega)$ such that

1) $x$ is a function with domain $\bigcup_{n \in \omega} A_n(x)$,

2) if $p \in A_n(x)$, $A_{n+1}(x)$, then $x(p) \in \prod_{m \leq n} T^n_m$, while if $p \in A_n(x)$ for all $n$, then $x(p) \in \prod_{m<\omega} T^n_m$ (as calculated in the ground model, of course),

3) if $p \geq p$ are in $\text{dom}(x)$, then $x(p) \supseteq x(p)$; that is, $x(p)$ is a sequence extending the sequence $x(p)$, and

4) if $\forall p \geq p (p \neq A_n(x))$, then there are $p_1, p_2 \leq p$ such that $p_1, p_2 \in A_n(x)$ and $x(p_1)(k) \neq x(p_2)(k)$ for some $k \leq n$.

Let $\Sigma = \bigcup_{\gamma \in \omega} \Sigma_\gamma$. If $x \in \Sigma$, set $\text{Spec}(x) = (A_n(x); n \in \omega)$, where $(A_n(x); n \in \omega)$ is as
above. For $z$ in the ground model, let $z \in V^B$ be the canonical name for $z$ in a $B$

generic extension.

**Notation.** Suppose $x \in \Sigma_\gamma$. Define $\chi : \{(k, t) : t \in T^*_\gamma\} \to B$ by $\chi((k, t)) = \sup\{p \in P : x(p)(k) = t\}$. Then $\chi \in V^B$.

The following can be verified by straightforward calculations:

**Claim 1.** Suppose $x \in \Sigma_\gamma$, $n \in \omega$, $s \in \prod_{m \leq n} T^*_\gamma$, and $p \in P$. Then $p \models x((n + 1) = s$ if and only if $\exists p' \in P (p' \geq p$ and $x(p') \supset s)$.

**Corollary.** If $x \in \Sigma_\gamma$, $\emptyset \models x \in \prod_{m \in \omega} T^*_m$. Furthermore, if $G$ is $B$ generic over $V$, then $\chi^{V[G]} = \bigcup\{x(p) : p \in G \cap \text{dom}(x)\}$.

The value of the terms $x$ for $x \in \Sigma$ is revealed by

**Claim 2.** Suppose $t \in V^B$ and $\emptyset \models t \in \prod_{m \in \omega} T^*_m$. Then there is an $x \in \Sigma_\gamma$ such that $\emptyset \models t = x$.

**Proof.** Define $A_n \subset 2^{<\omega}$ by

$$p \in A_n \text{ if and only if for some } s \in \prod_{m \leq n} T^*_m, p \models (n + 1) = s$$

and for all $p \geq p$, $p \in P$, $p \neq p$, $p \nvDash t \models (n + 1) = s$.

Then $(A_n : n \in \omega)$ is a spectrum.

Define a function $x$ with domain $\bigcup_{n \in \omega} A_n$ as follows. Pick any $p \in \bigcup_{n \in \omega} A_n$. For any $n$ such that $p \in A_n$, let $s \in \prod_{m \leq n} T^*_m$ be such that $p \models (n + 1) = s$ and set $x(p)(k) = s(k)$ for $k \leq n$. Then $x \in \Sigma_\gamma$. Suppose, for the sake of a contradiction, that $\emptyset \nvDash t = x$. Choose $p \in P$, $n \in \omega$, and $s, s' \in \prod_{m \leq n} T^*_m$ such that $s \neq s'$ and $p \models (n + 1) = s$ and $\models (n + 1) = s'$. Choose $p \in A_n$ such that $p \geq p$. Choose $k \leq n$ such that $s(k) \neq s'(k)$. Now $x(p)(k) = s(k)$ and so $x((k, s(k))) \geq p$. Consequently, $p \models (k, x(k)) \in x$ and so $p \nvDash x(k) = s(k)$. But $p \nvDash (n + 1) = s'$; hence $p \models x(k) = s'(k)$, contradicting that $s(k) \neq s'(k)$.

qed (Claim 2)

**Notation.** Suppose $S = (A_n : n \in \omega)$ and $S' = (A_n' : n \in \omega)$ are spectra. Define $\leq$, $\leq_v$, and $<_{ev}$ as follows:

$S \leq S'$ if and only if for all $n \in \omega$, if $p \in A_n'$, then there is a $\bar{p} \geq p$ such that $\bar{p} \in A_n$.

