A GRADIENT INEQUALITY AT INFINITY FOR TAME FUNCTIONS

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Abstract. Let f be a C^1 function defined over $\mathbb{R}^n$ and definable in a given o-minimal structure $\mathcal{M}$ expanding the real field. We prove here a gradient-like inequality at infinity in a neighbourhood of an asymptotic critical value c. When f is C^2 we use this inequality to discuss the trivialisation by the gradient flow of f in a neighbourhood of a regular asymptotic critical level.

1. Introduction

Given a C^1 function $f : U \subset \mathbb{R}^n \to \mathbb{R}$ a Łojasiewicz inequality in a neighbourhood of $x_0 \in \overline{U}$, the closure of U in $\mathbb{R}^n$, is, usually, an inequality that compares quantitatively the behaviour of $f(x)$ with $|\nabla f(x)|$ or $|x|$ in a neighbourhood of x_0.

When f is analytic and x_0 is a critical point of f, there are at least two well-known such inequalities. The standard Łojasiewicz's gradient inequality (cf. [Łoj]) states there exists a smallest rational number $\rho_f \in]0,1[$ and a positive constant C such that in a neighbourhood of x_0 we have

$$\textbf{(Ł)} \qquad |\nabla f(x)| \geq C|f(x) - f(x_0)|^{\rho_f}.$$

Another important inequality, called Bochnak-Łojasiewicz inequality, states that there is a constant C_f such that in a neighbourhood of $x_0 \in \overline{U}$, then

$$\textbf{(B-Ł)} \qquad |x - x_0| \cdot |\nabla f(x)| \geq C_f |f(x) - f(x_0)|.$$

These inequalities are very useful once we need to deal with quantitative behaviour of the function (these two inequalities are key elements in the proof of the gradient conjecture, see [KMP]).

The development in the last twenty years of so-called tame geometry, sharing many nice properties with semialgebraic geometry, lead many mathematicians to be interested in this sort of quantitative information about a tame function. One can find, for instance, more general Łojasiewicz inequalities in Pfaffian geometry (cf. [Li]) or in the o-minimal structure generated by semialgebraic sets and the exponential function (cf. [Loi]).

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Let us fix the framework of this note. Let $\mathcal{M}$ be a given o-minimal structure expanding the real field (see [vD] and [vDM] for the geometric meaning of this notion and some of its basic and important consequences). In the following by a definable set or a definable function we will mean a set or a function definable in the structure $\mathcal{M}$.

The first contribution of the notion of Łojasiewicz's gradient inequality in the o-minimal context was provided by Kurdyka in [Ku]. In the loc. cit. paper he was interested in the uniform behaviour of the trajectories of the gradient field ∇f where f is a C^1 definable function defined over a bounded open $U \subset \mathbb{R}^n$. Dealing with such a question needs to distinguish some values of the function where problems can occur. Of course the critical values have to be carefully considered (as usual) but other values may also be taken into account. Namely, we say that c is an asymptotic critical value of f if and only if there exists a sequence $\{x_\nu\}_\nu \in U$ such that $f(x_\nu) \to c$ and $\nabla f(x_\nu) \to 0$. The set of asymptotic critical values contains the set of critical values and the singular values on the boundary of U. We denote by $K_a(f)$ the set of asymptotic critical values. To control the behaviour of the trajectories of ∇f in the neighbourhood of asymptotic critical fibres Kurdyka established the following key result

Theorem 1.1 ([Ku]). *If $f : U \to \mathbb{R}+$ is a C^1 definable function, then for all $c \in K_a(f)$ there exist a constant $K_c > 0$ and a C^1 definable function $\Psi_c : [0, +\infty[\to \mathbb{R}$ such that $|\nabla(\Psi \circ f)(x)| \geq C$ for all $x \in U$ and $f(x)$ sufficiently close to c.*

