ON GRADIENT AT INFINITY OF REAL POLYNOMIALS

DIDIER D'ACUNTO AND VINCENT GRANDJEAN

Abstract. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a polynomial function. We discuss on different conditions to trivialise the graph of f by its level sets in the neighbourhood of a critical value at infinity via the gradient field of f. We also exhibit a Łojasiewicz type inequality which is useful to the present study. When $n = 2$, we are able to relate this Łojasiewicz type inequality with generic polar curves.

1. Introduction

Let $f : \mathbb{K}^n \to \mathbb{K}$ be a non constant polynomial function, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Many works were done in the thirty last years to understand the global topology of the levels of the function f. In the late sixties, Thom proved the existence of a smallest finite subset of $\mathbb{K}$, namely the bifurcation set of f denoted by $B(f)$, such that the restricted function $f : \mathbb{K}^n \setminus f^{-1}(B(f)) \to \mathbb{K} \setminus B(f)$ induces a locally trivial smooth fibration (see [Th]). But he did not give any effective way to find these bifurcation values. Obviously any critical values of f belongs to $B(f)$.

In [Br], Broughton exhibited his now on famous example, $f(x, y) := y(xy - 1)$, without any critical point but nevertheless with 0 as a unique bifurcation value ! At the same time Pham was exhibiting a sufficient condition to trivialise by the gradient field of f over a neighbourhood of a value $c \in \mathbb{K}$: the Malgrange condition [Ph]. Roughly speaking this condition means that the gradient ∇f is not too small in a neighbourhood of the germ at infinity of the given level $f^{-1}(c)$. As exhibited by Broughton a polynomial function without any critical point does not necessarily induce a locally trivial fibration over $\mathbb{K}$. The first idea to come to explain this, as did by Broughton, is that, working with the projective closure of the graph of f, the function should have singularities on the boundary of its domain, that is in this case, on the hyperplane at infinity. Some of its singularities at infinity may be indeterminate points of an extension of the function f on the hyperplane at infinity. Some of the limit values the function can take at such indetermination point have to be very special to create a change in

Date: 19th March 2004.

2000 *Mathematics Subject Classification.* Primary 32Bxx, 34Cxx, Secondary 32Sxx, 14P10.

Key words and phrases. gradient trajectories, polynomials.

Partially supported by the European research network IHP-RAAG contract number HPRN-CT-2001-00271.

the topology of the levels outside the set of the usual critical levels. This attempt to understand what can happen leads to the notion of asymptotic critical values, that is the values at which the Malgrange condition fails. It is now well known that, for any regular value which is not an asymptotic critical value, there is a neighbourhood of this regular value over which the function f induces a trivial fibration. That is any bifurcation value is either a critical value or an asymptotic critical value. Moreover there are at most finitely many asymptotic critical values (see for instance [KOS]).

In whole generality in the complex case, even now, we do not know if an asymptotic critical value is a bifurcation value. In the real case it is definitely not true.

Since the size of the gradient field ∇f in a neighbourhood at infinity of a level c is used as a sufficient criteria to trivialise the polynomial function over a neighbourhood of this value c, we would like to understand the connections between the geometry of the levels of the function f and the behaviour of the trajectories of the gradient field ∇f. To be more precise we are particularly interested in the trajectories leaving any compact subset of $\mathbb{K}^n$ and on a half-branch at infinity along which f tends to a finite value c. But this seems to be a quite difficult question, that is why we have restricted our attention to the plane situation. The paper is organised as follows.

Section 2 recalls the notion an asymptotic critical value.

Section 3 recalls an embedding theorem recenlty established by the first named author ([D'A2]). This enables to introduce the notion of trajectory of infinite length at a value c, the integral curves of ∇f we would like to understand.

Section 4 is devoted to, so-called by us, the Kurdyka-Łojasiewicz exponent at infinity at a real value c. This rational number, say ϱ carries important informations about the vanishing of ∇f at infinity. First of all $\varrho \leqslant 1$. We prove c is an asymptotic critical value if and only if $\varrho > 0$. Moreover we obtain that the trivialisation by ∇f over a neighbourhood of c is effective once $\varrho < 1$.

The last four sections only deal with the real plane polynomial situation. Section 5 recalls some background specific materials for this purpose. In Section 6, we study the behaviour of a connected component of a pencil at infinity of generic levels and are able to prove that $|\nabla f|$ uniformly tends to $+\infty$ on this pencil. Section 7 deals with trajectories of infinite length and principal polar curves. Section 8 illustrates our results through three simple but very representative examples.

Remark 1.1. All along the paper we do not say any single word about the analog results we could hope if we worked with a function definable in an o-minimal structure expanding the real numbers. Any o-minimal structure containing the exponential function brings many more technical difficulties. These natural extensions of the present work are the object of a current work ([DG]).

By the gradient ∇f of f we mean the vector field whose components are the partial derivative of f respective to x_i. We will denote by $\langle \cdot, \cdot \rangle$ the scalar product for the underlying real Euclidean structure on $\mathbb{K}^n$.

By a trajectory of the gradient field of f, we mean the underlying geometric object, that is the maximal curve (for the inclusion) which is tangent at each of its point to ∇f. This means that, unless specified, we do not distinguish the integral curves of ∇f or $-\nabla f$.

The only conventions we will use all along this paper are the following. Let u and v be two germs at infinity of single real variable functions. We write $u \sim v$ to mean that the ratio u/v has a non zero finite limit at infinity. We write $u \simeq v$ when the limit of u/v at infinity is 1.

2. Asymptotic critical values

Let $f : \mathbb{K}^n \to \mathbb{K}$ be a polynomial function of degree d, where $\mathbb{K}$ stands for $\mathbb{R}$ or $\mathbb{C}$. As said in the introduction, the fibres of this function describe only finitely many topological types (cf. [Th]). The values at which the topology changes are called the bifurcation values (or atypical values) of the function f. Any other value is called a typical value. The set of atypical values is finite and denoted by $B(f)$. In this set we distinguish two sorts of values: the usual critical values, denoted by $K_0(f)$, that is the values corresponding to the levels of f having singularities, and the asymptotic critical values. Since the projective closure of any level of the function f can have singularities at infinity, an asymptotic critical value is a special value (or limit value) the function takes at infinity because of the singularities of its levels on the hyperplane at infinity.

We denote by $K_\infty(f)$ the set of *asymptotic critical values* (also called *critical values at infinity*), to be any real number defined as follows

Definition 2.1. Let $f : \mathbb{K}^n \to \mathbb{K}$ be a polynomial function of degree d. A real number c belongs to $K_\infty(f)$ if and only if there exists a sequence $(x_m)_m \in \mathbb{K}^n$ satisfying the following conditions:
(1) $|x_m| \to \infty$
(2) $f(x_m) \to c$
(3) $|x_m| \cdot |\nabla f(x_m)| \to 0$.

Let $K(f)$ be the set of *generalised critical values* defined as $K_0(f) \cup K_\infty(f)$. Any value t not in $K(f)$ will be called a *generic value of* f. When f is a real or complex polynomial (in n variables), then the set $K_\infty(f)$, is finite (see [Pa] or [KOS]).

Remark 2.2. This definition is also making sense for any C^1 function defined on a non bounded open subset of $\mathbb{R}^n$.

If a value t is a generic value of f, then condition (3) is never satisfied for any sequence verifying conditions (1) and (2). So there exist constants $C > 0, R > 0, \eta > 0$ such that, for any $|x| \geqslant R$ with $|f(x) - t| \leqslant \eta$, then $|x| \cdot |\nabla f(x)| \geqslant C$.

This inequality is called the *Malgrange condition* and will be referred as condition **(M)** at t throughout the rest of this paper. Condition **(M)** at t is sufficient to ensure that the function f induces a locally trivial fibration over a neighbourhood of t (see [Ti, 2.15, 2.11, 2.6]). In the real case, condition **(M)** ensures the trivialisation via the gradient field ∇f (cf. e.g. [D'A2]).

In the complex domain, when the polynomial f has only isolated singularities at infinity (see [Pa]) the bifurcation values of f are exactly the critical values and the asymptotic critical values. But, in the real case, the situation is just more complicated: there are polynomials having an asymptotic critical value c which is not a bifurcation value. Moreover among such polynomials, there are examples where the trivialisation over a neighbourhood of c cannot be realised by any flow of the direction field provided by ∇f (see Section 9).