$S \leq_v S'$ if and only if there is an $n \in \omega$ such that for all $m \geq n$, if $p \in A_m'$, then there is a $\bar{p} \geq p$ such that $\bar{p} \in A_m$.

$S <_{ev} S'$ if and only if there is an $n \in \omega$ such that for all $m \geq n$, if $p \in A_m'$, then there is a $\bar{p} \geq p$ such that $\bar{p} \neq p$ and $\bar{p} \in A_n$.

Note that $\leq$ partially orders spectra, and that $\leq_v$ and $<_{ev}$ are transitive.

**Claim 3.** Suppose $x \in \Sigma_\gamma$ and $S$ is a spectrum such that $S \geq \text{Spec}(x)$ and $\gamma \geq \beta + \omega$. Then there is a $y \in \Sigma_\gamma$ such that $\text{Spec}(y) = S$ and $\emptyset \models x \leq y$.

**Proof.** Say $S = (A_n : n \in \omega)$. For each $p \in A_n(x)$, let $(p_i(p)) : i < w(p)) \leq \omega$ enumerate inductively $\{q \in A_n : q \geq p\}$, and for each $p \in A_n(x)$ choose $(t_i(p)) : i < w(p))$, a sequence of distinct elements in $T^*_\gamma$ such that $x(p)(n) = t_i(p)$ in $T^*$. (It is in order to insure branching in $T^*$ adequate to make this possible that we are seeking $y \in \Sigma_\gamma$, where $\gamma \geq \beta + \omega$, rather than in, say, $\Sigma_{\beta+1}$. By recursion on $n$, for $p \in A_n$, define $x(p)(n + 1)$; if $n > 0$, for $k < n$, let $p \geq p$ be such that $p \in A_{n-k}$ and set $y(p)(k) = y(p)(k)$. To define $y(p)(n)$, let $p \geq p$ be such that $p \in A_n(x)$ and set $y(p)(n) = t_i(p)$, where $i < w(p)$ such that $p_i(p)$.

We must show that $y \in \Sigma_\gamma$ and that $\emptyset \models x \leq y$.

In the definition of $\Sigma_\gamma$, (1), (2), and (3) are not difficult to establish. (4) can be argued as follows. Suppose $\forall \gamma \geq \beta p \neq A_n$. If $\forall \gamma \geq \beta p \neq A_n(x)$, then there are $p_1$,
\[ p_2 \in A_n(x) \text{ such that } p \geq p_1, p_2 \text{ and for some } k \leq n, x(p_1)(k) \neq x(p_2)(k). \] Choose \( p_1 \leq p \text{ and } p_2 \leq p \text{ such that } p_1, p_2 \in A_n \). Then, by construction, \( y(p_1)(k) > x(p_1)(k) \) in \( T^\circ \). Consequently, \( y(p_1)(k) \neq y(p_2)(k) \). On the other hand, if for some \( p \geq p, p \in A_n(x) \), choose \( p_1, p_2 \leq p \) such that \( p_1, p_2 \in A_n \). By construction, \( y(p_1)(n) \) and \( y(p_2)(n) \) are distinct elements in \( T^\circ \).

Finally, we claim that \( \emptyset \models \exists \chi \leq y \). Otherwise, there are \( n \in \omega \) and \( p, p' \in T^\circ \) such that \( s(n) \neq s(n) \) and \( p \models \exists \chi \models (n + 1) = s \) and \( p \models \exists \chi \models (n + 1) = s' \). In virtue of Claim 1, there are \( q_1, q_2 \) such that \( x(q_1) \models s \) and \( y(q_2) \models s' \). We may assume \( q_1 \in A_n(x) \) and \( q_2 \in A_n \). By construction, then, \( x(q_1)(n) < y(q_2)(n) \) in \( T^\circ \). This contradicts that \( s(n) \neq s(n) \) in \( T^\circ \). \( \text{qed(Claim 3)} \)

Claim 4. (a) Suppose \( S \) is a spectrum. Then there is a spectrum \( S' \) such that \( S \leq S' \) and \( S <_{\text{ev}} S' \).