Now let us assume that $f : \mathbb{R}^n \to \mathbb{R}$ is a C^1 definable function. Assume the o-minimal structure $\mathcal{M}$ is polynomially bounded.
From [DG] we know that for a value c such that $f^{-1}(c)$ is non compact there exist $C > 0$ and a smallest $\rho_c \leq 1$ such that for sufficiently large $|x|$ and sufficiently small $|f(x) - c|$ we have

$$\textbf{(K-Ł)} \qquad |x| \cdot |\nabla f(x)| \geq C|f(x) - c|^{\rho_c}.$$

We proved this inequality in the semialgebraic context but the proof extends easily with exactly the same arguments for C^1 functions defined in polynomially bounded o-minimal structures.

The aim of this note is to prove an analog of inequality **(K-Ł)** when $\mathcal{M}$ is not polynomially bounded. Such an inequality will be useful to decide whether we can trivialise the function f over a neighbourhood of a regular asymptotic critical value c by the gradient field.

Conventions. Let u and v be two continuous function of a single variable defined over $[1, +\infty[$. We will write $w \sim v$ to mean that u/v tends to a limit $l \in \mathbb{R}^*$ when x tends to $+\infty$. We will write $u \simeq v$ if $u \sim v$ and $l = 1$.

2. A Bochnak-Łojasiewicz inequality at infinity near an asymptotic critical value

Let f be a C^1 function defined over $\mathbb{R}^n$ and definable in $\mathcal{M}$. Let us denote by ∇f the gradient vector field of f for the standard Euclidean metric.

Definition 2.1. A real number c is an asymptotic critical value of the function f if there exists a sequence $\{x_\nu\}_\nu \in \mathbb{R}^n$ satisfying the following conditions when $\nu \to +\infty$

(1) $|x_\nu| \to +\infty$;
(2) $f(x_\nu) \to c$;
(3) $|x_\nu| \cdot |\nabla f(x_\nu)| \to 0$.

Let us denote by $K_\infty(f)$ the set of asymptotic critical values of f. Let $K_0(f)$ be the set of critical values of f. Then we recall

Theorem 2.2 ([D'A1])**.** *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a C^1 definable function. The set $K(f) := K_\infty(f) \cup K_0(f)$ is finite. Moreover, the function f induces a locally trivial continuous fibration over each connected component of $\mathbb{R} \setminus K(f)$.*

Unfortunately it is well known, even for a real polynomial function, that the set of bifurcation value of the function can be strictly contained in $K(f)$ (see for instance $f(x,y) = y(2x^2y^2 - 9xy + 12)$ in [TZ]). Nevertheless we know that any bifurcation value which is not a critical value is at least an asymptotic critical value (see the works of Némethi and Zaharia in [NZ] and Loi and Zaharia in [LZ] to shrink the set of asymptotic critical values where to find the regular bifurcation values).

On the other hand the attempts to better understand the behaviour of the trajectories of the gradient field ∇f nearby an asymptotic critical level c of f, lead us to find a gradient-like inequality at infinity nearby this level for semi-algebraic function:

Theorem 2.3 ([DG])**.** *Assume that f is semialgebraic. Let $c \in K_\infty(f)$. Then there exist a smallest rational number $\rho_c \in \mathbb{Q} \cap]0,1]$ and a positive constant K_c such that*

$$|x| \gg 1 \text{ and } |f(x) - c| \ll 1 \Longrightarrow |x| \cdot |\nabla f(x)| \geq K_c |f(x) - c|^{\rho_c}.$$

As a consequence (cf.[DG]), if f is C^2 and is ρ_c is strictly smaller than 1, we can trivialise by the gradient flow of f over a neighbourhood of c (and so shrink the set of asymptotic critical values that could be bifurcation value).