Remark 2.3. Working in the real or the complex plane, the Malgrange condition at t:

(M) $|(x,y)||\nabla f(x,y)| \geqslant C(>0)$, when $|(x,y)| \geqslant R$ and $|f(x,y)-t| \leqslant \eta$

is equivalent to require ([Ha])

$$|\nabla f(x,y)| \geqslant C'(>0), \text{ for any } |(x,y)| \geqslant R \text{ with } |f(x,y)-t| \leqslant \eta.$$

3. The embedding Theorem

In this section and in section 4, f denotes a C^2 semialgebraic function $f : \mathbb{R}^n \to \mathbb{R}$. The definition of asymptotic critical values still makes sense for f. Moreover the set $K(f)$ of generalised critical values is finite (see for instance [D'A1] or [KOS]). Let c be some value in $K_\infty(f) \setminus K_0(f)$. Let t be a generic value such that the interval $]c,t]$ contains neither critical value nor asymptotic critical value. The following result precise the behaviour of the trajectories of ∇f passing through the level set $f^{-1}(c)$.

Theorem 3.1 ([D'A2]). *There exists a C^1 embedding $\phi : f^{-1}(c) \to f^{-1}(t)$. More precisely, the flow of $\nabla f/|\nabla f|^2$ embeds each connected component of $f^{-1}(c)$ into a connected component of $f^{-1}(t)$.*

For a proof of Theorem 3.1, see [D'A2]. Note the same thing can be done for values $t < c$ in taking the opposite field $-\nabla f/|\nabla f|^2$. Theorem 3.1 remains true if f is a complex polynomial.

When c is not in $K(f)$, the same result is obviously true and its converse also is. So the only interesting values rising the problem of a converse of this statement are the asymptotic critical values.

The embedding Theorem is a key result for the work presented here. It says that any point on an asymptotic critical level is pushed by the gradient flow on any generic neighbouring level. But the embedding Theorem does not say anything about the possibility to push a whole generic neighbouring level in the asymptotic critical level.

The real setting provides examples of functions which are locally trivial fibrations near a value c which cannot be realised by the flow of ∇f. This means, given a value t arbitrarily close to c, there exists at least a maximal trajectory of ∇f (or of $-\nabla f$), from a point of the level t, which will never reach the level c. Such a trajectory has to leave any compact subset of $\mathbb{R}^n$. We call a connected component of the germ at infinity of such a trajectory a half-branch at infinity of the trajectory. This leads us to introduce the following

Definition 3.2. An integral curve of $\pm\nabla f$ leaving any compact subset of $\mathbb{R}^n$ such that the function f has finite limit c on a half-branch at infinity is called *an integral curve (or trajectory) of infinite length at c.*

Theorem 3.1 can be restated under the following form that we will use quite a lot throughout the rest of this paper.

Corollary 3.3. *Assume there is a trajectory of infinite length of ∇f at $c \in \mathbb{R}$. Then c is an asymptotic critical value of f.*

4. Kurdyka-Łojasiewicz exponent at infinity at an asymptotic critical value

Let us recall the well known Łojasiewicz inequality satisfied by any analytic function germ f at the origin O of $\mathbb{K}^n$ and let $c := f(O)$.
There exist a non negative number ϱ and a positive number C such that

$$|\nabla f| \geqslant C|f - c|^{\varrho}.$$

The supremum of the exponent e such that $|\nabla f||f-c|^{-e}$ has a positive limit along any sequence converging to O is called the Łojasiewicz exponent of f and is a rational number lying in $[0, 1[$.

When working with asymptotic critical value(s) of a polynomial function f, can we expect some similar inequalities satisfied by f in a neighbourhood of an asymptotic critical value?

Note that there is already a notion of Łojasiewicz exponent at infinity near a critical value for plane complex polynomial f, which is quite meaningful, but which compares $|\nabla f(x)|$ with $|x|$.

First, we recall, in the semialgebraic case, what we mean by a Łojasiewicz inequality at infinity near an asymptotic critical value:

Proposition 4.1 ([D'A2])**.** *Let f be a C^1 semialgebraic function. If $c \in \mathbb{R}$, then there exist real numbers $C, R, \tau > 0$ and a rational number $\varrho > 0$ such that for all $x \in \mathbb{R}^n$, $|x| > R$ and $|f(x) - c| < \tau$, then*

$$|x| \cdot |\nabla f(x)| \geqslant C|f(x) - c|^{\varrho}.$$

We can refine Proposition 4.1 in the following way:

Proposition 4.2. *The supremum of the exponent e such that $|\nabla f||f-c|^{-e}$ has a positive limit along any sequence going to infinity is a rational number and is always less or equal than 1.*

Proof. By the curve selection Lemma it suffices to prove this fact on semialgebraic curves. Let $\mathcal{C}$ be a semialgebraic curve satisfying the conditions of Definition 2.1. For simplicity we will only consider values $t < c$. The curve $\mathcal{C}$ is the image of the semialgebraic map $]c-\tau, c[\ni t \mapsto m(t) \in \mathbb{R}^n$ verifying $f \circ m(t) = t$ where τ is the real number of Proposition 4.1. Since the function $t \mapsto |m(t)|$ is semialgebraic and $|m(t)|$ increases to infinity as t goes to c, there exists a rational number $\alpha > 0$ and a positive real number K such that

$$|m(t)| \simeq K|t-c|^{-\alpha} \text{ when } t \text{ goes to } c.$$

By usual semialgebraic arguments, we get

$$|m'(t)| \simeq K\alpha|t-c|^{-(1+\alpha)} \text{ when } t \text{ goes to } c.$$

Taking derivatives with respect to t, we obtain

$$(f \circ m)'(t) = \langle \nabla f(m(t)), m'(t) \rangle = 1.$$

Thus we deduce $|\nabla f(m(t))| \geqslant \frac{1}{2K\alpha}|t-c|^{\alpha+1}$ and

$$|m(t)| \cdot |\nabla f(m(t))| \geqslant \frac{1}{4\alpha}|t-c| \tag{4.1}$$

By semialgebraicity, there exists a rational number ν such that $|m(t)| \cdot |\nabla f(m(t))| \sim |t-c|^{\nu}$. ¿From inequation (4.1) we obtain $\nu \leqslant 1$. Let ϱ be the supremum of these exponents ν. Let us define the following subset of $\mathbb{Q}$

$$E = \left\{ q \in \mathbb{Q} : \lim_{|x| \to +\infty} \frac{|x| \cdot |\nabla f(x)|}{|f(x)-c|^q} \in \mathbb{R}_+^*, \lim_{|x| \to +\infty} f(x) = c \right\}.$$

We easily verify that E is a semialgebraic subset of $\mathbb{R}$ contained in $\mathbb{Q}$, then it is finite (for details see [KMP, Proposition 4.2]). Thus ϱ is rational. □

Since there is yet a Łojasiewicz exponent at infinity, we will refer to ϱ as the *Kurdyka-Łojasiewicz exponent at infinity at c*.

The Malgrange condition corresponds to a value c of the given function for which the Kurdyka-Łojasiewicz exponent at infinity at c is less than or equal to 0.

Proposition 4.3. *Let f be a C^1 semialgebraic function. Let $c \in \mathbb{R}$. Then c is an asymptotic critical values of f if and only if the Kurdyka-Łojasiewicz exponent at infinity of f at c is positive.*

Proof. Obvious ! □

Let $c \in K_\infty(f) \setminus K_0(f)$ and let ϱ be the Kurdyka-Łojasiewicz exponent at infinity at c. This number contains an interesting information about the kind of value (typical or not) that c could be, as shown by the following

Theorem 4.4. *If $\varrho < 1$, then f is a locally trivial fibration over c, moreover the fibration can be realised by the flow of $\nabla f/|\nabla f|^2$.*

Proof. For simplicity we shall only work with values $t < c$. Let $t_0 < c$ be such that $[t_0, c] \cap K(f) = \{c\}$ and let $R, C > 0$ be real numbers such that Proposition 4.2 holds in $f^{-1}([t_0, c)) \cap \{|x| > R\}$ with constant C. Let $x_0 \in f^{-1}(t_0) \cap \{|x| > R\}$ and let Γ be a (maximal) trajectory of ∇f parametrised by the levels of f and so satisfying to the following differential equation

$$\gamma'(t) = \mathbf{X}(\gamma(t)), \text{with initial condition } \gamma(t_0) = x_0 \in f^{-1}(t_0) \tag{4.2}$$

where $\mathbf{X}$ is the vector field $\nabla f / |\nabla f|^2$. Thus for all t we obtain $f \circ \gamma(t) = t$.

