ON GRADIENT AT INFINITY OF SEMIALGEBRAIC FUNCTIONS

DIDIER D'ACUNTO AND VINCENT GRANDJEAN

Abstract. Let $f : U \subset \mathbb{R}^n \to \mathbb{R}$ be a C^2 semialgebraic function and let c be an asymptotic critical value of f. We prove that there exists a smallest rational number $\rho_c \leqslant 1$ such that $|x| \cdot |\nabla f|$ and $|f(x) - c|^{\rho_c}$ are separated at infinity. If $\rho_c < 1$ then f is a locally trivial fibration over c and the trivialisation is realised by the flow the gradient field of f.

1. Introduction

As a consequence of the fundamental paper of Thom (cf. [Th]) about conditions ensuring the local topological triviality of a smooth mapping, given a polynomial $f : \mathbb{C}^n \to \mathbb{C}$, there exists a finite subset of values $\Lambda \subset \mathbb{C}$ such that the function f induces a locally trivial fibration from $\mathbb{C}^n \setminus f^{-1}(\Lambda)$ onto $\mathbb{C} \setminus \Lambda$. The smallest such subset of $\mathbb{C}$, denoted by $B(f)$, is called the set of bifurcation values of the function f. It contains the usual critical values of f. But there could exist regular values that were also bifurcation values. But Thom did not give any way to find out these regular bifurcation values.

Few years later Pham, to ensure the convergence of oscillating integrals exhibited a condition to trivialise a complex polynomial f over a neighbourhood of a regular value $c \in \mathbb{C}$: the Malgrange condition (cf. [Ph]). Roughly speaking this condition means that the norm of the gradient is not too small in a neighbourhood of the germ at infinity of the given level $f^{-1}(c)$.

The set of values at which Malgrange condition is not satisfied is actually finite (see [Ti]). Moreover any bifurcation value that is also regular is a value at which Malgrange condition fails. Finally, Parusiński proved that any regular value of a given complex polynomial with isolated singularities at infinity at which Malgrange condition fails ([Pa]) is a bifurcation value. But in whole generality we still ignore if this property is true for any complex polynomial.

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Now, let us turn to the case of a real polynomial f. As in the complex case, the set of bifurcation values, as defined above, is finite as is also the set of values at which Malgrange condition is not satisfied (see [Ve], [Ti]). Again any regular bifurcation value does not satisfy Malgrange condition. As in the complex case, this hopefully ensures a fibration Theorem outside these special fibres and the critical fibres. But in the real case the result of Parusiński is no more true. A regular value of a real plane polynomial at which Malgrange condition fails is not necessarily a bifurcation value, we invite the interested reader to go straightforwardly to the King-Tibăr-Zaharia and Parusiński examples in Section 5.

When Malgrange condition is fulfilled at a regular value c, the function is locally trivial over a neighbourhood of c and this trivialisation can be realised by the flow of the gradient vector field ∇f.

In the early stage of this work, we expected that, at least in the real plane case, trivialising by the gradient ∇f in a neighbourhood of a regular value c and having Malgrange condition satisfied at c were equivalent conditions. But this belief was wrong, and once more, we invite the reader to deal with the Parusiński example in Section 5.

Nevertheless, these examples led us to try to understand more closely the connections between the behaviour of the trajectories of the gradient field ∇f, the asymptotic geometry of the neighbouring levels of the level c and the failure of the Malgrange condition at c. We have been particularly interested in the trajectories leaving any compact subset of $\mathbb{R}^n$ and along which f tends to a finite value c at infinity. We will not explore here the very difficult problem of the qualitative behaviour of such trajectories, but they lead us to the discovery of the Kurdyka-Łojasiewicz exponent at infinity related to c and its corresponding gradient-like inequality in a neighbourhood of the level c at infinity, a notion that actually improved a lot Malgrange condition and with a geometric content closely connected to the foliation by the levels of f.

In this article we will work with C^1 (or C^2 depending on the context) semialgebraic functions, since most of the result we are interested in, originally stated in the polynomial case, are also available in the semialgebraic frame.

