EFFECTIVE ŁOJASIEWICZ GRADIENT INEQUALITY FOR POLYNOMIALS

DIDIER D'ACUNTO AND KRZYSZTOF KURDYKA

Abstract. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a polynomial function of degree d with $f(0) = 0$ and $\nabla f(0) = 0$. Łojasiewicz's gradient inequality states there exist $C > 0$ and $\rho \in (0, 1)$ such that $|\nabla f| \geq C|f|^\rho$ in a neighbourhood of the origin. We prove that the smallest of such exponents ρ is not greater than $1 - R(n, d)^{-1}$ with $R(n, d) = d(3d - 3)^{n-1}$.

1. Introduction

Let f be an analytic function in a neighbourhood of the origin in $\mathbb{R}^n$ and assume $f(0) = 0$ and $\nabla f(0) = 0$. The well known Łojasiewicz's gradient inequality (cf [Ło1] or [Ło2]) states there exist an open neighbourhood U of the origin and two constants $C > 0$ and $\rho < 1$ such that for any $x \in U$ we have

$$|\nabla f(x)| \geq C|f(x)|^\rho. \tag{1.1}$$

The Łojasiewicz exponent of f at the origin, denoted by ρ_f, is the infimum of the exponents ρ satisfying the Łojasiewicz's gradient inequality. Bochnak and Risler (cf. [Bo-Ri]) proved that ρ_f is a rational number. Moreover, inequality (1.1) holds with exponent ρ_f and some constant $C > 0$. Knowing explicitly the exponent ρ_f is important in the study of the gradient flow near a singular point (cf. [Ło1] and [KMP]) .

We now assume that f is a polynomial of degree d in n variables. It is known that ρ_f can be bounded by some rational number $\rho(n, d) < 1$ depending only on n and d. If f has an isolated zero at the origin (that is f has a strict local extremum at 0) J. Gwoździewicz [Gw] proved that $\rho_f \leq 1 - \frac{1}{(d-1)^n+1}$.

In the present paper we consider the general case, that is f may have a non-isolated singularity at the origin. More precisely, for any integer $d \geq 2$ and for any polynomial f in n variables with $\deg f = d$ we have

Date: 2nd December 2004.

2000 *Mathematics Subject Classification.* Primary 32Bxx, 34Cxx, Secondary 32Sxx, 14P10.

Key words and phrases. polynomials, Łojasiewicz inequality, Łojasiewicz exponent, valley lines.

Supported by the EU research network IHP-RAAG contract number HPRN-CT-2001-00271.

Main Theorem. *The Łojasiewicz exponent ρ_f satysfies*

$$\rho_f \leq 1 - \frac{1}{d(3d-3)^{n-1}}.$$

Our approach, contrarily to [Gw] who uses polar curves, is based on the study of ridge and valley lines attached to the singularity. More precisely, in a fixed non-critical level hypersurface $f^{-1}(t)$, we detect the points where the restriction of the function $|\nabla f|$ to $f^{-1}(t)$ has a local minimum. We denote by $\Gamma(f)$ the collection of all these points when t varies in $\mathbb{R}$. As proved in [D'A-Ku] the set $\Gamma(f)$ is of dimension 1 for a generic polynomial of degree d. The set $\Gamma(f)$ is contained in the set of critical points of $|\nabla f|$ restricted to $f^{-1}(t)$ which is the set of points where ∇f is an eigenvector of H_f, the Hessian matrix of f.

The paper is organised as follows: first we explain the notion of ridge and valley lines and highlight some important properties of this set. Clearly the Łojasiewicz exponent ρ_f is reached on $\Gamma(f)$. Next we recall an elementary definition of multiplicity of intersection between a complex hypersurface $\{f = c\}$ and a complex algebraic curve containing $\Gamma(f)$. We state an important result on the semi-continuity of multiplicity of intersection. We give a sketch of its proof based on the existence of a stratification (in the complex algebraic case) satisfying Thom's (a_f) condition.