Integrating Equation (4.2) between t_0 ant $t < c$, we obtain

$$\int_{t_0}^{t} \gamma'(s) \mathrm{d}s = \int_{t_0}^{t} \mathbf{X}(\gamma(s)) \mathrm{d}s \tag{4.3}$$

From Equation (4.3), we get a first inequality

$$|\gamma(t)| \leqslant |\gamma(t_0)| + \int_{t_0}^{t} \frac{\mathrm{d}s}{|\nabla f(\gamma)|} \tag{4.4}$$

Then using Proposition 4.1 we have

$$|\gamma(t)| \leqslant |\gamma(t_0)| + \int_{t_0}^{t} \frac{|\gamma(s)|}{C|s-c|^{\varrho}} \mathrm{d}s \tag{4.5}$$

Then Gronwall Lemma gives

$$|\gamma(t)| \leqslant |\gamma(t_0)| \exp \int_{t_0}^{t} \frac{\mathrm{d}s}{C|s-c|^{\varrho}} \tag{4.6}$$

which actually provides

$$|\gamma(t)| \leqslant |\gamma(t_0)| \exp \frac{(c-t_0)^{1-\varrho} - (c-t)^{1-\varrho}}{C(1-\varrho)} \tag{4.7}$$

Then $|\gamma(t)|$ has a finite limit when t tends to c. This implies that the embedding ϕ of Theorem 3.1 is essentially a diffeomorphism from $f^{-1}(t)$ onto $f^{-1}(c)$. This ends the proof. □

Corollary 4.5. *If c is a regular value and a bifurcation value, then the Kurdyka-Łojasiewicz at infinity at c is equal to* 1.

Proof. Since we cannot trivialise the function f over a neighbourhood of c, from Theorem 4.4, the exponent has to be at least 1. □

The following result is an interesting corollary, because it provides an information about the geometry of the fibres near a value c whose Kurdyka-Łojasiewicz exponent at infinity is strictly smaller than 1:

Corollary 4.6. *Let c be a value in $K_\infty(f) \setminus B(f)$. Assume that, ϱ, the Kurdyka-Łojasiewicz exponent at infinity at c is < 1. Let $\mathcal{C}$ be a semialgebraic curve with a half branch at infinity along which f tends to c at infinity. Then*

$$\lim_{x \in \mathcal{C}, |x| \to +\infty} \left\langle \frac{x}{|x|}, \frac{\nabla f(x)}{|\nabla f(x)|} \right\rangle = 0$$

Proof. For simplicity let us assume that f is increasing along the curve. Let us parametrise this curve by $m : r \in [R_0, +\infty[\to \mathbb{R}^n$ such that $|m(r)| = r$ and $R_0 > 0$ is chosen such that $|m(r)|$ is strictly increasing. When r tends to $+\infty$, this implies $|m'(r)|$ tends to 1 and by semialgebraicity $m'(r)$ becomes radial. Because of the semialgebraic data, there exist a non zero real number A and a positive rational number α such that

$$f(m(r)) - c \simeq Ar^{-\alpha} \text{ and so } \langle m'(r), \nabla f(m(r))\rangle \simeq \alpha A r^{-(\alpha+1)}.$$

Since $r|\nabla f(m(r))| \geqslant C|f(m(r)) - c|^{\rho}$, there exists a positive constant B such that

$$\left|\left\langle m'(r), \frac{\nabla f(m(r))}{|\nabla f(m(r))|}\right\rangle\right| \leqslant Br^{-\alpha(1-\varrho)} \to 0 \text{ when } r \to +\infty.$$

This ends the proof. □

From Theorem 3.1 and Theorem 4.4, when c belongs to $K_\infty(f) \setminus B(f)$, the function f induces a trivial fibration over a neighbourhood of c. Moreover this trivialisation is provided by the flow of ∇f when the Kurdyka-Łojasiewicz exponent at infinity at c is.

Corollary 4.7. *Let* Γ *be a trajectory of* ∇f *of infinite length at* c*. Then the Kurdyka-Łojasiewicz exponent at infinity at* c *is equal to* 1*.*

Proof. Let us also denote by Γ, the half-branch at infinity of this trajectory along which f tends to c. Assume that f is increasing along this half-branch when going to infinity. Let $\gamma(t)$ be $\Gamma \cap f^{-1}(t)$, with $t < c$ and $[t, c[\cap K(f) = \emptyset$. If the Kurdyka-Łojasiewicz exponent at infinity at c is < 1, then from Theorem 4.4 we deduce, $|\gamma(t)|$ is bounded when t tends to c, which contradicts the definition of Γ. □

There is an important subset of $\mathbb{R}^n$ that plays an important part in the study of a singular point. This set, which we call *Talweg*, is defined as the locus of points inside a typical fibre $f^{-1}(t)$ where the function $|\nabla f|^2$ restricted to $f^{-1}(t)$ attains a local minimum. Doing this for all typical value t produces an algebraic set. One can prove, see [DK], that the Talweg set is generically of dimension one. As we shall see it further, in dimension 2, the function $|\nabla f|^2$ restricted to $f^{-1}(t)$ goes to infinity on the half-branches of $f^{-1}(t)$ provided that t is not a complex asymptotic critical value. What we prove is the following

Proposition 4.8. *Assume* f *is a polynomial in two variables. Let* $c \in K_{infty}(f)$ *and let* ϱ *be the the Kurdyka-Łojasiewicz exponent at infinity associated to* c*. Then the Kurdyka-Łojasiewicz is reached on a connected component of the Talweg set of* f*.*

Proof. If c belongs to $K_{infty}(f)$ then there is a half-branch at infinity, say $t\gamma(t)$ with $f \circ \gamma(t) = t$ and $|gamma(t)| \to \infty$ when $t \to c$, that belongs to the Talweg set of f. Among the (finite) family of such curves, let us take one where the local minimum of $|\nabla f|^2$ restricted to $f^{-1}(t)$ is actually a global minimum. We still call γ this curve even after reparameterising this curve by the distance to the origin. We thus have defined a curve $r \mapsto \gamma(r)$ with $\gamma(r)| = r$ and $f \circ \gamma(r) = t(r) \to c$ when $r \to \infty$. Then there exists $R > 0$ such that the function $r \mapsto |\nabla f(\gamma(r))|$ is strictly decreasing on $[R, +\infty)$. And for all $x \in f^{-1}(f(\gamma(r)))$ we have

$$|\nabla f(x)| \geq |\nabla f(\gamma(r))|$$

This imply, for all $x \in f^{-1}(f(\gamma(r)))$ with $|x| \geq R$, the inequality

$$|x||\nabla f(x)| \geq r|\nabla f(\gamma(r))|$$

This imply that the Kurdyka-Łojasiewicz is reached on the curve γ. □

5. Background material for the real plane situation

We recall here several results we will use quite often in the following of our paper.

Let A be a germ at infinity of a subset of $\mathbb{R}^2$. In compactifying $\mathbb{R}^2$ by a line $\mathbb{RP}^1$ at infinity, denoted by $\mathbb{RP}^1_\infty$ we define the cone at infinity of A by $C_\infty(A) = \overline{A} \setminus A$ where the closure $\overline{A}$ of A is taken in $\mathbb{R}^2 \cup \mathbb{RP}^1_\infty$ for the usual topology.

Lemma 5.1. [FP, Lemma 2.1]) *Let A be the germ at infinity of a semialgebraic band in $\mathbb{R}^2$, that is $A := \{(x, y) : g_1(x) < y < g_2(x)\}$ and g_1 and g_2 are continuous semialgebraic functions, and let f be a real polynomial of degree $d > 1$ defined on $\mathbb{R}^2$. If f is bounded on A, then $C_\infty(A)$ consists in exactly one point.*

We will often say the germ of a curve, or a connected semialgebraic set, has $[a : b] \in \mathbb{RP}^1$ as asymptotic direction, or limit point, to mean that the cone at infinity of this geometric object is exactly $[a : b : 0]$.

Proposition 5.2. [FP, Lemma 2.1]) *Let A be the germ at infinity of a semialgebraic band in $\mathbb{R}^2$, and let f be a real polynomial of degree $d > 1$ defined on $\mathbb{R}^2$. If f has a finite limit on A at infinity, then $g_2 - g_1$ tends to* 0 *at infinity.*

The last key result is about the nature of the integral curves. Let us consider the following differential form $\omega := -\partial_y f \mathrm{d}x + \partial_x f \mathrm{d}y$, and let $\mathcal{F}$ be the foliation defined by ω on $\mathbb{R}^2$. Then $\mathcal{F}$ is just the phase portrait of the differential equation defined by the gradient ∇f.

By the usual Poincaré compactification of $\mathbb{R}^2$ as the open north hemisphere of $\mathbb{S}^2$, we extend the foliation $\mathcal{F}$ to an algebraic foliation $\tilde{\mathcal{F}}$on the whole $\mathbb{S}^2$. On the open south hemisphere of $\mathbb{S}^2$, the foliation is given by the differential 1-form $\omega_1(x, y) := -\omega(-x, -y)$ (after having diffeomorphically mapped the open south hemisphere on $\mathbb{R}^2$). If we denote by $\mathbb{S}^1$ the boundary of $\mathbb{R}^2$ in $\mathbb{S}^2$, then $\mathbb{S}^1$ is a finite union of stationary points and leaves of $\tilde{\mathcal{F}}$.