(b) Suppose \( S_0 \) and \( S_1 \) are spectra. Then there is an \( S \) such that \( S_0, S_1 \leq S \).

(c) If \( (S_n : n \in \omega) \) is a sequence of spectra, then there is a spectrum \( S \) such that, for all \( n \in \omega \), \( S_n \leq_{\text{ev}} S \).

Proof. (a) Suppose \( S = (A_n : n \in \omega) \). Set \( S' = (A'_n : n \in \omega) \), where \( A'_n = \{ p \models i : p \in A_n \text{ and } i = 0, 1 \} \).

(b) Suppose, for \( i = 0, 1 \), \( S_i = (A'_n : n \in \omega) \), where \( A_n \) is defined by recursion on \( n \) as follows: \( A_0 \) is a maximal antichain in \( P \) such that if \( p \in A_0 \), then there are \( p, p' \geq p \) such that \( p \in A_0^0 \) and \( p' \in A_0^1 \); and \( A_{n+1} \) is a maximal antichain in \( P \) such that if \( p \in A_{n+1} \), then there are \( p, p' \in A_{n+1}^0, p' \in A_{n+1}^1 \) and \( p' \models p \).

(c) Suppose \( S_\omega = (A'_n : n \in \omega) \). Set \( S = (A_n : n \in \omega) \), where \( A_n \) is defined by recursion on \( n; A_0 = A_0^0 \), and \( A_{n+1} \) is a maximal antichain in \( P \) such that if \( p \in A_{n+1} \), then there are \( p, p_0, \ldots, p_n \geq p \) such that \( p \in A_n \) and, for \( k \leq n, p_k \in A_{n+1} \).

Claim 5 (CH). There is a sequence of spectra \( (S_\alpha : \alpha < \omega_1) \) that satisfy the following conditions:

1) Let \( S_0 = (A_n : n \in \omega) \). Then the \( A_n \)'s are pairwise disjoint.
2) If \( \alpha < \beta < \omega_1 \), then \( S_{\alpha} \leq_{\text{ev}} S_{\beta} \).
3) If \( S \) is any spectrum, then there is an \( \alpha < \omega_1 \) such that \( S \leq S_{\alpha} \).

Proof. Using CH to enumerate all spectra in order type \( \omega_1 \) and Claim 4, it is straightforward to construct such a sequence by recursion on \( \alpha \). \( \text{qed(Claim 5)} \)

We next seek to define \( (A_\alpha : \alpha < \omega_1) \) for which the following conditions hold:

1) \( A_\alpha \subseteq \Sigma, |A_\alpha| = \kappa \).
2) If \( \alpha \neq \beta < \omega_1 \), then \( \emptyset \models \exists \chi \models \chi \).
3) If \( t \models \exists \chi \leq t \models \exists \chi \).

This will suffice to prove the theorem.

Fix \( \alpha < \omega_1 \). First let \( (x_\beta : \beta < \kappa) \) enumerate, with repetitions unbounded in \( \kappa \), the set \( \{ x \in \Sigma : \text{Spec}(x) \leq S_\alpha \} \) in such a way that if \( x_\beta \in \Sigma \), then \( \gamma \leq \beta \). This is possible because of our hypotheses that \( \kappa^{\omega_1} = \kappa \) and that the levels of the \( T^\circ \) have cardinality \( \leq \kappa \). Next, for \( \beta < \kappa \), using Claim 3, choose \( z_\beta \models \Sigma \models \alpha \mid z_\beta \leq z_\beta \). Finally, again using Claim 3, choose \( z_\beta \models \Sigma \models z_\beta \leq z_\beta \) such that \( \emptyset \models z_\beta \leq y_\beta \) and \( A_n(y_\beta) = \{ p_\models i : p \in A_n \text{ and } i = 0, 1 \} \) (where \( S_\alpha = (A_n : n \in \omega) \)). Set \( A_\alpha = \{ y_\beta : \beta < \kappa \} \). Goal (1) is then evident.
Claim 6. If $\beta < \beta$, then $\emptyset \models y_{\beta}^\alpha \approx y_{\beta}^\alpha$.

