Gradient trajectories and geometric bounds of definable sets

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TRAJECTORIES OF ∇f

Let $f : U \to \mathbb{R}$ be an analytic function and consider the following ODE :

$$\text{(E)} \quad \dot{x}(t) = \nabla f(x(t))$$

A solution of (E) is called an *integral curve* or a *trajectory* of the vector field ∇f.

Examples.

– $f(x,y) = -(x^2 + y^2)$ trajectories = half-lines attracted by the origin

– $f(x,y) = y^2 - x^3$ *(cusp)* trajectories= left half-plane and right $x-$axis

Question. How does a trajectory approach a singular point ($\nabla f(x_0) = 0$) of f ?

KNOWN RESULTS

Theorem.(Łojasiewicz, 1960) *Let* $x_0 \in U$ *s.t.* $\nabla f(x_0) = 0$ *and let* γ_x *be a trajectory of* ∇f *attracted by* x_0 *then*

$$\text{Length}(\gamma_x) < +\infty \ .$$

Theorem.(Kurdyka, 1998) *Let* $f : U \to \mathbb{R}$ *be* C^2 *definable in an o-minimal structure, then* $\exists M(f) > 0$ *s.t.* $\forall x \in U$,

$$\text{Length}(\gamma_x) \leq M(f).$$

Theorem.(Kurdyka, Mostowski, Parusiński, 2000) [Gradient conjecture of R. Thom]

Let $f : U \to \mathbb{R}$ *be an analytic function,* $x_0 \in U$ *s.t.* $\nabla f(x_0) = 0$ *and let* γ_x *be a trajectory of* ∇f *s.t.* $\gamma_x(t) \to x_0$ *when* $t \to +\infty$, *then* γ_x *has a tangent at* x_0.

Remark. The last result also holds in polynomially bounded o-minimal structures.

Gradient trajectories have good finiteness properties even if they are in general transcendant w.r.t. the geometric category in which the function lives.

Goals of the talk.

1. Parametrized version of Kurdyka's Theorem.
2. Effective bound when f is a polynomial in n variables of degree d.
3. Use trajectories of some gradient vector field to join two points in a connected semialgebraic set.

UNIFORM BOUND FOR THE LENGTH OF GRADIENT TRAJECTORIES

Let $\mathcal{F} = \{f_p\}_{p \in \mathbb{R}^k}$ be a definable familly of C^2 functions $f_p : B^n(0, r) \to \mathbb{R}$.

Theorem 1. *There exists $M_{\mathcal{F}} > 0$ such that for all $p \in \mathbb{R}^k$ and for all trajectories $t \mapsto x_p(t)$ of ∇f_p*

$$\textit{Length}(x_p) \leq M_{\mathcal{F}}.$$

Immediate consequence :

Corollary 2. *Let n, d be two integers, then there exists $A(n, d) > 0$ s.t. for any polynomial f in n variables of degree d, the length of the trajectories of ∇f is bounded by $A(n, d)$.*

Ideas of the proof of Theorem 1.

1– Define $K_a(f_p)$ the set of *generalized critical values of* f by :

$$K_a(f_p) = \{c \in \mathbb{R} : \exists \{x_\nu\}, f(x_\nu) \to c, \nabla f(x_\nu) \to 0\}$$

⋆ Then $K_a(f_p)$ is **finite** and the set

$$K(\mathcal{F}) = \{(p, c) \in \mathbb{R}^{k+1} : p \in \mathbb{R}^k, c \in K_a(f_p)\}$$

is definable.

2– Fix $p \in \mathbb{R}^k$ and $t \in \text{Im } f_p \setminus K_a(f_p)$ and define

$$\Delta_p(t) = \{x \in f_p^{-1}(t) : |\nabla f_p(x)| \leq 2\varphi_p(t)\}$$

where $\varphi_p(t) = \inf\{|\nabla f_p(y)| : f(y) = t\}$.

⋆ The sets $\Delta_p = \bigcup_t \Delta_p(t)$ and $\Delta = \bigcup_{p \in \mathbb{R}^k} \Delta_p$ are definable.

3– Using *definable choice with parameters* we choose in each set Δ_p a piecewice C^1 definable curve Γ_p such that for a generic level t :

(a) $\Gamma_p \cap f^{-1}(t) = \{\text{one point}\}$,

(b) $\Gamma_p \pitchfork f^{-1}(t)$,

(c) The familly $\Gamma = \{\Gamma_p\}_{p\in\mathbb{R}^k}$ is definable.

4– If $t \mapsto x_p(t)$ is a trajectory of ∇f_p then

$$\text{Length}(x_p) \leq 2\text{Length}(\Gamma_p)$$

5– Bound the length of the curves Γ_p independenty on p using :

(a) *Uniform finiteness Lemma*, i.e. $\exists N_{\mathcal{F}}$ s.t. for $p \in \mathbb{R}^k$ and H an affine hyperplane, if $H \cap \Gamma_p$ is finite then $\#(H \cap \Gamma_p) \leq N$.

(b) *Cauchy-Crofton Formula* (in the unit ball, i.e. $r = 1$)

$$\text{Length}(\Gamma_p) = \int_{H \in \mathcal{A}(1,n)} \#(H \cap \Gamma_p) \mathrm{d}H$$

(c) $\text{Length}(\Gamma_p) \leq N_{\mathcal{F}} \int_{H \in \mathbb{A}(1,n)} \mathrm{d}H = N_{\mathcal{F}} \cdot \nu(n)$
where $\mathbb{A}(1,n)$ is the set of hyperplanes that cut the unit ball.