We then compute Łojasiewicz exponent first for a generic polynomial and finally, using semi-continuity of intersection multiplicity, for any polynomial of fixed degree.

2. Generalised valley lines

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a polynomial with $\deg f = d \geq 2$. In a given level hypersurface of f there are points of particular interest, namely the ridge and valley lines of f. A naive but important way of describing the ridge and valley lines of f is the following: let us fix a non-critical level hypersurface $f^{-1}(t)$ and let us pick the points $x \in f^{-1}(t)$ such that for all s sufficiently close to t the Euclidean distance dist $(x, f^{-1}(s))$ is greater than or equal to dist $(x', f^{-1}(s))$ for all $x' \in f^{-1}(t)$ sufficiently close to x. We now give a more rigorous definition of the ridge and valley lines of f.

Definition 2.1. We say that a point $x \in \mathbb{R}^n$ belongs to the ridge and valley set of f if the function $|\nabla f|^2$ restricted to $f^{-1}(f(x))$ has a local minimum at x. We denote by $\Gamma_1(f)$ the ridge and valley set of f.

The terminology "*ridge and valley lines*" used in the present paper is motivated by its analogy with ridges and valleys encountered in nature when looking for example at the Earth landscape.

Clearly, the ridge and valley set of f is contained in the set

$$\Theta_1(f) = \{x \in \mathbb{R}^n : \mathrm{d}(|\nabla f|^2) \wedge \mathrm{d}f = 0\},$$

the set of critical points of the function $|\nabla f|^2$ restricted to the level sets of f. Observe that a point x belongs to $\Theta_1(f)$ if and only $\nabla f(x)$ is an eigenvector of $H_f(x)$, the Hessian matrix of f at x. Note that $\Theta_1(f)$ is a real algebraic set while the ridge and valley set is semi-algebraic. The set $\Theta_1(f)$ is the set of common zeros of at most $n-1$ coefficients of the differential form $\omega = \mathrm{d}(|\nabla f|^2) \wedge \mathrm{d}f$.

Let $B^n(r_0)$ be the open ball of radius r_0 centered at the origin. Then the infimum of the function $|\nabla f|^2$ restricted to the hypersurfaces $f^{-1}(t) \cap B^n(r_0)$ is not necessarily reached inside $B^n(r_0)$ but maybe on the sphere $S^n(r_0) = \partial B^n(r_0)$. This can occur when the origin is not an isolated singularity of f. One has to take into account some boundary effects. We thus introduce the *boundary ridge and valley set of* f, denoted by $\Gamma_2(f)$, as the set of points at which the function $|\nabla f|^2$ restricted to $f^{-1}(t) \cap S^n(r_0)$ has a local minimum. Clearly, the set $\Gamma_2(f)$ is contained in the set

$$\Theta_2(f) = \{x \in S^n(r_0) : \mathrm{d}(|\nabla f|^2) \wedge \mathrm{d}f \wedge \mathrm{d}r = 0\},$$

where $r(x) = |x|^2 - r_0^2$. Then we define *generalised ridge and valley set of* f *associated to* $B^n(r_0)$ as $\Gamma(f) = \Gamma_1(f) \cup \Gamma_2(f)$. Clearly

$$\Gamma(f) \subset \Theta(f) = \Theta_1(f) \cup \Theta_2(f).$$

The dimension of $\Theta(f)$ is not always equal to 1. Nevertheless for a "generic" polynomial the set $\Theta(f)$ is an algebraic curve. Let $\mathbb{R}_d[\mathbf{X}]$ be the set of polynomials in n variables of degree less than or equal to d and $\mathbf{X} = (X_1, \ldots, X_n)$. Then we have

Proposition 2.2 ([D'A-Ku]). *Let us fix integers $d, n \geq 2$. There is a semi-algebraic set $G_d \subset \mathbb{R}_d[\mathbf{X}]$, with* codim $G_d \geq 1$, *such that for any polynomial $f \in \mathbb{R}_d[\mathbf{X}] \setminus G_d$, the set $\Theta(f)$ is of dimension* 1.