Since $\mathbb{S}^2$ is simply connected and $\tilde{\mathcal{F}}$ is an algebraic foliation, it is actually a Rolle foliation (see [Ch]). The following result exhibits an important property of any leaf of a Rolle foliation

Theorem 5.3. ([Sp], applied to our situation) *Let P be any stationary point of $\tilde{\mathcal{F}}$ lying on $\mathbb{S}^1$. Let $\tilde{\omega}_P$ be a local equation of $\tilde{\mathcal{F}}$ at P, defined in an semialgebraic open neighbourhood of P. Then there exists an o-minimal structure expanding $\mathbb{R}$, the Pfaffian closure of $(\mathbb{R}, +, \cdot, <)$, such that any leaf $L \subset U$ of $\tilde{\omega}_P$ is definable in this o-minimal structure.*

Any stationary point of $\mathbb{S}^1$ is a limit point of a trajectory of ∇f (of infinite length), this result actually implies that any such trajectory is the graph of a function definable in the Pfaffian closure of $(\mathbb{R}, +, \cdot, <)$. If $\Gamma \subset \mathbb{R}^2$ is a trajectory of infinite length, then it admits at least a half-branch at infinity, which coincides, for R large enough with the graph of a smooth definable function $h : [R, +\infty[\to \mathbb{R}$.

If we had compactified $\mathbb{R}^2$ just by a projective line $\mathbb{RP}^1$ at infinity, then we could have extended the foliation $\mathcal{F}$ to a foliation of $\mathbb{RP}^2 = \mathbb{R}^2 \cup \mathbb{RP}^1_\infty$, say $\mathcal{F}_1$. It is well known that $\tilde{\mathcal{F}}$ is the lift $\Pi^*(\mathcal{F}_1)$ by the usual projection $\Pi : \mathbb{S}^2 \to \mathbb{RP}^2$. If P is any stationary point of $\mathcal{F}_1$ lying on $\mathbb{RP}^1_\infty$, then any trajectory of ∇f, $\Gamma \subset \mathbb{R}^2$, with limit point P, can again be parametrised by a smooth definable function.

6. Asymptotic behaviour of $|\nabla f|$ on the generic fibre

From now to the end of this paper, unless specified, the function f is a real plane polynomial. We assume there are at most finitely many levels of f that contain an affine line.

Let t_0 be a generic value of f. Assume that $f^{-1}(t_0)$ is not compact. So $f^{-1}(t_0)$ has a half-branch, say F_{t_0}, with $[a : b : 0]$ as limit point in $\mathbb{RP}^2$ and assume this half-branch is contained (as germ at infinity) in a right half plane. By a rotation we can assume that $[a : b : 0] = [1 : 0 : 0]$.

Let $\varepsilon > 0$ such that $[t_0 - \varepsilon, t_0 + \varepsilon] \cap K(f) = \emptyset$. Let us denotes by Λ the germ at infinity of the connected component of $f^{-1}([t_0 - \varepsilon, t_0 + \varepsilon])$ that contains F_{t_0}. Then for all $t \in [t_0 - \varepsilon, t_0 + \varepsilon]$, the fibre $f^{-1}(t)$ has a unique half branch $F_t := f^{-1}(t) \cap \Lambda$. By Lemma 7.1, there is no component of the polar curve $\{\partial_y f = 0\}$ contained in Λ. From the semialgebraic version of the implicit function theorem ([BCR]), we deduce there exists $R_0 \gg 1$ such that, for each t, F_t is the graph of a Nash function g_t defined on $[R_0, +\infty[$.

By the change of coordinate at $+\infty$, $s = x^{-1}$, the germ at 0 of the function $sg_t(s^{-1})$ can be written as a Puiseux expansion. Hence there exists a polynomial function P_t of degree d_t such that $g_t(x^{d_t}) = P_t(x) + r_t(x^{d_t})$, where r_t is a Nash function such that $r_t(x) \to 0$ when $x \to +\infty$. Remember that d_t divides d.

Since $[t_0 - \varepsilon, t_0 + \varepsilon] \cap K(f) = \emptyset$, in taking a bigger R_0 if needed, there is a positive constant C such that, for all $(x, y) \in \Lambda \cap \{x \geqslant R_0\}$, the equivalent condition to condition **(M)** in the plane case, that is, $|\nabla f(x,y)| \geqslant C$ is satisfied (remark 2.3). Then the non oriented angle between the gradient and the horizontal direction at any point of $\Lambda \cap \{x \geqslant R_0\}$, has a non zero infimum, say $\theta_0 \in]0, \frac{\pi}{2}[$ ([Pa]). Since f is increasing along the trajectories of ∇f, there exists $(x_0, y_0) \in F_{t_0-\varepsilon}$ such that the trajectory of ∇f through this point will reach the level $t_0+\varepsilon$ in staying in Λ. Then, taking any point $(x,y) \in F_{t_0-\varepsilon}$ with $x \geq x_0$, the trajectory passing through (x,y) also reach the level $t_0+\varepsilon$ in staying in Λ. We then integrate the vector field $\nabla f/|\nabla f|^2$ between the levels $t_0-\varepsilon$ and $t_0+\varepsilon$ for all such trajectories. Using Malgrange inequality we obtain that the length of all the trajectories (more precisely the part of the trajectories lying between the levels $t_0-\varepsilon$ and $t_0+\varepsilon$) that stay in Λ is bounded by $2\varepsilon/C$. Thus we have proved the following

Lemma 6.1. *Under the above hypotheses*

$$\lim_{x\to+\infty} g_{t_0+\varepsilon}(x) - g_{t_0-\varepsilon}(x) \in \mathbb{R}\,.$$

Then there exists $l(t_0,\varepsilon) \geq 0$ such that for any $[t_1,t_2] \subset [t_0-\varepsilon, t_0+\varepsilon]$, the function $|g_{t_2}-g_{t_1}|$ has limit $l \leq l(t_0,\varepsilon)$ when $x \to +\infty$. Since the polynomial function $(P_{t_2}-P_{t_1})(x^q)$ has a finite limit at $+\infty$, with $q = d_{t_1}d_{t_2}$, we deduce the following

Corollary 6.2. *There exists a non negative integer N, smaller than or equal to d, such that for every t in $[t_0-\varepsilon, t_0+\varepsilon]$, there exists $a_t \in \mathbb{R}$ such that $P_t(x) = P(x^N) + a_t$.*

Since there are several choices of the constant in the polynomial P, we take P such that $a_{t_0} = 0$.

Proposition 6.3. *Under the previous hypotheses, for every t and t', the half branches F_t and $F_{t'}$ are asymptotic.*

Proof. Assume this assertion is not true. There exist $t_1 < t_2$ in $[t_0-\varepsilon, t_0+\varepsilon]$, such that $a_{t_1} - a_{t_2} > 0$ (the other case is similar).

Claim. *For each a in $]a_{t_1}, a_{t_2}[$, there exists a value $t \in [t_1,t_2]$, such that $a_t = a$.*

Let $\mathcal{C}_a$ be the semialgebraic curve contained in $\Lambda \cap f^{-1}([t_1,t_2])$ defined as $\{y = P(x^{\frac{1}{d}}) + a\}$, where $a \in]a_{t_1}, a_{t_2}[$. Then this curve is neither asymptotic to F_{t_1} nor to F_{t_2}. Let t_a be the limit of $f\,|_{\mathcal{C}_a}$ at infinity. Since there is just a connected component of $f^{-1}(a)$ in Λ, by Proposition 5.2, we deduce F_{t_a} is asymptotic to $\mathcal{C}_a$. The function $a \mapsto t_a$ is semialgebraic and injective (by Proposition 5.2), so the claim is proved.

A consequence of this Claim is that necessarily $f(x, P(x^{\frac{1}{d}})+a_t) \to t$ when $x \to +\infty$ (Proposition 5.2). Since $f(x, P(x^{\frac{1}{d}}) + a_t)$ is a polynomial in the single variable $x^{\frac{1}{d}}$ with a finite limit, it is constant, and thus $F_t = \{(x,y) : y = P(x^{\frac{1}{d}}) + a_t\}$. Then F_t contains an open half-line, but we assumed there were only but finitely many levels whose connected components could contain open lines. Thus we get a contradiction and so F_{t_1} and F_{t_2} have to be asymptotic. $\square$

Proposition 6.4. *Let t_0 and ε and Λ as above. Assume moreover that any $t \in [t_0 - \varepsilon, t_0 + \varepsilon]$ is not a complex asymptotic critical value of the function f, when seen as defined on $\mathbb{C}^2$. Then for every $M > 0$, there exists $R > 0$ such that*

$$\forall (x,y) \in \Lambda \cap \{|(x,y)| \geqslant R\},\ |\nabla f(x,y)| \geqslant M.$$

That is, $|\nabla f| \to +\infty$ on any sequence of points going to infinity and on which f tends to a generic value.