Let $\mathbb{R}_{\geq 0}$ be $[0, +\infty[$. In the definable context the first gradient-like inequality at infinity is given by the following

Lemma 2.4 ([D'A2])**.** *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a C^1 definable function. Let c be an asymptotic critical value of f. There exists a (germ at 0 of a) definable and non constant function $\Psi :]0,1[\to \mathbb{R}_{\geq 0}$, such that*

$$|x| \gg 1 \text{ and } 0 < |f(x) - c| \ll 1 \Longrightarrow |x| \cdot |\nabla(\Psi \circ f)(x)| \geq 1$$

This inequality can be rephrased as

Lemma 2.5. *Under the hypotheses of Lemma 2.4, taking* $\nu :]0,1[\to \mathbb{R}_{\geq 0}$ *such that* $\nu(t) = |\Psi'(t)|^{-1}$ *we obtain*

$$|x| \gg 1 \text{ and } |f(x) - c| \ll 1 \Longrightarrow |x| \cdot |\nabla f(x)| \geq \nu(|f(x) - c|))$$

Since $c \in K_\infty(f)$, we necessarily must have that ν tends to 0 near 0. The first issue about this inequality is to be able to find the "best" function ν (i.e. the biggest). The second is knowing the "best" such function is to be able to have a quantitative information about its asymptotic behaviour when we get close to 0.

For this purpose we will use an elementary Lemma about the growth properties of germs at infinity of a single real variable definable functions, which here is an analog of what was done by Kurdyka and Parusiński in [KP].

Let us denote by $\mathbb{R}_{\gg 1}$ the germ at infinity of $[1, +\infty[$. Let us recall that if φ is the germ of a definable function in a single variable r at infinity, and if $\tilde{\varphi}$ is a representative of φ over $[R, +\infty[$, then for any positive integer k, there exists $R_k \geq R$ such that $\tilde{\varphi}$ is C^k on the interval $]R_k, +\infty[$. The real number R_k can be chosen so that each derivative, if not constant, is strictly monotone.

Lemma 2.6. *Let* φ *and* ψ *be definable functions* $\mathbb{R}_{\gg 1} \to (\mathbb{R}_{\geq 0}, 0)$*, non identically zero. Assume that* $\varphi > \psi$*. Let* $K > 1$ *be given. Then for* r *large enough we get*

$$\varphi'(r) \leq \psi'(r) \text{ and } K\frac{\psi'}{\psi} \leq \frac{\varphi'}{\varphi}$$

Proof. Since $\varphi - \psi$ is a positive function and tends to 0 at $+\infty$ then, by monotonicity the derivative $(\varphi - \psi)'$ increases to 0 at infinity, hence we get the first inequality.

Let us first choose $K = p/q > 1$ to be a rational number such that p and q are positive and relatively primes. Let v be the definable function defined as $v := \psi^p \varphi^{-q}$. Since $0 < v \leq \psi^{p-q}$, we get that v tends to 0 at infinity and thus $v' \leq 0$, which provides

$$\frac{\psi^p}{\varphi^q}\left[p\frac{\psi'}{\psi} - q\frac{\varphi'}{\varphi}\right] \leq 0,$$

and thus get the second inequality when K is a rational and by density the proof is over. □

Let us come know to the main result of this section, that is a Bochnak-Łojasiewicz inequality at infinity nearby c.

Proposition 2.7. *Let* $f : \mathbb{R}^n \to \mathbb{R}$ *be a* C^1*-definable function defined over* $\mathbb{R}^n$*. Let* c *be an asymptotic critical value of* f*. Then there exist positive constants* R*,* ε *and* K *such that*

$$\text{If } |x| > R \text{ and } |f(x) - c| < \varepsilon, \text{ then } |x| \cdot |\nabla f(x)| \geq K|f(x) - c|.$$

Proof. Assume this does not hold. Then, by the Curve Selection Lemma at infinity there exists a definable curve $\gamma : \mathbb{R}_{\gg 1} \to \mathbb{R}^n$ such that $|\gamma(r)| = r$ and

$$\lim_{r\to+\infty} f(\gamma(r)) = c \text{ and } \lim_{r\to+\infty} \frac{|\gamma(r)| \cdot |\nabla f(\gamma(r))|}{|f(\gamma(r)) - c|} = 0. \tag{2.1}$$

Let v be the definable function defined as $v(r) := |f \circ \gamma(r) - c|$.