The proof of this proposition is based on transversality arguments and a detailed study of the space of symmetric matrices. One may refer to [D'A-Ku] for a detailed proof. In the sequel we shall call a polynomial f of degree d *generic* if $f \in \mathbb{R}_d[\mathbf{X}] \setminus G_d$.

When $\dim \Theta_1(f) = 1$, the curve $\Theta_1(f)$ is described by the common zeros of exactly $n-1$ independent polynomials. As mentioned before a point x belongs to $\Theta_1(f)$ if and only if there exists $\lambda \in \mathbb{R}$ such that $H_f(x)\nabla f(x) = \lambda \nabla f(x)$. Thus the $n-1$ polynomials describing $\Theta_1(f)$ have degree at most $3d-4$. In the same way the set $\Theta_2(f)$ is described by the common zeros of $n-2$ polynomial equations of degree at most $3d-3$ and one polynomial equation of degree 2.

In the sequel we shall need the following Corollary of Proposition 2.2.

Corollary 2.3. *For any polynomial $f \in \mathbb{R}_d[\mathbf{X}]$ there exists a polynomial mapping $\varphi : \mathbb{R} \to \mathbb{R}_d[\mathbf{X}]$ such that $\varphi(0) = f$ and for all but finitely many $t \in \mathbb{R}$ the polynomial is generic that is $\varphi(t) \notin G_d$.*

Proof. Let Z_d be the Zariski closure of G_d. Since $\dim Z_d = \dim G_d < n$, so Z_d is a proper algebraic subset of $\mathbb{R}_d[\mathbf{X}]$. By the classical curve selction

Lemma [Lo1] there exists an analytic mapping $\tilde{\varphi} : (-a, a) \to \mathbb{R}_d[\mathbf{X}]$ such that $\tilde{\varphi}(0) = f$, and $\tilde{\varphi}(0, s) \not\subset Z_d$. As φ we can take a truncation of $\tilde{\varphi}$ up to a sufficiently high order. □

3. Multiplicity of intersection

To prove that the estimate of the main Theorem for the Łojasiewicz exponent holds true both for generic and also non-generic polynomials we will use some basic facts from elementary intersection theory. First we precise what we shall mean by multiplicity of intersection of an algebraic curve $\Gamma \subset \mathbb{C}^n$ with an algebraic hypersurface $f^{-1}(z)$, where $f : \mathbb{C}^n \to \mathbb{C}$ a polynomial. Assume that f is non-constant on any irreducible component of Γ. Then it is well known that for all but finitely many $z \in \mathbb{C}$ the number of points of $\Gamma \cap f^{-1}(z)$ is constant. We call this number the *multiplicity of intersection of the curve* Γ *with polynomial* f, and we denote it by $m(\Gamma, f)$. Note that we don't need to specify which particular hypersurface $f^{-1}(z)$ we consider.

The next Lemma easily follows from the Rouché's Theorem.

Lemma 3.1. *Let $\Gamma \subset \mathbb{C}^n$ be an algebraic curve and let $f : \mathbb{C}^n \to \mathbb{C}$ be a polynomial. Assume that f is non-constant on any irreducible component of Γ. Let $\gamma : D \to \Gamma \subset \mathbb{C}^n$ be an injective holomorphic function defined in an open disc $D \subset \mathbb{C}$. Then the order of $f \circ \gamma$ at any point $s \in D$ is not greater $m(\Gamma, f)$.*