Proof. Assume this is not true. Then there exists a semialgebraic curve G, graph of a Nash function g contained in Λ, and a constant $C_1 > 0$, such that, when $x \to +\infty$

$$f(x, g(x)) \to t_1 \in [t_0 - \varepsilon, t_0 + \varepsilon] \text{ and } |\nabla f(x, g(x))| \to C_1.$$

Since $G \subset \Lambda$, we deduce

$$\frac{|\partial_y f|}{|\partial_x f|}(x, g(x)) \to +\infty \text{ and } |\partial_y f| \to C_1, \text{ as } |x| \to +\infty.$$

Let $f_0(x,y) = f(x, g(x) + y)$. Then

$$f_0(x,y) = t_1 + a_0(x) + y a_1(x) + \ldots + y^{d-1} a_{d-1}(x) + a_d y^d.$$

Since the coordinate system is not necessarilly generic we may have $a_d = 0$. Note that $|a_1(x)| \to C_1$. Since the functions a_k are Nash, for each k, there exist a rational number δ_k and non zero real number α_k such that $x^{-\delta_k} a_k(x) \to \alpha_k$ when $x \to +\infty$.
We also write

$$f_0(x, y+z) = t_1 + A_0(x,z) + y A_1(x,z) + \ldots + y^{d-1} A_{d-1}(x,z) + a_d y^d,$$

with

$$A_k(x,z) = a_k(x) + C_k^{k+1} z a_{k+1}(x) + \ldots + C_k^{d-1} z^{d-l-k} a_{d-1}(x) + C_k^d z^{d-k} a_d.$$

Lemma 6.5. *There exists $i \in \{1, \ldots, d\}$ such that $|a_i(x)| \to +\infty$ when $x \to +\infty$.*

Proof. Assume there exists a positive constant B such that, for each $i = 1, \ldots, d$ and $x \geqslant R$, then $|a_i(x)| \leqslant B$. We obtain

$$A_0(x,z) = \sum_{k=1}^{d} z^k a_k(x) \text{ and } A_1(x,z) = a_1(x) + \sum_{k=2}^{d} k z^{k-1} a_k(x)$$

Since all the functions a_i are bounded, for x large enough, if $h : [R, +\infty[\to \mathbb{R}$ is any function tending to 0 at infinity, then we get

$$f_0(x, h(x)) = A_0(x, h(x)) \to t_0 \text{ when } x \to +\infty.$$

Let $t_1 \in [t_0-\varepsilon, t_0+\varepsilon]\setminus\{t_0\}$. If we take $h = g_{t_1}-g_{t_0}$, we obtain a contradiction, since $t_1 = f(x, g_{t_0}(x) + h(x)) = f_0(x, h(x)) \to t_0$, when $x \to +\infty$. □

Let $E := \{k : \delta_k > 0\}$. By the previous Lemma, E is not empty. Let $m := \max\{(\delta_k - \delta_0)k^{-1} : k \in K \cup \{1\}\}$. Then $m \geqslant -\delta_0 > 0$. Let M be $\max\{\delta_k/(k-1)^{-1} : k \in K\}$. Then we easily verify that $M \geqslant m$. Let E_M be the set $\{k : \delta_k = (k-1)M\}$. For any $\lambda \in \mathbb{R}$ we obtain

$$A_1(x, \lambda x^{-M}) = \alpha_1 + \sum_{k \in E_M} k\lambda^{k-1}\alpha_k + r(x, \lambda),$$

with $r(x, \lambda) \to 0$ when $x \to +\infty$. Then there exists a complex number λ_1, such that $\sum_{k\in E_M} k\lambda_1^{k-1}\alpha_k = \alpha_1$. So we deduce

$$\partial_y f(x, g(x) + \lambda_1 x^{-M}) \to 0 \text{ when } x \to +\infty.$$

We just have to verify that $f(x, g(x) + \lambda_1 x^{-M}) \to t_1$ when $x \to +\infty$. For each $k = 1, \ldots, d$, $kM > \delta_k$ and since

$$f(x, g(x) + \lambda_1 x^{-M}) = t_1 + a_0(x) + \textstyle\sum_{k=1}^{d} \lambda_1^{-kM} x^{-kM} a_k(x),$$

then $f(x, g(x) + \lambda_1 x^{-m}) \to t_1$ when $x \to +\infty$ and thus $|\nabla f(x, g(x) + \lambda_1 x^{-m})| \to 0$ when $x \to +\infty$ which is impossible since we have assumed that t_1 was not a complex asymptotic critical value. Then $|a_1(x)| \to +\infty$ when $x \to +\infty$. □

Remark 6.6. Let us recall that for real polynomial there can be some real numbers that are not real critical values, while they are complex critical values of this polynomial seen as function on a complex affine space (take $f(x, y) = x^3 + y^3 + 3x + 3y$, then 0 is a complex critical value but a regular real value). Thus, the restriction made in the hypotheses of Proposition 6.4, seems to be necessary to get the desired result, since a real number can be a complex asymptotic critical value of a real polynomial function while this number is a real generic value of the same polynomial function as a real function.

Corollary 6.7. *Let G be semialgebraic half-branch at infinity such that the limits of f and $|\nabla f|$ along G are finite. If c is the limit of f, then c is a complex critical value of f.*

Proof. The only case to deal with is $|\nabla f| \to C > 0$, and the demonstration is given in the proof of Proposition 6.4 □

Remark 6.8. After having established this result, we were informed that Kuo and Parusiński obtained the same result in a rather similar way, see [KP, Theorem 4.1 (3)].

Assume now $c \in K_\infty(f) \setminus B(f)$. Assume there exists a connected component, say Λ_c, of $f^{-1}([c - \varepsilon_0, c]$ such that, as germs, $\Lambda \subset \Lambda_c$ and $f^{-1}(c)$ has a half-branch at infinity, denoted by F_c contained in Λ_c. Also assume that

$t_0 - \varepsilon_0$ is such that $[t_0 - \varepsilon_0, c] \cap K_\infty(f) = \{c\}$. Since f is bounded on Λ_c, by Lemma 5.1, $[1:0:0]$ is the limit point of the curve F_c at infinity. Thus F_c is the graph of a Nash function g_c in the variable x.

Proposition 6.9. *Under the previous hypotheses, F_c is asymptotic to any F_t, for $t_0 - \varepsilon_0 \leqslant t < c$.*

Proof. Let $t' \in (t, c)$ be close enough to c such that $|g_t(x) - g_c(x)| \to \lambda \in]0, +\infty]$ when $x \to +\infty$.
Assume that $g_t > g_c$. Since $g_t - g_{t'} \to 0$ at $+\infty$, then $f(x, g_c(x) + \alpha)$ tends to c for each $\alpha \in]0, \lambda[$. Let $\mathcal{H}_\alpha$ be the curve defined as $\{y = h_\alpha(x) = g_c(x) + \alpha\}$. For given α and for any $\eta > 0$, there exists $R_\eta \gg 1$ such that $c - \eta < f(x, y) < c$ for each $(x, y) \in \{x > R_\eta, 0 < y - g_c(x) < \alpha\}$. But by Proposition 5.2, we deduce that $h_\alpha - g_c$ has to tend to 0 at infinity, which is impossible. The case $g_t - g_c < 0$ is similar and we deduce that $g_t - g_c$ tends to 0 at $+\infty$. □

7. Sufficient conditions to trivialise by ∇f

In this section we investigate obstructions to trivialise by ∇f, near a value $c \in K_\infty(f) \setminus B(f)$, in terms of geometric objects related to the polar properties of f.

¿From the work done in Section 4, we just have to deal with the case of an asymptotic critical value c whose Kurdyka-Łojasiewicz exponent at infinity is exactly 1.

We assume here that f is now written in a generic (orthonormal) coordinate system, that is $f(x, y) = a_0 y^d + a_1(x) y^{d-1} + \ldots + a_d(x)$, where $a_i(x)$ are polynomial in the single variable x of degree at most i and with $a_0 \neq 0$. We assume as before that there are at most finitely many levels of f that are lines.