Conventions. Let u and v be two germs at infinity of single real variable functions. We write $u \sim v$ to mean that the ratio u/v has a non zero finite limit at infinity. We write $u \simeq v$ when the limit of u/v at infinity is 1.

2. Asymptotic critical values and the embedding Theorem

Let $f : U \subset \mathbb{R}^n \to \mathbb{R}$ be a C^1 semialgebraic function. For our purpose, we assume that U is a non bounded open set.

What has been said in the introduction, the fibres of f describe only finitely many topological types, is still available here ([Ve] or [KOS]. The values at which the topology changes are called the bifurcation values (or atypical values) of the function f. Any other value is called a typical value. The set of atypical values is finite and denoted by $B(f)$. In this set we distinguish two sorts of values: the usual critical values, denoted by $K_0(f)$ and the asymptotic critical values.

We denote by $K_\infty(f)$ the set of *asymptotic critical values*; the set of values at which Malgrange condition fails:

Definition 2.1. The function f satisfies Malgrange condition **(M)** at a value $t \in \mathbb{R}$ if and only if there exists a constant $C > 0$ such that for sufficiently large x and $f(x)$ sufficiently close to t the following inequality holds :

$$\textbf{(M)} \quad |x| \cdot |\nabla f(x)| \geqslant C$$

Equivalently, if $c \in K_\infty(f)$ then there exists an unbounded sequence $\{x_\nu\}_\nu \in \mathbb{R}^n$ such that $f(x_\nu) \to c$ and $|x_\nu| \cdot |\nabla f(x_\nu)| \to 0$.

Remark 2.2. The previous definition and the notion of critical values at infinity are also making sense as well for C^1 real function defined on a non bounded open subset of $\mathbb{R}^n$ as for complex polynomials.

Let $K(f) = K_0(f) \cup K_\infty(f)$ be the set of *generalised critical values.* Any value $t \notin K(f)$ will be called a *generic value of f.*

In the real case, condition **(M)** ensures the trivialisation via the gradient field ∇f. To be more precise let us come closer to the embedding Theorem ([D'A2]) and let us assume here that f denotes a C^2 semialgebraic function. Let Φ be the local flow of the gradient ∇f defined as the mapping satisfying the following conditions

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t}(x,t) = \frac{\nabla f}{|\nabla f|^2} \circ \Phi(x,t) \text{ and } \Phi(x,0) = x.$$

Let us begin by stating this embedding Theorem, which is absolutely fundamental in this work. Let c be a regular value of f. Let t be any regular value such that $[t,c[\cap K(f) = \emptyset$, if $t < c$ or $]c,t] \cap K(f) = \emptyset$, if $t > c$. Then we have:

Theorem 2.3 ([D'A2])**.** *There exists a C^1 injective open immersion ϕ : $f^{-1}(c) \to f^{-1}(t)$. More precisely, the flow of $\nabla f/|\nabla f|^2$ embeds each connected component of $f^{-1}(c)$ into a connected component of $f^{-1}(t)$.*

Remark 2.4. Such an embedding ϕ maps diffeomorphically the compact connected components of $f^{-1}(c)$ onto those of $f^{-1}(t)$.

If we cannot trivialise by the flow of ∇f, over a neighbourhood of a regular value c, then this means there is at least a trajectory of ∇f that never reaches the level c. More precisely we have then introduced the following

Definition 2.5. An integral curve of ∇f leaving any compact subset of $\mathbb{R}^n$ such that the function f has finite limit c on a half-branch at infinity of the trajectory is called **an integral curve (or trajectory) of infinite length at** c.

3. Kurdyka-Łojasiewicz exponent at infinity at an asymptotic critical value

Let us recall the well known Łojasiewicz inequality satisfied by any analytic function germ f at the origin O of $\mathbb{R}^n$ and let $c := f(O)$. Also assume that f has a singularity at the origin.
There exist positive numbers ϱ and C such that

$$|\nabla f| \geqslant C|f-c|^{\varrho}.$$

The infimum of the exponents ϱ such that $|\nabla f||f-c|^{-\varrho}$ has a positive limit along any sequence converging to O is called the Łojasiewicz exponent of f and is a rational number lying in $]0,1[$.