<u>Conclusion</u> $\forall p$ any traj. of ∇f_p is bounded by

$$\boxed{M_{\mathcal{F}} = 2r\nu(n)N_{\mathcal{F}}}$$

EFFECTIVE BOUND FOR POLYNOMIALS

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a polynomial of degree d. We want to estimate the upper bound $A(n,d)$ of the length of traj of ∇f inside the unit ball $\mathbb{B}^n$.

Theorem 3. *Any trajectory of ∇f is bounded by*

$$\nu(n)((3d-4)^{n-1} + 2(3d-3)^{n-2}).$$

Ideas of the proof of Theorem 3.

1– Determine explicitly the curve Γ of the proof of Theorem 1 :

We fix $t \in \mathbb{R}$ and we search the points $x \in \overline{\mathbb{B}^n}$ such that the function $|\nabla f|^2$ restricted to $f^{-1}(t) \cap \overline{\mathbb{B}^n}$ has a local minimum at x. Denote by Γ this set of points.

2– We decompose $\Gamma = \Gamma_1 \cup \Gamma_2$ in the following way :

(a) $\Gamma_1 \subset \mathbb{B}^n$

(b) $\Gamma_2 \subset \overline{\mathbb{B}^n} \setminus \mathbb{B}^n = \mathbb{S}^{n-1}$

3– $\Gamma_1 \subset \theta_1(f)$ and $\Gamma_2 \subset \theta_2(f)$ where

(a) $\theta_1(f) = \{x \in \mathbb{R}^n : \mathrm{d}(|\nabla f|^2) \wedge \mathrm{d}f = 0\}$

(b) $\theta_2(f) = \{x \in \mathbb{S}^{n-1} : \mathrm{d}(|\nabla f|^2) \wedge \mathrm{d}f \wedge \mathrm{d}r = 0\}$ where $r = |x|^2$.

4– For a generic polynomial $f \in \mathbb{R}_d[X]$ the sets $\theta_1(f)$ and $\theta_2(f)$ are a finite union of algebraic curves and points. More precisely :

(a) $\theta_1(f)$ is described by $n-1$ polynomials of degree $\leq 3d-4$ ($\nabla(|\nabla f|^2) = 2H(f)\nabla f$) ;

(b) $\theta_2(f)$ is described by $n-2$ polynomials of degree $\leq 3d-3$ and one polynomial of degree 2 ($|x|^2 = 1$).

5– Bezout's Theorem :

(a) $\#(H \cap \theta_1(f)) \leq (3d-4)^{n-1}$

(b) $\#(H \cap \theta_2(f)) \leq 2(3d-3)^{n-2}$

for a generic hyperplane H.

6– The length of any traj of ∇f inside $\mathbb{B}^n$ is bounded by

Length$(\theta_1(f) \cap \mathbb{B}^n) +$ Length$(\theta_2(f) \cap \mathbb{S}^{n-1})$

Cauchy-Crofton formula for generic polynomials :

$A(n,d) = \nu(n)((3d-4)^{n-1} + 2(3d-3)^{n-2})$.

7– The bound holds for any polynomial of degree $\leq d$ by approximating a non-generic polynomial with a sequence of generic polynomials.

JOINING TWO POINTS IN A COMPACT SEMIALGEBRAIC SET

Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a polynomial of degree d. Let $M \subset B^n(a, r)$ be a compact connected component of $\varphi^{-1}(0)$.

Theorem 4. *Any two points in M can be joined by a curve of length bounded by*

(1) $2r\nu(n)d(4d-5)^{n-2}$ *if M is smooth or* codim $M > 1$

(2) $4r\nu(n+1)d(8d-5)^{n-1}$ *if M is singular of codimension* 1*.*

Ideas of the proof of Theorem 4 in the smooth case

1– Define on M a vector field X in the following way

(a) Choose a line L and let $\pi : \mathbb{R}^n \to L$ be the orthogonal proj on L.

(b) For a generic choice of L the function

$$f = \pi_{|M} : M \to L$$

is a Morse function.

(c) Define $X = \nabla_M f = \nabla\pi - \langle \nabla\pi, \frac{\nabla\varphi}{|\nabla\varphi|}\rangle \frac{\nabla\varphi}{|\nabla\varphi|}$

2– Compare the length of the traj. of $\nabla_M f$ with the length of the "curve"

$$\theta_M(f) = \{x \in M : \mathrm{d}(|\nabla_M f|^2) \wedge \mathrm{d}\pi \wedge \mathrm{d}\varphi = 0\}.$$

3– For a generic polynomial φ and a generic choic of L, $\theta_M(f)$ is a curve. Similar computations show that

$$\text{Length}(\theta_M(f) \leq r\nu(n)d(4d-5)^{n-2}.$$

4– The bound holds for any polynomial φ.

5– The proof of Theorem 4 uses induction on the number of critical points of f and the fact that each connected component of $f^{-1}(t)$ contains a point of $\theta_M(f)$.

The bound is quite sharp, i.e :

Denote by $D(d, n)$ the supremum of geodesic diameters of connected components – included in a unit ball in $\mathbb{R}^n$– of sets $\varphi^{-1}(0)$.

Theorem 5. *For any* $n, d \in \mathbb{N}$

$$D(d, n) \geq n^{-1/2}(2d)^{n-1}, \ d \in \mathbb{N}.$$