Let $\tilde{f}$ be the homogeneous polynomial made from f. Let

$$\mathbf{Y} := \{([x:y:z]; t) \in \mathbb{RP}^2 \times \mathbb{K} : \tilde{f}(x:y:z) - tz^d = 0\}.$$

Then $\mathbf{Y}$ is the projective closure of the graph of f. Let $\mathbf{Y}_\infty$ be $\mathbf{Y} \cap \mathbb{RP}^1_\infty$, where $\mathbb{RP}^1_\infty$ is the hyperplane at infinity with equation $\{x_0 = 0\}$. Let $\hat{f} := t \,|_{\mathbf{Y}}$.

Let $v := (v_1, v_2)$ be any point of $\mathbb{R}^2 \setminus (0,0)$ and let L_v be the linear form $\mathbb{R}^2 \to \mathbb{R}$ defined by $L_v(x) = xv_1 + yv_2$.

Let $\mathbf{PV}_v := \{(x, y) \in \mathbb{R}^2 : \nabla f \wedge v = 0\}$ be the affine polar variety in the direction of v. This algebraic set depends only on v seen as a direction in $\mathbb{RP}^1$. By [Ti, Theorem 3.1], there is an open dense subset Ω_f of $\mathbb{RP}^1$ such that for all $v \in \Omega_f$ the set $\mathbf{PV}_v$ is a curve or empty. A direction $v \in \Omega_f$ is called generic. Let $\overline{\mathbf{PV}}_v$ be the projective closure of $\mathbf{PV}_v$.

Let $P = ([v_P : 0]; c)$ be a point in $\mathbf{Y}_\infty$ where c belongs to $\mathbb{R}$. Let $\mathbf{PV}_P(t, x_0)$ be the germ at P of the closure of $\mathrm{Crt}_P(t, x_0) \setminus \mathbf{Y}_\infty$ in $\mathbf{Y}$, where $\mathrm{Crt}_P(t, x_0)$ denotes the critical locus of the map germ $(t, x_0) : (\mathbf{Y}, P) \to (\mathbb{R}^2, c \times 0)$.

The following conditions are equivalent [Ti, Proposition 3.5]

(1) $\mathbf{PV}_P(t, x_0) \neq \emptyset$.
(2) There exists $v \in \mathbb{RP}^1$ such that $P \in \overline{\mathbf{PV}}_v$.
(3) For any $v \in \mathbb{RP}^1$ such that $L_{v_P}(v) \neq 0$ then $P \in \overline{\mathbf{PV}}_v$.

An immediate consequence of this result is that the Kurdyka-Łojasiewicz exponent at c is equal to 1 since it is reached on any half-branch at infinity of a polar variety with limit point P verifying the above equivalent conditions.

Lemma 7.1. *Let $P := ([v_P : 0], c) \in \mathbf{Y}_\infty$. Let $v \in \mathbb{RP}^1$ such that $L_{v_P}(v) \neq 0$. Assume that the germ at P of $\mathbf{PV}_v$ is not empty. Then c is an asymptotic critical value and the Kurdyka-Łojasiewicz exponent at infinity at c is equal to* 1.

Proof. After an appropriate change of coordinates we may assume $v_P = [1 : 0 : 0]$ and without loss of generality, we assume there is a half-branch at infinity, say G, of $\mathbf{PV}_v$ with limit point $[1 : 0 : 0]$. So f tends to c along G. Let $g : [R, +\infty[\to \mathbb{R}^2$ be a parametrisation of G by the distance at the origin r, that is $|g(r)| = r$. Then if $g = (g_1, g_2)$ we deduce that $g_2/g_1 \to 0$ when $r \to +\infty$. Thus there exists a positive rational number α such that $f(g(r)) - c \sim r^{-\alpha}$ and so taking the derivative in r provides

$$r^{-(\alpha+1)} \sim \frac{\mathrm{d}}{\mathrm{d}r}(f \circ g)(r) = \langle g'(r), \nabla f(g(r)) \rangle.$$

By hypotheses we have $|g'| \simeq |g_1'| \simeq 1$, and since $v := [1 : v_2]$ we find

$$\frac{\mathrm{d}}{\mathrm{d}r}(f \circ g)(r) \sim g_1'(r) \frac{\partial f}{\partial x_1}(g(r)) \sim r^{-(\alpha+1)}.$$

Since $|\nabla f|(g(r)) \sim |\partial_{x_1} f|(g(r))$ we deduce immediately that $c \in K_\infty(f)$ and the Kurdyka-Łojasiewicz exponent at infinity at c is 1. □

Definition 7.2. Let $[v_P : 0]$ be a point on the line at infinity of $\mathbb{RP}^2$ belonging to the closure in $\mathbb{RP}^2$ of $f^{-1}(c)$. Any non compact connected component $\mathcal{C}$ of $\{\nabla f \wedge v_P = 0\}$, such that,
(i) the point $[v_P : 0]$ is the limit point of one of the half-branches, say G, at infinity of the curve $\mathcal{C}$,
(ii) the restriction of the function f to G tends to c at infinity,
is called *a principal polar curve at c.*

Let $c \in K_\infty(f) \setminus B(f)$. Let ε be a given positive real number such that $[c - \varepsilon, c] \cap K(f) = \{c\}$. We can assume there exists a non-empty connected component of $f^{-1}[c-\varepsilon, c]$, say $\Lambda(\varepsilon)$, contained in a right half plane $\{x \geqslant R_0\}$ (as germ at infinity). Let $\Lambda(R, \varepsilon) := \Lambda(\varepsilon) \cap \{x \geqslant R\}$. Let $[1 : \delta : 0] \in \mathbb{RP}^1_\infty$ be the limit point of $\Lambda(R, \varepsilon)$. For each $t \in [c-\varepsilon, c]$, there exists $R_t \geqslant R_0$ such that $F_t := f^{-1}(t) \cap \Lambda(R, \varepsilon)$ is the graph of a Nash function $g_t : [R_t, +\infty[\to \mathbb{R}$ such that $\lim_{x\to+\infty} g_t(x)/x = \delta$. In the following, given ε, we will always assume that $R \geqslant R_{c-\varepsilon}$. Let $P \in \mathbb{RP}^2 \times \mathbb{R}$ be the point $([1 : \delta : 0], c)$ and let us denote $v_P := [1 : \delta]$.

The following result precises the close relations between the exponent and the polar varieties, moreover it is a converse to Lemma 7.1.

Proposition 7.3. *Let $c \in K_\infty(f) \setminus B(f)$. Then Kurdyka-Łojasiewicz exponent at infinity at c is equal to 1, if and only if there is a principal polar curve at c.*

Proof. Assume there exists a semialgebraic half-branch at infinity, say G, contained in $\Lambda(R,\epsilon)$, for R large enough and ε small enough, such that the Kurdyka-Łojasiewicz exponent at infinity at c is reached on G. Let $g : [R,+\infty[\to \mathbb{R}^2$ be a parametrisation of G by r, the distance to the origin. Then we have $r|\nabla f(g(r))| \sim |f(g(r)) - c|$. For simplicity in the proof we assume $\delta = 0$. Let $v \in \mathbb{RP}^1$ be the limit direction of $\nabla f(g(r))$ when r goes to infinity. Then from Corollary 4.6 we deduce that $\langle v_P, v\rangle \neq 0$. So $v := [1 : \nu]$. Let $g(r(t))$ be $G \cap F_t$. $r(t)$ is strictly increasing if R and ε are well chosen. Let $v(t) := [1 : \nu(t)] \in \mathbb{RP}^1$ be the direction of $\nabla f(g(r(t)))$. Since $[v_P : 0] \in \overline{F_t}$, between $g(r(t))$ and P, the direction $\nabla f\,|_{F_t}$ contains an interval of the form $[\nu(t),+\infty[$ or $]-\infty,\nu(t)]$. Assume that it is $[\nu(t),+\infty[$ for each $t \in [c-\varepsilon, c[$. Since $\nu(t)$ tends to $\nu \in \mathbb{R}$ when t tends to c, this means there exist κ (depending on R and ε) such that $[\kappa,+\infty[\subset [\nu(t),+\infty[$ for each $t \in [c-\varepsilon, c[$. Thus for any $v := [1,a]$ with $a \in [\kappa,+\infty[$, there is a half-branch at infinity of $\mathbf{PV}_v$ contained in $\Lambda(R,\epsilon)$. To conclude there is just to apply Tibăr's result to find a principal polar curve whose germ at P is not empty. $\square$

Let us now define S_f.

Definition 7.4. A real number c belongs to the set S_f if and only if there exists a sequence $((x_n,y_n))_{n\in\mathbb{N}}$ satisfying the following conditions

(1) $|(x_n,y_n)| \to \infty$
(2) $f(x_n,y_n) \to c$
(3) there exists $\lambda_n \in \mathbb{R}^*$ such that $\nabla f((x_n,y_n)) = \lambda_n(x_n,y_n)$.