There is a notion, quite meaningful in the complex setting, of Łojasiewicz exponent at infinity but, this exponent is used to compare the size of $|\nabla f(x)|$ with $|x|$. From our point of view it is interesting to compare $|\nabla f(x)|$ with $|f(x)-c|$ for an asymptotic critical value c. The following result provides an analogue at infinity of the standard Łojasiewicz gradient inequality stated above. This result is the very first important one of this article and was to our best knowledge not known before.

Proposition 3.1. *Let f be a C^1 semialgebraic function. If $c \in \overline{\mathrm{Im}\ f}$, then there exist real numbers $C, R, \tau > 0$ and a smallest rational number $\rho_c \leqslant 1$ such that for all $x \in \mathbb{R}^n$, $|x| > R$ and $|f(x)-c| < \tau$, then*

$$|x| \cdot |\nabla f(x)| \geqslant C|f(x)-c|^{\rho_c}.$$

Proof. By the curve selection Lemma it suffices to prove this fact on semialgebraic curves that admit a half-branch at infinity. For simplicity we will only consider values $t < c$. Let G be a semialgebraic half-branch at infinity along which f tends to $c \in \mathbb{R}$ at infinity. We can assume that f is increasing along G. Let $[c-\tau, c[\ni t \mapsto g(t) \in \mathbb{R}^n$ be a semialgebraic parametrisation of the germ of G at infinity verifying $f \circ g(t) = t$, for $0 < \tau \ll 1$. Then there exist a rational number $\eta > 0$ and a positive real number K such that

$$|g(t)| \simeq K|t-c|^{-\eta} \text{ when } t \text{ goes to } c.$$

By usual semialgebraic arguments, we get

$$|g'(t)| \simeq K\eta|t-c|^{-(1+\eta)} \text{ when } t \text{ goes to } c.$$

Taking derivatives with respect to t, we obtain

$$(f \circ g)'(t) = \langle \nabla f(g(t)), g'(t) \rangle = 1.$$

Thus we deduce $|\nabla f(g(t))| \geqslant \frac{1}{2K\eta}|t-c|^{\eta+1}$ and

$$|g(t)| \cdot |\nabla f(g(t))| \geqslant \frac{1}{4\eta}|t-c| \tag{3.1}$$

Since the function $t \mapsto f(g(t))$ is semialgebraic, there exists a rational number ν such that

$$|g(t)| \cdot |\nabla f(g(t))| \sim |t-c|^{\nu}.$$

From inequality (3.1) we obtain $\nu \leqslant 1$.

Let ρ_c be the infimum of these exponents ν. Let us define the following subset of $\mathbb{Q}$

$$E_c = \left\{ q \in \mathbb{Q} : \lim_{|x|\to+\infty} \frac{|x| \cdot |\nabla f(x)|}{|f(x)-c|^q} \in \mathbb{R}_+^*, \lim_{|x|\to+\infty} f(x) = c \right\}.$$

We easily verify that E_c is a semialgebraic subset of $\mathbb{R}$ contained in $\mathbb{Q}$, then it is finite (for details see [KMP, Proposition 4.2]). Thus ρ_c is rational. □

Since there is yet a Łojasiewicz exponent at infinity, we will refer to ρ_c as the **Kurdyka-Łojasiewicz exponent at infinity for the value** c.

Remark 3.2. Let us mention that Theorem 3.1 also holds when $f : V \to L$ is a semialgebraic C^1 function defined on a semialgebraic (closed and connected) C^1 submanifold of $\mathbb{R}^n$ equipped with a semialgebraic Riemannian metric.