We now explain a kind of semi-continuity of the multiplicity of intersection. Let us consider an algebraic family C_t, $t \in \mathbb{C}^*$ of algebraic curves in $\mathbb{C}^n$. That is we assume that the set

$$C = \{(x, t) \in \mathbb{C}^n \times \mathbb{C}^* : x \in C_t\}$$

is algebraic in $\mathbb{C}^n \times \mathbb{C}^*$. Let C_0 be the limit when $t \to 0$ of this family, precisely

$$C_0 \times 0 = \bar{C} \setminus C,$$

where the closure is taken in the Zariski topology in $\mathbb{C}^n \times \mathbb{C}$. In fact, since C is constructible in $\mathbb{C}^n \times \mathbb{C}$ the closures of C in the strong and the Zariski topologies are the same. Hence C_0 is an algebraic set in $\mathbb{C}^n$ of dimension $1 = \dim C - 1$.

Now let $f_t : \mathbb{C}^n \to \mathbb{C}$, $t \in \mathbb{C}$ be a family of polynomials such that coefficients of f_t are polynomials in t. Assume that, for any $t \in \mathbb{C}^*$, f_t is non-constant on any irreducible component of C_t.

Lemma 3.2. *Let Γ_0 be the union of all irreducible components of C_0 on which f_0 is non constant, then*

$$m(\Gamma_0, f_0) \le m(C_t, f_t), \tag{3.1}$$

for any $t \neq 0$ sufficiently close to $0 \in \mathbb{C}$.

Proof. This result is a particular case of the general intersection theory (see for instance Chapter 11 in [Fu]). However, we shall give below a sketch of a simple geometric argument based on stratification theory. There exists (see for instance [Ha-Lê] or [HMS]) a stratification of $(C_0 \times 0, C)$ which satisfies so called Thom's condition (a_f). In our case it means that there exists a finite set $B \subset C_0$ such that:

(∗) $C_0 \setminus B$ is smooth,
(∗) for any $x \in C_0 \setminus B$ and any $\varepsilon > 0$ there exist U a neighbourhoodof x and $\delta > 0$, such that if $y \in C_t \cap U$ and $0 < |t| < \delta$, then C_t is smooth at y and the distance between the tangent space $T_y(C_t)$ and the tangent space $T_x(C_0)$ is less than ε.

Now let us choose $z \in \mathbb{C}$ in such a way that the hypersurface $\{f_0 = z\}$ meets transversally Γ_0 at the points of $\Gamma_0 \setminus B$. So by the definition, $m(\Gamma_0, f_0)$ is equal to the cardinality of $\Gamma_0 \cap \{f_0 = z\}$. Fix a point $x \in C_0 \cap \{f_0 = z\}$ and its small neighbourhood U. By transversality, if t is close to 0 and z' is close to z, then $\{f_t = z'\}$ and C_t have at least one common point in U. This proves inequality (3.1). □

4. Proof of the main Theorem

In this section we use the generalised ridge and valley lines to bound the Łojasiewicz exponent ρ_f of a polynomial $f : \mathbb{R}^n \to \mathbb{R}$ of degree d.

Remark 4.1. Gwoździewicz [Gw] proved that, if an analytic function f has an isolated zero at the origin, then the Łojasiewicz exponent for the gradient inequality (1.1) is reached on all polar curves $P_v = (\nabla f)^{-1}(\mathbb{R}v)$ provided v belongs to the complement in $\mathbb{R}^n$ of a proper linear subspace L. Moreover he gave examples showing that this is no longer true for non-isolated singularities. In particular he proved that if f is a polynomial of degree d with an isolated zero at the origin then ρ_f is bounded by $1 - ((d-1)^n + 1)^{-1}$.

From now on we suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is a polynomial of degree $d \geq 2$. We still assume that $f(0) = 0$ and $\nabla f(0) = 0$. Then we have

Theorem 4.2. *For any polynomial $f : \mathbb{R}^n \to \mathbb{R}$ of degree d the Łojasiewicz exponent ρ_f at 0 is less than or equal to $1 - \dfrac{1}{d(3d-3)^{n-1}}$. More precisely if $f(0) = 0$ and $\nabla f(0) = 0$, then for any $r_0 > 0$ there exist $\varepsilon > 0$ and $C > 0$ such that*

$$|\nabla f(x)| \geq C|f(x)|^{\rho}$$

holds for any $x \in B^n(r_0)$, $|f(x)| < \varepsilon$, with $\rho = 1 - (d(3d-3)^{n-1})^{-1}$.