The following inclusions are true: $B(f) \subset S_f \subset K_\infty(f)$ (see for instance [D'A1]). There are quite elementary examples (real algebraic) that prove that any two of these sets of values are different.

Since f is actually a polynomial, this statement can be improved in the following way. Let us define $M(f)$ to be the real algebraic subset of $\mathbb{R}^2$

$$M(f) := \left\{(x,y) : \nabla f(x,y) \wedge \frac{\partial}{\partial r}(x,y) = 0\right\} \text{ with } \frac{\partial}{\partial r}(x,y) = \frac{(x,y)}{|(x,y)|}.$$

A real number c belongs to S_f if and only if $M(f)$ has a half-branch at infinity along which f tends to c. This is obviously obtained by a curve selection Lemma argument at infinity.

Proposition 7.5. *If $c \in S_f$,then there exists a half-branch of a principal polar curve contained in $\Lambda(R,\eta)$ along which f tends to c at infinity.*

Proof. Here since on any half-branch at infinity of $M(f)$ the gradient field becomes radial, this means that the Kurdyka-Łojasiewicz exponent at infinity at c is actually 1. So from now there is just to apply the previous Proposition. □

The other result we are interested in is

Proposition 7.6. *Let Γ be a trajectory of infinite length of ∇f at c, contained in any $\Lambda(R,\eta)$. Then there exists a half-branch of a principal polar curve at c.*

Proof. It works exactly in the same way as in the previous proof, since by Theorem 5.3, the Γ can be parametrised as the graph of a smooth definable function in the variable u ($u = x$ or $u = y$) on an interval $[A, +\infty[$, say γ, which means that $\gamma'(u)/|\gamma'(u)|$ and $\gamma(u)/|\gamma(u)|$ have the same limit when u tends to $+\infty$. □

Remark 7.7. By now, we were not able to provide the converse statements to the previous two Propositions in every cases despite many partial converses. The difficulty comes from the fact we are not able to say there are "enough" polar curves contained in $\Lambda(R,\varepsilon)$, since this is obvioulsy much more precise than having all, but at most the one in the normal direction, the polar curves which are not empty as germ at $[v_P : 0] \in \mathbb{RP}^2$, and along any half-branch (germ at $[v_P : 0]$) of which the function f is tending to c when tending to $[v_P : 0]$. If we assume $v_P := [1 : 0]$, *enough* means, there is a positive real number ν_c, such that for any $\nu \in]-\nu_c, +\infty[$ or for any $\nu \in]-\infty, \nu_c[$ the polar curve in the direction $[1 : \nu]$, (that is the gradient ∇f is colinear to that direction),has a connected component which is not empty as germ at P and which is contained in $\Lambda(R,\varepsilon)$, ε small enough.

Nevertheless, we can state, in a very special case, that having a Kurdyka-Łojasiewicz at infinity at c equal to 1, is equivalent to have a trajectory of inifinite length at c.

Let $c \in K_\infty(f) \setminus B(f)$, and let $[v_1 : 0], \ldots, [v_k : 0]$ be the limit points of $f^{-1}(c)$ on the line at infnity $\mathbb{RP}^1_\infty$.

Proposition 7.8. *Assume v_1 is such that any connected component of $f^{-1}(c)$ whose limit point at infinity is $[v_1 : 0]$ is a line. Then there is a trajectory of infinite length at c, with limit point $[v_1 : 0]$ if and only if the Kurdyka-Łojasiewicz exponent at infinity at c is equal to* 1.

Proof. Assume that this exponent is equal to 1. For simplicity we can assume that $v_1 := [1 : 0]$ and that the germ of $f^{-1}(c)$ at $[1 : 0 : 0]$ is just a line, whose equation in $\mathbb{R}^2$ is $y = 0$ (after a translation if needed). We then deduce there exists a principal polar curve at c. We can suppose that a half-branch at infinity, say G, of this principal polar curve is contained in $\Lambda(R,\varepsilon)$ as previoulsy defined. Thus G, as germ at infinity, is the graph of a Nash function $g : [R_G, +\infty[\to \mathbb{R}$. Let us assume that $\Lambda(R,\varepsilon)$ is contained in the upper half-plane $\{y \geqslant 0\}$. Thus g decreases to 0 at infinity. If $(x(t), g(x(t)))$

is the unique intersection point $G \cap F_t$, where $t \in [c-\varepsilon, c[$ (if ε is small enough), we can assume that G is an odd root of $\partial_y f$, since the limit direction of ∇f on any F_t is $[0:1]$ while the limit point of F_t is $[1:0:0]$. Again for simplicity we assume that G is the only connected component of $\{\partial_y f = 0\}$ contained in $\Lambda(R,\varepsilon)$. Since we have assumed $\Lambda(R,\varepsilon) \subset \{y \geqslant 0\}$, we deduce that for any point $(x,y) \in \Lambda(R,\varepsilon)$ with $y > g(x)$ we have $\partial_y f(x,y) < 0$, and so we also deduce that $\partial_x f(x(t), g(x(t))) > 0$, by continuity of ∇f along the fiber F_t. Let Γ_t be the trajectory of ∇f through the point $(x(t), g(x(t)))$. Since $\partial_y f(x(t), g(x(t)))) = 0$ and $\partial_x f(x(t), g(x(t))) > 0$ and g decreases to 0 so that we can assume that $g'(x(t)) < 0$, we necessarily deduce that the curve $\Gamma_t \cap \{x > x(t)\} \cap G = \emptyset$. So Γ_t is a trajectory of infinite length at c.

The key point of this proof is that there is always a connected component of $\{\partial_y f = 0\}$, contained in $\Lambda(R,\varepsilon)$ which is an odd root of $\partial_y f = 0$.

The case of several (parallel) lines is the same since there is somewhere a principal polar curve which asymptotic to one of these lines. ☐

8. Examples

In this section we produce some examples that illustrate the results stated before. All the polynomials presented below have an asymptotic critical value. Each example describes a different phenomenon.

Example 8.1 (Broughton example)**.**
Let f be the polynomial $f(x,y) = y(xy-1)$. We immediately observe that f has no critical point. The set $\{\partial_y f = 0\}$ is the algebraic curve $\{2xy - 1 = 0\}$ and $f(x, 1/2x) \to 0$ when $x \to +\infty$, and $0 \in K_\infty(f)$. Thus the Kurdyka-Łojasiewicz exponent associated to the value 0 is equal to 1. It can easily be shown that $B(f) = S_f = K_\infty(f) = \{0\}$. If ϕ denotes the embedding of Theorem 3.1 then the complement of $\phi(f^{-1}(0))$ in $f^{-1}(t)$ is non empty for all $t > 0$. In taking $-f$ instead of f, we have a similar result for all $t < 0$. Many questions arise from this example: one can ask about the dynamic of ∇f at infinity, or the nature of $f^{-1}(t) \setminus \phi(f^{-1}(0))$.

In this example we can prove that *in the upper half plane, there is a unique integral curve of ∇f which is of infinite length at* 0.

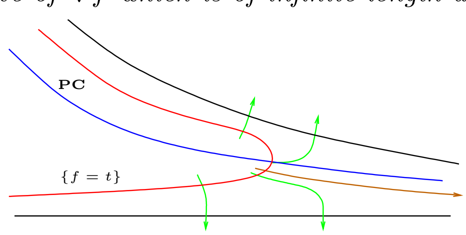

Fig.1: Phase portrait of ∇f

Example 8.2 (King, Tibăr & Zaharia example)**.**
Let f be the polynomial $f(x,y) = -y(2x^2y2 - 9xy + 12)$. This function induces a smooth locally trivial fibration (see [TZ, Proposition 2.6]).

We obtain $K(f) = K_\infty(f) = S_f = \{0\}$, and $B(f)$ is empty. Any level $\{-y(2x^2y2 - 9xy + 12) = t\}$ is homeomorphic to a line.

The point $[1:0:0]$ is the point at infinity of all the fibres of f and there are two principal polar curves, the algebraic curves $\mathbf{PC}_1 := \{xy-1=0\}$ and $\mathbf{PC}_2 := \{xy-2=0\}$. Then the Kurdyka-Łojasiewicz exponent at infinity at 0 is equal to 1. From Proposition 7.8, we deduce there are infinitely many integral curves of going to infinity without reaching the zero level of f, that is infinitely may trajectory of infinite length at 0.

This means *the triviality of* $f = y(2x^2y2 - 9xy + 12)$ *over the value* 0 *cannot be realised by the gradient vector field of* f.