Under the hypothesis (2.1) the following holds true.

Lemma 2.8. *$rv(r) \to 0$ when $r \to +\infty$.*

Proof. Taking the derivative in r, provides $|\gamma'(r)| \cdot |\nabla f(\gamma(r))| \geq |v'(r)|$.

Let u be the function defined as $u(r) := rv(r)$. Then u is definable. If $\lim_{+\infty} u = l \in]0, +\infty[$, we deduce $r^2|v'(r)| \simeq l$ and since $|\gamma'(r)| \to 1$ as $r \to +\infty$, we find

$$r \cdot |\nabla f(\gamma(r))| \geq l|v(r)|$$

which contradicts the hypotheses (2.1).

Assume now that $\lim_{+\infty} u = +\infty$. Then the derivative u' is positive. If $r \cdot u'(r) \simeq u(r)$ we would obtain $u(r) \simeq r$. But this is impossible since $\lim_{+\infty} v = 0$. Thus there is a positive constant $C > 1$ such that $u' > Cv$ or $v > Cu'$. Hence there is a constant $M < 1$ such that

$$r \cdot |\nabla f(\gamma(r))| \geqslant r \cdot |v'(r)| = |u'(r) - v(r)| \geqslant Mv(r),$$

which again contradicts (2.1). The lemma is proved. □

We can end the proof of Proposition 2.7. Since $|\gamma'(r)| \to 1$ as $r \to +\infty$, there exists a positive constant $M' < 1$ such that

$$\frac{|\gamma(r)| \cdot |\nabla(f)(\gamma(r))|}{|f(\gamma(r)) - c|} \geq M' \frac{r \cdot |v'(r)|}{v(r)}.$$

Let $w(r) = 1/r$. Then

$$r\frac{|v'(r)|}{v(r)} = \frac{w(r)}{|w'(r)|} \frac{|v'(r)|}{v(r)}.$$

Since v and w are definable and $v/w \to 0$ at infinity, for r large enough, $w - v$ is positive and decreases to 0. From Lemma 2.6, there exists a positive constant $A < 1$ such that

$$\left|\frac{w(r)}{w'(r)}\right| \cdot \left|\frac{v'(r)}{v(r)}\right| \geq A,$$

which contradicts (2.1). □

Remark 2.9. The proof of Lemma 2.7 is straightforward when $\mathcal{M}$ is polynomially bounded. In this case we can conclude without Lemma 2.6 because for all non ultimately zero definable function v in the single variable r, the function rv'/v has a non zero limit as r goes to infinity. The reason is that the Hardy field of $\mathcal{M}$ has rank one.

3. MAIN RESULT

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a C^1 definable function and assume $c \in K_\infty(f)$. There is an explicit way to produce the "best" function ν satisfying Lemma 2.5. For $|x| \gg 1$ and $0 < |f(x) - c| = t \ll 1$ let us define the function m_c as

$$m_c(t) = \inf\{|x| \cdot |\nabla f(x)| : |f(x) - c| = t\}.$$

This function is well defined and positive since any such t is a regular value and not an asymptotic critical value. Moreover m_c is definable in $\mathcal{M}$, and thus continuous over a small interval of the form $]0, b]$ and satisfies the inequality of Lemma 2.5.

As a consequence of the inequality of Lemma 2.7, we actually get

Proposition 3.1. *Under the previous hypotheses we have*

(1) there exists $A > 0$ such that for $0 < t \ll 1$ we have $m_c(t) \geq At$;
(2) if $|x| \gg 1$ and $|f(x) - c| \ll 1$ then $|x| \cdot |\nabla f(x)| \geq m_c(|f(x) - c|)$;
(3) any definable function v satisfying points (1) and (2) instead of m_c satisfies $m_c(t) \geq v(t)$ for all sufficiently small $t > 0$.