The proof of Theorem 4.2 is based on the estimate of the order of contact of the hypersurface $f^{-1}(0)$ with a suitable parametrisation of the half-branches of $\Gamma(f)$, the generalised ridge and valley set of f in the ball $\overline{B}^n(r_0)$.

Precisely the following proposition is crucial.

Proposition 4.3. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a polynomial of degree d. Assume that $(0,b) \subset f(B^n(r_0))$ for some $b > 0$. Then there exists an analytic arc $\theta : (-a,a) \to \mathbb{R}^n$ such that:*

(1) $f(\theta(0)) = 0$ and $f \circ \theta((0,a)) \subset (0,b)$;
(2) if $s > 0$ and $x \in f^{-1}(f(\theta(s))) \cap \overline{B}^n(r_0)$, then $|\nabla f(x)| \geq |\nabla f(\theta(s))|$;
(3) $ord_0(f \circ \theta) \leq d(3d-3)^{n-1}$.

Proof of Theorem 4.2. Assume we have proved Proposition 4.3 and let θ be an analytic curve satisfying the Proposition 4.3.

We write $\theta(t) = a_m t^m + a_{m+1} t^{m+1} + \ldots$, with $a_m > 0$, $m \geq 1$. For small $s > 0$ we put

$$\gamma(s) = \frac{1}{a_m}\theta(s^{1/m}) = s + \frac{a_{m+1}}{a_m} s^{(m+1)/m} + \ldots,$$

which is a convergent Puiseux series. Note that $|\gamma'(s)| \to 1$ as $s \to 0$. Let us write the Puiseux expansion of $f \circ \gamma$:

$$(f \circ \gamma)(s) = \alpha_\nu s^\nu + \alpha_k s^k + \cdots,$$

with $\nu \leq \frac{1}{m} d(3d-3)^{n-1}$, $k \in \frac{1}{m}\mathbb{N}$, $k > \nu$. Then we have $(f \circ \gamma)'(s) = \langle \nabla f(\gamma(s)), \gamma'(s) \rangle$ and $|\gamma'(s)| \simeq 1$ as $s \to 0$. This implies

$$|\nabla f(\gamma(s))| \geq c\nu\alpha_\nu s^{\nu-1} \simeq f(\gamma(s))^{\frac{\nu-1}{\nu}}$$

for some positive constant $c > 0$ and small $s > 0$. So there exists $C > 0$ such that

$$|\nabla f(\gamma(s))| \geq C|f(\gamma(s))|^{\frac{\nu-1}{\nu}}.$$

Recall that if $x \in f^{-1}(f(\gamma(s)))$, then $|\nabla f(x)| \geq |\nabla f(\gamma(s))|$. Thus we have

$$|\nabla f(x)| \geq C|f(x)|^{\frac{\nu-1}{\nu}}$$

for any $x \in B^n(r_0)$ such that $f(x) > 0$ is small enough. Replacing f by $-f$ we obtain the result also for $f(x) < 0$. □

Remark 4.4. Note that, for fixed multiplicity of intersection $m(\Gamma(f), f)$, the exponent $\rho = \frac{\nu-1}{\nu}$ is the biggest possible when the curve $\Gamma(f)$ is smooth.

To complete the proof of Theorem 4.2 it remains to prove Proposition 4.3. We shall distinguish two cases. Namely, we first consider the case of a generic polynomial and then, using the results of section 3, we extend our bound on the Łojasiewicz exponent to non-generic polynomials.