The Malgrange condition corresponds to a value c of the given function for which the Kurdyka-Łojasiewicz exponent at infinity for c is less than or equal to 0. The following proposition is just a rewriting of condition **(M)**:

Proposition 3.3. *Let f be a C^1 semialgebraic function. Let $c \in \overline{\mathrm{Im}\ f}$. Then c is an asymptotic critical values of f if and only if the Kurdyka-Łojasiewicz exponent at infinity of f at c is positive.*

Let $c \in K_\infty(f) \setminus K_0(f)$ and let ρ_c be the Kurdyka-Łojasiewicz exponent at infinity for c. This number contains interesting information about the kind of value (typical or not) that c could be, as shown by the following

Theorem 3.4. *Let f be a C^2 semialgebraic function. If $\rho_c < 1$, then f is a locally trivial fibration over c, moreover the fibration, can be realised by the flow of $\nabla f/|\nabla f|^2$.*

Proof. For simplicity we shall again only work with values $t < c$. Let $t_0 < c$ be such that $[t_0, c] \cap K(f) = \{c\}$ and let $R, C > 0$ be real numbers such that Proposition 3.1 holds in $f^{-1}([t_0, [)) \cap \{|x| > R\}$ with constant C. Let $x_0 \in f^{-1}(t_0) \cap \{|x| > R\}$ and let γ be a (maximal) trajectory of ∇f parametrised by the levels of f and so satisfying to the following differential equation

$$\gamma'(t) = \mathbf{X}(\gamma(t)), \text{with initial condition } \gamma(t_0) = x_0 \in f^{-1}(t_0) \tag{3.2}$$

where $\mathbf{X}$ is the vector field $\nabla f/|\nabla f|^2$. Thus for all t we obtain $f \circ \gamma(t) = t$.

Integrating Equation (3.2) between t_0 ant $t < c$, we obtain

$$\int_{t_0}^{t} \gamma'(s)\mathrm{d}s = \int_{t_0}^{t} \mathbf{X}(\gamma(s))\mathrm{d}s \tag{3.3}$$

From Equation (3.3), we get a first inequality

$$|\gamma(t)| \leqslant |\gamma(t_0)| + \int_{t_0}^{t} \frac{\mathrm{d}s}{|\nabla f(\gamma(s))|} \tag{3.4}$$

Then using Proposition 3.1 we have

$$|\gamma(t)| \leqslant |\gamma(t_0)| + \int_{t_0}^{t} \frac{|\gamma(s)|}{C|s-c|^{\rho_c}} \mathrm{d}s \tag{3.5}$$

Then Gronwall Lemma gives

$$|\gamma(t)| \leqslant |\gamma(t_0)| \exp \int_{t_0}^{t} \frac{\mathrm{d}s}{C|s-c|^{\rho_c}} \tag{3.6}$$

which actually provides

$$|\gamma(t)| \leqslant |\gamma(t_0)| \exp \frac{(c-t_0)^{1-\rho_c} - (c-t)^{1-\rho_c}}{C(1-\rho_c)} \tag{3.7}$$

Then $|\gamma(t)|$ has a finite limit when t tends to c. This implies that the embedding ϕ of Theorem 2.3 is essentially a diffeomorphism from $f^{-1}(t)$ onto $f^{-1}(c)$. This ends the proof. □

Remark 3.5. Note that Theorem 3.4 also holds under the statements of Remark 3.2 replacing only the C^1 regularity of f by the C^2 regularity of f.

Corollary 3.6. *If c is a regular value and a bifurcation value, then the Kurdyka-Lojasiewicz at infinity for c is equal to* 1.

Proof. Since we cannot trivialise the function f over a neighbourhood of c, from Theorem 3.4, the exponent has to be at least 1. □

When c belongs to $K_\infty(f) \setminus B(f)$, the function f induces a locally trivial fibration over of c. Moreover this trivialisation is provided by the flow of ∇f when the Kurdyka-Łojasiewicz exponent at infinity at c is strictly less than 1. From the view point of Definition 2.5, Theorem 3.4 can be stated otherwise:

Corollary 3.7. *Let Γ be a trajectory of ∇f of infinite length at c. Then the Kurdyka-Lojasiewicz exponent at infinity of c is equal to* 1.