Let $\mathbf{PC}_v = \{4xy - 9 = 0\}$ be the polar curve in the vertical direction. These three polar curves give enough information on the dynamics at infinity of the gradient field. A trajectory has a unique intersection point with each of the polar curves $\mathbf{PC}_*$ (with $* = 1, 2, v$). Now the phase portrait of ∇f is organised around two special integral curves which actually are branching points of the space of leaves of the foliation by ∇f. For any level $t > 0$ since, the same kind of phenomenon occur because of the symmetry of f.

A quick study of the signs of the $\partial_x f$ and $\partial_y f$, and the study of the inflection points of the trajectories give enough informations to draw the following phase portrait

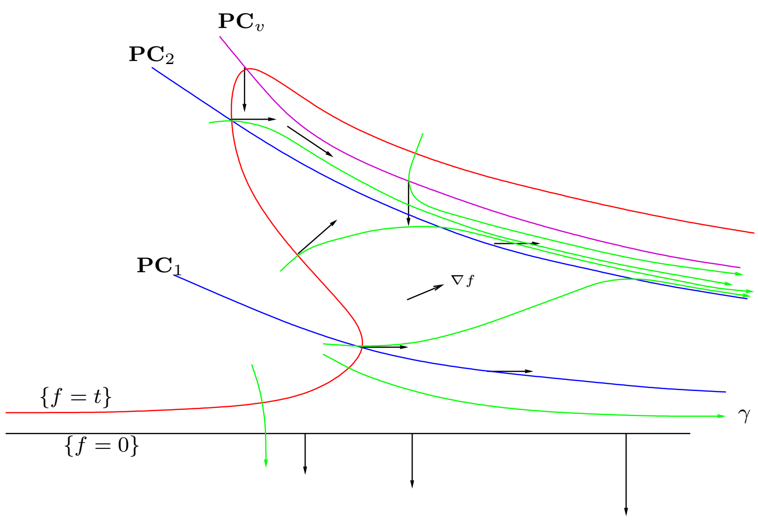

Fig.2: Phase portrait of ∇f

Example 8.3 (The Parusiński example)**.**

Let f be the polynomial $f(x, y) = y^{11} + (1 + (1 + x^2)y)^3$. All the fibres of f have $[1:0:0]$ as the unique point at infinity and there is no principal polar curve.

On the curve $\mathbf{PC}_v := \{1 + (1 + x^2)y = 0\}$, we get that 0 belongs to $K_\infty(f)$. Moreover we find $K(f) = K_\infty(f) = \{0\}$ but the set S_f is empty. All the fibres of this function are homeomorphic to a line. And so by [TZ], f is a locally trivial fibration.

In this case *the gradient field realises the trivialisation.* The gradient vector field of f is given by

$$\nabla f(x,y) = 6xy(1+y+x^2y)^2\frac{\partial}{\partial x} + \left(11y^{10}+3(1+x^2)(1+y+x^2y)^2\right)\frac{\partial}{\partial y}$$

Note that any level $f=t$ is actually a graph in y of some function x_t, and we have

$$x_t(y) = \sqrt{\frac{t^{\frac{1}{3}}-1-y}{y}} + \text{h.o.t} \simeq \frac{k(t)}{y^{1/2}},$$

with $k(t) < 0$.

Let ϱ be the Kurdyka-Łojasiewicz exponent at infinity at 0. Let G be any semialgebraic curve along which f is negative and tends to 0. The curve G is the graph of a function, say g, in the variable x. Thus we have $g(x) \sim x^\alpha$. We necessarily have $g(x) \simeq -x^{-2}$. So there exists $\beta > 1$ such that $\partial_x f(x,g(x)) \sim x^{-\beta}$, then $\partial_y f(x,g(x)) \geqslant 3x^{3-\beta}$, and so

$$|(x,g(x)|\cdot|\nabla f(x,g(x))| \simeq x\partial_y f(x,g(x)).$$

We verify that there is a positive constant C such that

If $\beta \geqslant 23$ then

$$x\partial_y f(x,g(x)) \geqslant C|f(x,g(x))|^{19/22}.$$

If $\beta \in]47/3, 23[$ then

$$x\partial_y f(x,g(x)) \geqslant C|f(x,g(x))|^{(4-\beta)/22} \geqslant C|f(x,g(x))|^{19/22}.$$

If $\beta \in]1, 47/3]$ then

$$x\partial_y f(x,g(x)) \geqslant C|f(x,g(x))|^{(8-2\beta)/(3-3\beta)} \geqslant C|f(x,g(x))|^{2/3}.$$

Taking $g(x)) := -(1+x^2)^{-1}$, we verify that along $y = g(x)$

$$x\partial_y f(x,g(x)) \sim |f(x,g(x))|^{19/22},$$

and thus $\varrho = 19/22$. So the flow of $\nabla f/|\nabla f|^2$ realises the trivialisation.

References

[BCR] J. Bochnak, M. Coste, M.F. Roy, *Real algebraic geometry,* E.M.G vol. 36 (1998) Springer.

[BeRi] R. Benedetti, J. J. Risler, *Real algebraic and semialgebraic sets,* Actualités Mathmatiques, Hermann, Paris (1990).

[Br] S.A. Broughton, *On the topology of polynomial hypersurfaces,* Proc. A.M.S. Symp. in Pure Math., vol 40, Part 1, (1983), 165-178.

[Ch] F. Chazal, *Sur les feuilletages de Rolle* Thèse de l'universit de Bourgogne (1997).

[CP] M. Coste & M.J. de la Puente , *Atypical values at infinity of a polynomial function on the real plane: an erratum, and an algorithmic criterion,* Journal of Pure and applied Algebra 162 (2001) 23-35.

[D'A1] D. D'Acunto, *Valeurs Critiques Asymptotiques d'une Fonction Définissable dans une Structure o-minimale,* Ann. Pol. Math, 35 (2000), pp. 35-45.

[D'A2] D. D'Acunto, *sur la topologie des fibres d'une fonction définissable dans une structure o-minimale,* C. R. Acad. Sci.Paris, Ser. I 337 (2003).

[DG] D. D'Acunto & V. Grandjean, *Trajectoires à l'infini de gradients de fonctions definissables,* En préparation.

[DK] D. D'Acunto & K. Kurdyka, *bounds for gradient trajectories of polynomial and definable functions with applications,* Preprint Univ-Savoie (2004).

[FP] J. Ferrera & M.J. de la Puente, *The asymptotic values of a polynomial function on the real plane,* Journal of Pure and applied Algebra 106 (1996) 263-276.

[Ha] H. V. Ha, *Nombres de Lojasiewicz et singularités à l'infini des polynômes de deux variables complexes,* C.R.A.S. t. 311, Série I, (1990) 429-432.

[KP] T.Z. Kuo & A. Parusiński, *Newton Polygon relative to an arc,* Real and complex singularities, J.W. Bruce & F. Tari editors, Chapman and Hall/CRC Research Notes in Mathematics, 412, pp. 76-93.

[Ku] K. Kurdyka, *On gradient of functions definable in o-minimal structures,* Ann. Inst. Fourier, 48-3 (1998), 769-783.

[KMP] K. Kurdyka, T. Mostowski, A. Parusiński, *Proof of the gradient conjecture of R. Thom,* Annals of Math.,152 (2000) 763-792.

[KOS] K. Kurdyka, P. Orro & S. Simon, *Semialgebraic Sard theorem for generalized critical values,* J. Differential Geometry **56** (2000) 67-92.

[Pa] A. Parusiński, *On the bifurcation set of complex polynomial with isolated singularities at infinity,* Compositio Mathematica, 97 (1995) 369-384.

[Ph] F. Pham, *La descente des cols par les onglets de Lefschetz, avec vues sur Gauss-Manin,* in Systèmes différentiels et singularités, Juin-Juillet 1983, Astérisque 130 (1983), 11-47.

[Sp] P. Speisseger, *The Pfaffian closure of an o-minimal structure,* J. reine. angew. Math **508** (1999), 189-211.

[Th] R. Thom, *Ensembles et morphismes stratifiés,* bul. Amer. Math. Soc., 75 (1969) 240-282.

[Ti] M. Tibăr, *Regularity at infinity of real and complex polynomial functions,* Singularity Theory, Edited by Bill Bruce & David Mond, LMS Lecture Notes, 263, Cambridge University Press, 1999, 249-264.

[TZ] M. Tibăr & A. Zaharia, *Asymptotic behaviour of families of real curves,* Pub. IRMA, LILLE 1998, Vol. 46, N°XII.

D. D'Acunto, Departamento de Álgebra, Facultad de Matematicas, Universidad Complutense, Avda. Complutense, 28040 Madrid, (ESPAÑA)
E-mail address: didier.dacunto@univ-savoie.fr

V. Grandjean, Departement of Computer Science, University of Bath, BATH BA2 7AY, England,(UNITED KINGDOM)
E-mail address: cssvg@bath.ac.uk