Proof of Proposition 4.3. We will use notations and results from section 2. Clearly the arc θ must be chosen from the algebraic set $\Theta(f)$.

Case 1. The polynomial f is generic. that is $\Theta(f)$ is actually a curve. We have

Lemma 4.5. *Assume that f is generic. Let $\theta : (-a,a) \to \Theta(f) \subset \mathbb{R}^n$ be an analytic arc. Suppose that the complexification of θ is injective in a small disk around the origin in $\mathbb{C}$. Then $ord_0(f \circ \theta) \leq d(3d-3)^{n-1}$.*

Recall that $\Theta(f) = \Theta_1(f) \cup \Theta_2(f)$ is the union of two algebraic sets. So the image of θ entirely lies in one of them.

First we assume that $\theta(s) \in \Theta_1(f)$, $s \in (-a, a)$. Recall that if f is generic then set $\Theta_1(f)$ is of dimension one and is described by the common zeros of $n-1$ independent coefficients of the 2-form $\omega = \mathrm{d}(|\nabla f|^2) \wedge \mathrm{d}f$. Let us denote them by $g_1, \ldots, g_{n-1}$, recall that $\deg g_i \le 3d-4$. Note that the gradients $\nabla g_1(x), \ldots, \nabla g_{n-1}(x)$ are linearly independent for all, but finitely many, $x \in \Theta_1(f)$. From the Bezout's Theorem (cf. e.g. [Be-Ri],[Fu]) we obtain

$$m(\Theta_1(f), f) \le \deg f \Pi_{i=1}^{n-1} \deg g_i \le d(3d-4)^{n-1} \le d(3d-3)^{n-1}. \tag{4.1}$$

Hence by Lemma 3.1

$$\mathrm{ord}_0(f \circ \theta) \le m(\Theta_1(f), f) \le d(3d-3)^{n-1}. \tag{4.2}$$

Assume now $\theta(s) \in \Theta_2(f)$, for all $s \in (-a, a)$. Recall that the set $\Theta_2(f)$ is described by the common zeros of $n-2$ polynomial equations of degree at most $3d-3$ and one polynomial equation of degree 2. So again by Bezout's Theorem

$$m(\Theta_2(f), f) \le 2d(3d-3)^{n-2} \le d(3d-3)^{n-1}, \tag{4.3}$$

and by Lemma 3.1

$$\mathrm{ord}_0(f \circ \theta) \le 2d(3d-3)^{n-2} \le d(3d-3)^{n-1}. \tag{4.4}$$

This proves Lemma 4.5.

We now continue the proof of Proposition 4.3 in the first case. Let us choose $\varepsilon >$ such that $(0, \varepsilon) \subset f(B^n(r_0))$. By the classical curve selection lemma (cf. e.g. [BCR]) there exists a semialgebraic curve $\tilde{\theta} : (0, \varepsilon) \to \overline{B}^n(r_0)$ such that

$$|\nabla f(x)| \ge |\nabla f(\tilde{\theta}(s))|, \quad x \in f^{-1}(\tilde{\theta}(s)) \cap \overline{B}^n(r_0).$$

It follows from Puiseux's Theorem (cf. e.g. [Ło3]) that there exists a rational number $q > 0$ such that $\theta(s) = \tilde{\theta}(s^q)$, $s > 0$ has an analytic (holomorphic) extension which is actually injective on a small disk around the origin.

Observe that $\theta(s) \in \Gamma(f) \subset \Theta_1(f) \cup \Theta_2(f)$. But θ is analytic and $\Theta_1(f)$, $\Theta_2(f)$ are algebraic sets, so

$$\theta(s) \in \Theta_1(f),\ s \in (0, \varepsilon) \quad \textit{or} \quad \theta(s) \in \Theta_2(f), s \in (0, \varepsilon)$$

for an $\varepsilon > 0$ small enough. From (4.2) and (4.4) we obtain

$$\mathrm{ord}_0(f \circ \theta) \le d(3d-3)^{n-1}. \tag{4.5}$$

This proves Proposition 4.3 in Case 1 that is for a generic polynomial.