4. Kurdyka-Łojasiewicz exponent of complex polynomials

Let $f : \mathbb{C}^n \to \mathbb{C}$ be a complex polynomial. As mentioned in section 2, we can define the set $K_\infty(f)$ of asymptotic critical values. This set is a finite subset of $\mathbb{C}$. Again we denote $K(f) = K_0(f) \cup K_\infty(f)$ the set of generalised critical values. If $t \in \mathbb{C} \setminus K(f)$ then f is a locally trivial fibration over t.

There is also an analog of the Embedding Theorem 2.3 in the complex case. Namely, if $c \in K_\infty(f) \setminus K_0(f)$ and $t \in \mathbb{C} \setminus K(f)$ then

Theorem 4.1 ([D'A2])**.** *There exists an embedding $\varphi_{c,t} : f^{-1}(c) \to f^{-1}(t)$.*

Let us denote by grad $f(z) \in \mathbb{C}^n$ the polynomial vector field whose components are $(\partial f(z)/\partial z_1, \ldots, \partial f(z)/\partial z_1)$. The proof of Theorem 4.1 (see [D'A2] for details) can be mixed up with the proof of Proposition 3.1 to obtain the following

Proposition 4.2. *There exist $C > 0$ and a rational number $0 < \rho \leqslant 1$ such that for sufficiently large $|z|$ and sufficiently small $|f(z) - c|$ we have*

$$|z| \cdot |\mathrm{grad} f(z)| \geqslant C|f(z) - c|^{\rho}$$

We again denote by ρ_c the smallest such exponent ρ and call it the Kurdyka-Łojasiewicz exponent at infinity related to c.

The complex situation is more rigid than the real one. When $n = 2$ or f has only isolated singularities at infinity we actually know the value of ρ_c. We obtain

Theorem 4.3. *Under these hypotheses, if $c \in K_{\infty}(f) \setminus K_0(f)$, then the Kurdyka-Łojasiewicz exponent at infinity ρ_c is equal to 1.*

Proof. If $c \in K_{\infty}(f) \setminus K_0(f)$ then Malgrange condition (**M**) fails at c. By hypothesis, Parusiński (cf. [Pa]) proved that c is a bifurcation value of f. Thus the embedding $\varphi_{c,t} : f^{-1}(c) \to f^{-1}(t)$ is not a diffeomorphism for any typical value $t \in \mathbb{C}$.

Let us identify $\mathbb{C}$ with $\mathbb{R}^2$. Let us write $f = P + iQ$, where P and Q are respectively the real and the imaginary part of f.

Assume now that $\rho_c < 1$ and pick a typical value t, such that the real line $L \subset \mathbb{R}^2$ through c and t does not meet any other generalised critical value than c, that is $L \cap K(f) = \{c\}$. Let $V_L = f^{-1}(L)$. This is a smooth real algebraic hypersurface of $\mathbb{R}^{2n}$. Let f_L be the restriction of f to V_L. Then it is a submersion. We equip $\mathbb{C}^n$ identified to $\mathbb{R}^{2n}$ with the usual Euclidean structure. We also equip V_L with the Riemannian structure induced by the Euclidean structure of $\mathbb{R}^{2n}$. Note that f_L smooth semialgebraic function. Thus $K_{\infty}(f_L)$ is finite.

From Remark 3.2 there exists $\rho_c^L \leqslant \rho_c \leqslant 1$, the Kurdyka-Łojasiewicz exponent of f_L at c. If $\rho_c < 1$ then $\rho_c^L \leqslant \rho_c < 1$ and by Remark 3.5 the fibre $f_L^{-1}(c)$ is diffeomorphic to the fibre $f_L^{-1}(t)$. Thus $f^{-1}(c)$ is also diffeomorphic to $f^{-1}(t)$. This is impossible because c is a bifurcation value. Then $\rho_c = 1$. □

5. Examples

In this section we produce some examples that illustrate the results stated before. All the polynomials presented below have an asymptotic critical value. Each example describes a different phenomenon.

Example 5.1 (Broughton example)**.**

Let f be the polynomial $f(x, y) = y(xy - 1)$. We immediately observe that f has no critical point. The set $\{\partial_y f = 0\}$ is the algebraic curve $\{2xy - 1 = 0\}$ and $f(x, 1/2x) \to 0$ when $x \to +\infty$, and $0 \in K_{\infty}(f)$. Thus the Kurdyka-Łojasiewicz exponent associated to the value 0 is equal to 1.