Proof. Suppose not. Say $p \models y_{\beta}^\alpha \not\approx y_{\beta}^\alpha$. As $y_{\beta}^\alpha \in \Sigma_{\omega^2 \cdot \beta + \omega + \omega}$, $y_{\beta}^\alpha \in \Sigma_{\omega^2 \cdot \beta + \omega + \omega}$ and $\omega^2 \cdot \beta + \omega + \omega < \omega^2 \cdot \beta + \omega + \omega$, it follows that $p \models y_{\beta}^\alpha \approx y_{\beta}^\alpha$. Furthermore, as $\emptyset \models z_{\beta} \leq y_{\beta}^\alpha$ and $z_{\beta} \in \Sigma_{\omega^2 \cdot \beta + \omega + \omega}$ and $\omega^2 \cdot \beta + \omega + \omega < \omega^2 \cdot \beta + \omega + \omega$, it follows that $p \models y_{\beta}^\alpha < z_{\beta}$. Now, Spec($z_{\beta}$) = $S_\alpha \approx S_\rho$. Thus, using (1) of Claim 5, there is an $n \in \omega$ and a $p \leq p$ such that $p \in A_n$ (where $S_\alpha = \{ A_n : n \in \omega \}$). Then $p \models 0$, $p \not\models 1 \in A_n(y_{\beta}^\alpha)$. By (4) in the definition of $\Sigma$, there is a $k \leq n$ such that $y_{\beta}^\alpha(p \cdot i)(k) \neq y_{\beta}^\alpha(p \cdot i)(k)$. Choose $i = 0$, 1 such that $y_{\beta}^\alpha(p \cdot i)(k) \neq z_{\beta}(p)(k)$. Then $p \models 0 \cdot i = z_{\beta}$, contradicting that $p \models y_{\beta}^\alpha < z_{\beta}$. qed (Claim 6)

Claim 7. Suppose $t \in V^\emptyset$ and $\emptyset \models t \in \prod_{n \in \omega} \mathbb{T}^n$. Then there is an $\alpha < \omega_1$ such that

$$\models \{ y \in A_\alpha : \emptyset \models \overline{y} \geq t \} = \kappa.$$

Proof. Choose $x \in \Sigma$ such that $\emptyset \models \overline{x} = t$. Choose $\alpha < \omega_1$ such that Spec($x$) $\leq S_\beta$. Choose any $\beta < \kappa$ such that $x_{\beta} = x$ (where $x_{\beta}$; $\beta < \kappa$ is as in the construction of $A_\beta$). There are $\kappa$ many such $\beta$ by our choice of $(x_{\beta}; \beta < \kappa)$. By construction, $\emptyset \models x_{\beta} \leq y_{\beta}^\alpha$. qed (Claim 7)

This completes the proof of the theorem.

Question. Is it possible to find a sequence $(T_n)_{n \in \omega}$ of $\omega_2$ trees such that $\forall n \prod_{m < n} T_m$ is $\omega_2$-cc, but the iteration of the $T_n$ with countable support collapses $\omega_2$? This would show that there is no way to extend Shelah’s notion of properness to preserve higher cardinals — more precisely, that there is no property $H$ such that

1) if $P$ is $\omega_2$-cc then $P$ has the property $H$,
2) $H$ is preserved by countable support iteration, and
3) if $P$ has the property $H$ then $\omega_2$ is preserved in any $P$ generic extension.

§III. Specialization of Suslin trees.

Definition 1. Let $T$ be a $\kappa^+$ tree such that any level (except the least one) has cardinality $\geq \kappa$. Define $A_T$ as follows: $p \in A_T$ iff $T \models p$, card($p$) $\leq \kappa$ and $p$ is an antichain in $T$ and the root of $T$ is not in $p$; $p \leq q$ iff $p \models q$.

Clearly $A_T$ is $\kappa$-closed (for every regular $\kappa$). It is well known (see, for example, [Je]) that when $T$ is an $\omega_1$ Suslin tree (in fact $T$ Aronszajn is enough) $A_T$ is $\omega_1$-cc. This is no longer true for $\omega_2$ (see [LS]) However.

Proposition 2. Let $\kappa$ be a regular cardinal, and let $T$ be the canonical $\kappa^+$ Suslin tree. Then $A_T$ is $\kappa^+$-cc.