Case 2. The general case. We now consider an arbitrary polynomial $f : \mathbb{R}^n \to \mathbb{R}$ of degree $d \ge 2$. By Proposition 2.2 there exists an algebraic family of polynomials f_t, $t \in \mathbb{R}$ of degree at most d such that: $f_0 = f$ and for all sufficiently small $t \ne 0$ the polynomial f_t is generic in the sense of section 2. Actually, by Corollary 2.3 we may choose the family in the way that the coefficients of f_t are polynomials in t.

Hence f_t is a well defined polynomial on $\mathbb{C}^n$ of degree at most d. So we have two associated algebraic family of curves $C_t = \Theta_1(f_t)$, $t \in \mathbb{C}^*$ and $D_t = \Theta_2(f_t)$, $t \in \mathbb{C}^*$.

More precisely C_t or D_t is a curve for all but finitely many $t \in \mathbb{C}^*$. This follows from the fact that the genericity in Proposition 2.2 comes from a transversality condition which is valid also in the complex case.

Let C_0 and D_0 be the respective limits , as $t \to 0$, of the families $C_t = \Theta_1(f_t)$, $t \in \mathbb{C}^*$ and $D_t = \Theta_2(f_t)$, $t \in \mathbb{C}^*$. Note C_0 and D_0 are algebraic curves in $\mathbb{C}^n$. Let us denote

$$\Gamma_0(f) = \mathbb{R}^n \cap (C_0 \cup D_0).$$

Lemma 4.6. *Let $y \in \mathbb{R}$ and assume that $f^{-1}(y) \cap \overline{B}^n(r_0) \neq \emptyset$. Then there exists a point $x_0 \in \Gamma_0(f) \cap \overline{B}^n(r_0)$ such that*

$$|\nabla f(x)| \geq |\nabla f(x_0)|, \quad x \in f^{-1}(y) \cap \overline{B}^n(r_0).$$

To prove the lemma observe that $\Gamma_0(f) \cap \overline{B}^n(r_0)$ is the Hausdorf limit, as $t \to 0$, of the family $(C_t \cup D_t) \cap \overline{B}^n(r_0)$, $t \in \mathbb{R}^*$. Recall that if $f_t^{-1}(y) \cap \overline{B}^n(r_0)$ is nonempty then $|\nabla f_t|$, restricted to $f_t^{-1}(y) \cap \overline{B}^n(r_0)$, has a minimum at a point which belongs to $(C_t \cup D_t) \cap \overline{B}^n(r_0) = \Theta(f_t) \cap \overline{B}^n(r_0)$. We leave the details to the reader.

We now are in the position to finish the proof of Proposition 4.3 in the second case. As in the case 1 we can choose in $\Gamma_0(f)$ an analytic arc $\theta(s)$, $s \in (-a, a)$ which satisfies condition 2 in Proposition 4.3. Clearly as in the case 1 there are two possibilities: the image of θ is included in C_0 or in D_0. Recall that, by formula (4.1),