Since the value 0 is the only generalised critical value, we deduce $B(f) = K_{\infty}(f) = \{0\}$.

If ϕ_t denotes the embedding of Theorem 2.3 then the complement of $\phi_t(f^{-1}(0))$ in $f^{-1}(t)$ is non empty for all $t > 0$. In taking $-f$ instead of f, we have a similar result for all $t < 0$.

In this example the following is true

in the upper half plane, there is a unique integral curve of ∇f *which is of infinite length at* 0.

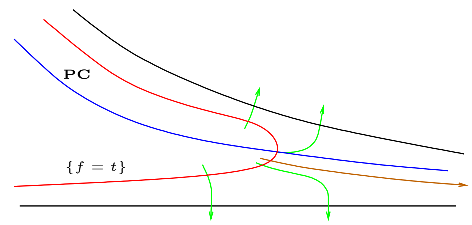

Fig.1: Phase portrait of ∇f

Example 5.2 (King, Tibăr & Zaharia example)**.** Let f be the polynomial $f(x,y) = -y(2x^2y2 - 9xy + 12)$. This function induces a smooth locally trivial fibration (see [TZ, Proposition 2.6]).

We obtain $K(f) = K_\infty(f)$, and $B(f)$ is empty. Any level $\{-y(2x^2y2 - 9xy + 12) = t\}$ is homeomorphic to a line.

The point $[1 : 0 : 0]$ is the point at infinity of all the fibres of f and $\{\partial_y f = 0\}$ is the union of the algebraic curves $\mathbf{PC}_1 := \{xy - 1 = 0\}$ and $\mathbf{PC}_2 := \{xy - 2 = 0\}$. Then the Kurdyka-Łojasiewicz exponent at infinity at 0 is equal to 1. This means *the triviality of* $f = y(2x^2y2 - 9xy + 12)$ *over the value* 0 *cannot be realised by the gradient vector field of* f. In this case there are infinitely many integral curves of going to infinity without reaching the zero level of f, that is infinitely many trajectory of infinite length at 0.

Let $\mathbf{PC}_v = \{4xy - 9 = 0\}$ be the polar curve in the vertical direction. These three polar curves give enough information on the dynamics at infinity of the gradient field. A trajectory has a unique intersection point with each of the polar curves $\mathbf{PC}_*$ (with $* = 1, 2, v$). Now the phase portrait of ∇f is organised around two special integral curves which actually are branching points of the space of leaves of the foliation by ∇f. For any level $t > 0$, the same kind of phenomenon occur because of the symmetry of f.

A quick study of the signs of the $\partial_x f$ and $\partial_y f$, and the study of the inflection points of the trajectories give enough information to draw the following phase portrait

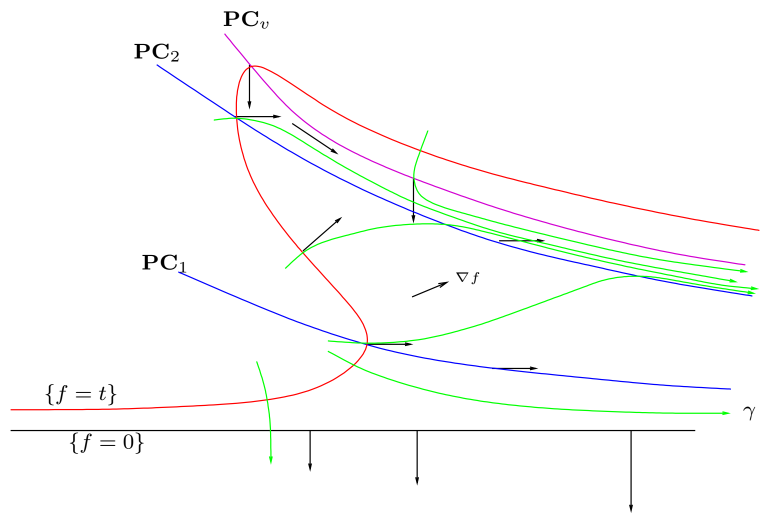

Fig.2: Phase portrait of ∇f

Example 5.3 (The Parusiński example)**.**

Let f be the polynomial $f(x, y) = y^{11} + (1 + (1 + x^2)y)^3$. All the fibres of f have $[1 : 0 : 0]$ as the unique point at infinity.