Proof. Point (1) is a direct consequence of Lemma 2.7 while point (2) is just the definition of the function m_c. Since $m_c(t)$ is the infimum of the function $|x| \cdot |\nabla f(x)|$ taken on the level hypersurface $f^{-1}(t)$ outside a (given) ball of large radius, point(3) is true otherwise the existence of such a function $v > m_c$ would contradict this infimum property. □

Remark 3.2. If $\mathcal{M}$ is polynomially bounded, then the function $1/m_c$ of Proposition 3.1 is of the form $1/m_c(r) \simeq Cr^{-\rho}$, with C a positive constant and $0 < \rho \leq 1$ is an exponent lying in the field of the exponents of the Hardy field of $\mathcal{M}$.

The next consequence is the analog in the current context of [DG, Theorem 4.4]. We here assume f is actually C^2 and $c \in K_\infty(f) \setminus K_0(f)$. Then

Theorem 3.3. *If the function $1/m_c$ is integrable on an interval of the type $]0, b]$, then we can trivialise the function f over a neighbourhood of c by the gradient flow of f.*

Proof. A careful reading of the proof of [DG, Theorem 4.4] shows that the key ingredient is the use of Gronwall Lemma combined with the integrability in a small neighbourhood of c of the function $t \mapsto (t - c)^{-\rho_c}$, when $\rho_c < 1$. Thus each trajectory through a point of the level t_0 is of finite length between the levels t_0 and c, once $[t_0, c[\cap K(f) = \emptyset$ (of course the same is true over intervals of the type $]c, t_0]$ with $]c, c_0] \cap K(f) = \emptyset$).

The same conclusion is again valid here whence we require that $1/m_c$ is integrable in a neighbourhood of c. This implies that the flow of ∇f maps injectively the whole level t_0 into the level c. To conclude we use an embedding theorem proved by the first named author [D'A2], stating that any connected component of the level c is injectively mapped into a connected component of the level t_0 by the flow of $-\nabla f$. Thus we have proved the trivialisation by the gradient near c. $\square$

4. The Riemannian case

In this section we assume that $f : M \to \mathbb{R}$ is a C^1 definable function defined on a C^1 definable submanifold $M \subset \mathbb{R}^n$ equipped with the definable Riemannian metric g induced by the standard Euclidean metric of $\mathbb{R}^n$. We also assume that M is closed, connected, unbounded and without boundary. We respectively denote by $|\cdot|_g$ and ∇_g the norm and the gradient with respect to the metric g.

In this setting it makes sense to study the function $|x| \cdot |\nabla_g f(x)|_g$ and again to define the set $K_\infty(f)$ of asymptotic critical values of f with it. It was proved in [D'A1] that $K_\infty(f)$ is finite and in [D'A2] that Lemma 2.4 holds with this setting.
In this context, it is easy to verify that the results stated in the present paper are also true. Let $c \in K_\infty(f)$. For sufficiently small $t > 0$, let m_c be the function defined as $m_c(t) := \inf\{|x| \cdot |\nabla_g f(x)|_g : |f(x) - c| = t\}$, then the following holds true

Proposition 4.1.
(1) There exists $A > 0$ such that for $0 < t \ll 1$ we have $m_c(t) \geq At$;
(2) if $|x| \gg 1$ and $|f(x) - c| \ll 1$ then $|x| \cdot |\nabla_g f(x)|_g \geq m_c(|f(x) - c|)$;
(3) any definable function v satisfying points (1) and (2) instead of m_c satisfies $m_c(t) \geq v(t)$ for all sufficiently small $t > 0$.

Similarly, the main result of this paper extends in this setting as

Theorem 4.2. *If the function $1/m_c$ is integrable on an interval of the type $]0, b]$, then we can trivialise the function f over a neighbourhood of c by the flow of $\nabla_g f$.*

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