$$m(\Theta_1(f_t), f_t) \leq d(3d-4)^{n-1} \leq d(3d-3)^{n-1},\ t \neq 0.$$

So, by semi-continuity of intersection Lemma 3.2, we have

$$m(C_0, f) \leq d(3d-4)^{n-1} \leq d(3d-3)^{n-1}. \tag{4.6}$$

Hence if the image of θ is included in C_0 it follows from Lemma 3.1

$$\mathrm{ord}_0(f \circ \theta) \leq m(C_0, f) \leq d(3d-3)^{n-1}. \tag{4.7}$$

Analogously by formula (4.3) we have

$$m(\Theta_2(f_t), f_t) \leq 2d(3d-3)^{n-2} \leq d(3d-3)^{n-1},\ t \neq 0.$$

Again, by semi-continuity of intersection Lemma 3.2, we have

$$m(D_0, f) \leq 2d(3d-3)^{n-1} \leq d(3d-3)^{n-1}. \tag{4.8}$$

Hence if the image of θ is included in D_0 it follows from Lemma 3.1

$$\mathrm{ord}_0(f \circ \theta) \leq m(D_0, f) \leq d(3d-3)^{n-1}. \tag{4.9}$$

This ends the proof of Proposition 4.3. □

Remark 4.7. Note that, contrarily to multiplicity of intersection, Łojasiewicz's exponent is not upper semi-continuous. Consider for instance the family

$$f_t(x) = tx^2 + x^d.$$

Clearly $\rho_{f_t} = \dfrac{1}{2}$ for $t \neq 0$ and $\rho_{f_0} = \dfrac{1}{d}$.

So in the proof of Theorem 4.2 we cannot claim that we first prove the estimate for a generic polynomial and then we extend it by "continuity" for a general polynomial. Actually a generic polynomial is a Morse function hence its Łojasiewicz's exponent is equal to $\frac{1}{2}$. What we can control is the multiplicity of intersection of the polynomial with an algebraic curve on which the Łojasiewicz's exponent is reached.

References

[Be-Ri] R. Benedetti, J-J. Risler, *Real algebraic and semialgebraic sets,* Actualités Mathématiques, Hermann, Paris (1990).

[BCR] J. Bochnak, M. Coste, M.F. Roy, *Real algebraic geometry,* E.M.G vol. 36 (1998) Springer.

[Bo-Ri] J. Bochnak & ; J-J. Risler, *Sur les exposants de Lojasiewicz,* Comment. Math. Helv. 50 (1975), no. 4, 493–507.

[D'A-Ku] D. D'Acunto & K. Kurdyka, *Bounds for gradient trajectories of polynomial and definable functions with applications,* Preprint Université de Savoie (2004).

[Fu] W. Fulton, *Intersection Theory,* Springer (1998).

[Gw] J. Gwoździewicz, *The Lojasiewicz exponent of an analytic function at an isolated zero,* Comment. Math. Helv. 74 (1999), no. 3, 364–375.

[Ha-Lê] H. A. Hamm, & D. T. Lê, *Un thorme de Zariski du type de Lefschetz,* Ann. Sci. cole Norm. Sup. (4) 6 (1973), 317–355.

[HMS] J. P. G. Henry, M. Merle, C. Sabbah *Sur la condition de Thom stricte pour un morphisme analytique complexe,* Ann. Sci. Ec. Norm. Sup. 17 (1984), 227–268. definable in o-minimal structures,

[KMP] K. Kurdyka, T. Mostowski, A. Parusiński, *Proof of the gradient conjecture of R. Thom,* Annals of Math.,152 (2000) 763-792.

[Ło1] S. Łojasiewicz *Ensembles semi-analytiques,* preprint IHES, 1965.

[Ło2] S. Łojasiewicz ,*Sur les trajectoires du gradient d'une fonction analytique,*Geometry seminars, 1982–1983 , 115–117, Univ. Stud. Bologna, Bologna, (1984).

[Ło3] S.Łojasiewicz, *Introduction to complex analytic geometry.* Birkhäuser Veralg. Basel, (1991).

[Mo] R. Moussu, *Sur la dynamique des gradients. Existence de varits invariantes,* Math. Ann. 307 (1997), no. 3, 445–460.

D. D'Acunto, Dipartimento di Matematica, Università degli Studi di Pisa, Via Filippo Buonarroti, 2 56127 Pisa, ITALY

E-mail address: `didier.dacunto@univ-savoie.fr`

K. Kurdyka, LAMA, UMR 5127 CNRS, Le Bourget-du-Lac Cedex, (FRANCE)

E-mail address: `krzysztof.kurdyka@univ-savoie.fr`