Proof. Using the standard condensation argument, it is enough to show the following: let cof($\mu$) = $\kappa$, $p \in A_T$, $T \models X$, $X \in L_{\omega_1}$, and $X$ a maximal antichain in $A_p = A_T \cap L_{\omega_1}$. Then $p$ is compatible with some element in $X$:

Set $a = b \cap T \mu$ and $b = p - a$. Let $I < \kappa$, card($I$) $\leq \kappa$, be such that for every $x$ in $b$ there is $x \in I$ such that $B_i \leq x$ (recall that $T_\mu = \{ B_\xi : \xi < \kappa \}$, where the $(B_\xi)_{\xi < \kappa}$ are $C(\mu)$-generic). It is enough to show that for any $f \in C(\mu)$ and $g \in C(\mu)$, $g \leq f$, there is an $h \leq g$ such that $h \models_{\text{cond}} p$ is compatible with some $d$ in $X$, where $p$ is the canonical name for $a \cup \{ B_\xi : \xi \in I \}$. I may assume dom $q \models I$ and $\forall \xi \in I \text{ dom } g \forall x \in a |g(\xi)| > |x|$, for $\xi \in I$ choose an immediate successor $g' (\xi)$ of $g(\xi)$. Let $c = a \cup \{ g' (\xi) : \xi \in I \}$. Then $c \in A_p$. Let $d \in X$ be compatible with $c$. Then define $h$ as follows: dom $h = \text{ dom } g$ and, for $\xi \in \text{ dom } g$, $h(\xi) = \text{ some } u \geq g(\xi)$ distinct and above all the elements in $d$. It is easy to check that $h$ is as needed.
The following propositions show that a variety of situations are possible.

**Proposition 3.** There is an $\omega_1$ normal tree $T$ such that $T$ is not Suslin but $A_T$ is $\omega_2$-cc. Moreover, we may assume that forcing with $T$ collapses $\omega_2$ on $\omega_1$.

**Proof.** The proof looks like that of Proposition II.2. Let $F$ be as there. Define the tree essentially as the canonical $\omega_2$ Suslin tree, together with the $(c_f; f \in F)$ as in Proposition II.2 except that $c$ is replaced by

$$\forall x \in T, \forall f: \omega_1 \rightarrow \omega_2 \exists x x \notin c_f[x].$$

The following induction hypothesis $I(\mu)$ is maintained during the construction:

$$\forall x(|x| < \mu \Rightarrow \exists y |y| = \mu \text{ and } \forall f (c_f \leq x \text{ iff } c_f \leq y)).$$

The forcing $C(\mu)$ (to build $T_\mu$) when $\text{cof}(\mu) = \omega_1$ is as in the canonical tree with the additional requirement that

$$d \leq d \Rightarrow \forall i \in \text{dom } d, \forall f \in F (c_f \leq d(i) \text{ iff } c_f \leq d(i)).$$

The definition of the $c_f$ with the mentioned properties is straightforward. The sequence $(c_f)$ ensures that forcing with $T$ collapses $\omega_2$ on $\omega_1$. Following the proof of Proposition III.2, it is easy to check that $A_T$ is $\omega_2$-cc.

**Proposition 4.** There are $\omega_2$ Suslin trees $T_1$ and $T_2$ such that forcing with $A_{T_1} \times T_2$ is not equivalent to forcing with $A_{T_1} \times A_{T_2}$.

**Proof.** Let $(T_1, T_2)$ be the canonical pair of $\omega_2$ Suslin trees. It is easy to check (as in Proposition 2) that $A_{T_1} \times T_2$ is $\omega_2$-cc. But since $A_{T_1} \times T_1$ is clearly not $\omega_2$-cc, $A_{T_1} \times A_{T_2}$ cannot be isomorphic to $A_{T_1} \times A_{T_2}$.

(*Note. I suspect that the proposition is true for every pair of tree, since the converse seems to imply a partition property on $|T|$. But I do not know how to prove it.*)

**Proposition 5.** 1) Let $T$ be the canonical $\omega_2$ Suslin tree. Then forcing with $A_T \times T$ preserves the cardinals.

2) There is an $\omega_2$ Suslin tree $T$ such that $A_T$ is $\omega_2$-cc but forcing with $A_T \times T$ collapses $\omega_2$.