On the curve $\mathbf{PC}_v := \{1+(1+x^2)y = 0\}$, we get that 0 belongs to $K_\infty(f)$. Moreover we find $K(f) = K_\infty(f) = \{0\}$. All the fibres of this function are homeomorphic to a line. And so by [TZ], f is a locally trivial fibration.

Note that there is no polar curve $\{\partial_y f - a\partial_x f\}$, with $a \in \mathbb{R}$, having a half-branch at infinity with limit point $[1 : 0 : 0]$ and along which the function f tends to 0. In this case *the gradient field realises the trivialisation*. The gradient vector field of f is given by

$$\nabla f(x, y) = 6xy(1 + y + x^2y)^2 \frac{\partial}{\partial x} + \left(11y^{10} + 3(1 + x^2)(1 + y + x^2y)^2\right) \frac{\partial}{\partial y}$$

Note that any level $f = t$ is actually a graph in y of some function x_t, and we have

$$x_t(y) = \sqrt{\frac{t^{\frac{1}{3}} - 1 - y}{y}} + \text{h.o.t} \simeq \frac{k(t)}{y^{1/2}},$$

with $k(t) < 0$.

Let ρ_0 be the Kurdyka-Łojasiewicz exponent at infinity at 0. Let G be any semialgebraic curve along which f is negative and tends to 0. The curve G is the graph of a function, say g, in the variable x. Thus we must have $g(x) \sim -x^\nu$ for a rational number $\nu < 1$. We assume $x \gg 1$.

If $\nu \neq -2$, It is easy to verify that

$$|(x, g(x)| \cdot |\nabla f(x, g(x))| \geqslant x^3.$$

Then the Kurdyka-Łojasiewicz exponent along any such curve is non positive.

Assume $\nu = -2$. Then we deduce $g(x) \simeq -x^{-2}$. So there exists $\eta > 1$ such that $\partial_x f(x, g(x)) \sim x^{-\eta}$, then $\partial_y f(x, g(x)) \geqslant 3x^{3-\eta}$, and so

$$|(x, g(x)| \cdot |\nabla f(x, g(x))| \simeq x\partial_y f(x, g(x)).$$

We verify that there is a positive constant C such that

If $\eta \geqslant 23$ then

$$x\partial_y f(x, g(x)) \geqslant C|f(x, g(x))|^{19/22}.$$

If $\eta \in]47/3, 23[$ then

$$x\partial_y f(x, g(x)) \geqslant C|f(x, g(x))|^{(4-\eta)/22} \geqslant C|f(x, g(x))|^{19/22}.$$

If $\eta \in]1, 47/3]$ then

$$x\partial_y f(x, g(x)) \geqslant C|f(x, g(x))|^{(8-2\eta)/(3-3\eta)} \geqslant C|f(x, g(x))|^{2/3}.$$

Taking $g(x)) := -(1 + x^2)^{-1}$, we verify that along $y = g(x)$

$$x\partial_y f(x, g(x)) \sim |f(x, g(x))|^{19/22},$$

and thus $\rho_0 = 19/22$. So the flow of $\nabla f/|\nabla f|^2$ realises the trivialisation.

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D. D'Acunto, Dipartimento di Matematica Università degli Studi di Pisa, Via Filippo Buonarotti 2 56127 Pisa, (ITALIA)

E-mail address: dacunto@mail.dm.unipi.it

E-mail address: didier.dacunto@univ-savoie.fr

V. Grandjean, Departement of Computer Science, University of Bath, BATH BA2 7AY, England,(UNITED KINGDOM)

E-mail address: cssvg@bath.ac.uk