**Proof.** 1) It is easy to check that $A_T \times T$ satisfies

$$\forall p \exists q \leq p \{r \in A_T \times T/r \leq q\} \text{ is } \omega_2\text{-cc}$$

(if $p = (a, x)$, choose $q = (b, y)$ where $y$ is above some element in $b$).

2) The idea is to build a tree $T$ such that

a) $x \in T \Rightarrow x: |x| < \omega_2 \rightarrow \omega_1$,

b) $T$ is $\omega$-closed, and

(*) $\forall x, y \in T \forall \gamma \leq |x| \gamma \times x \in T$ where $\gamma \times x$, is defined as follows: $z = \gamma \times x$, iff $|z| = |y| + (|x| - \gamma)$ and $\forall \zeta < |y| \gamma \zeta = y(\zeta)$ and $\forall \zeta = |x| + \eta, \eta < |x| - \gamma, z(\zeta) = x(\gamma + \eta)$.

The construction of $T$ is that of the canonical Suslin tree except that the forcing construction $C(\mu)$ is made only when $\text{cof}(\mu) = \omega_1$ and $\forall x < \mu \mu = \mu - x$ (for the other $\mu$, branches are added in some canonical way so that the tree is normal and satisfies (*)). The points in $T(\mu)$ are branches added by some generic on $C(\mu)$ and the branches having the same tail as one in the generic. Noting that then every branch in $T_\mu$ still is generic over $T|\mu$, we can easily show that $T$ and $A_T$ are $\omega_2$-cc.

In $L[A]$ (where $A \subset T$ is $A_T$-generic) we can define the family $(c_s; s \in X)$ for
some $X \subset F'$, where $F' = \{ s : |s| < \omega_1 \rightarrow \omega_2 \}$, as follows. Let $(x(x)/x < \omega_2)$ be an enumeration of $T$; then

$$c_{x \rightarrow x} = c_{x \rightarrow x} \cdot x(x) \quad \text{if } x(x) \in A, \quad c_{x} = \bigcup_{x \rightarrow x} c_{x} \quad \text{for limit } |s|.$$

It is enough (and easy) to check (by using the genericity of $A$) that for $s$ such that $c_{x}$ is defined, $\{ c_{x \rightarrow x} : x \in A \}$ is a maximal antichain of size $\omega_2$ above $c_{x}$, and that $\forall x \in T \forall x \rightarrow x \exists y (x \not\in c_{f_{x \rightarrow y}}$ or $c_{f_{x \rightarrow y}}$ is undefined).

As a final fact, the following variation of $A_T$ (namely the forcing to specialize the tree) can be used in the [JJJ] construction (or its extension to higher cardinals) to produce a $\Pi^1_2$ singleton.

Let $T$ be an $\omega_1$ tree. Set $B_T = \{ p : T \supset \text{dom } p; \text{card(dom } p) < \omega_2; \forall x \in \text{dom } p \ p(x) \in \omega; \forall x, y \in \text{dom } p \ x < y \Rightarrow p(x) \neq p(y) \}$. It is not difficult to check that forcing with $B_T$ adds a specializing function to $T$ and that if $T$ is the canonical Suslin tree then $B_T$ is $\omega_1$-cc (the proof follows that of Proposition 2).

Let $(T_n)_{n \in \omega}$ be the canonical sequence of $\omega_1$ Suslin trees. Using the $B_{T_n}$, we can define a forcing notion $P$ that produces a real $a$, for $n \in \omega$ a cofinal branch in $T_n$, and for $n \notin \omega$ a specializing function in $T_n$. This forcing notion preserves the cardinals. Moreover, $a$ can "code" these branches and specializing functions.

This will be developed and extended in a forthcoming paper.

**Question.** Is it possible to define in $L$ a sequence of $\omega_1$ Suslin trees $T_n$ and to generically add a real $a$ that preserves the cardinals and such that there is in $L(a)$ for $n \in \omega$ a cofinal branch in $T_n$ and for $n \notin \omega$ a specializing function in $T_n$?

**REFERENCES**


DÉPARTEMENT DE MATHÉMATIQUES
UNIVERSITÉ DE SAVOIE
73101 CHAMBERY, FRANCE