L^∞ BOUNDS OF STEKLOV EIGENFUNCTIONS AND SPECTRUM STABILITY UNDER DOMAIN VARIATION

DORIN BUCUR, ALESSANDRO GIACOMINI, AND PAOLA TREBESCHI

ABSTRACT. We give a practical tool to control the L^∞-norm of the Steklov eigenfunctions in a Lipschitz domain in terms of the norm of the BV-trace operator. The norm of this operator has the advantage to be characterized by purely geometric quantities. As a consequence, we give a spectral stability result for the Steklov eigenproblem under geometric domain perturbations and several examples where stability occurs. In particular we deal with geometric domains which are not equi-Lipschitz, like vanishing holes, merging sets, approximations of inner peaks.

CONTENTS

1. INTRODUCTION

In the present paper we are interested to obtain a control of the L^∞-norm of Steklov eigenfunctions in terms of the associated eigenvalues and *some* geometric information of the domain, with the aim to obtain stability results for the Steklov eigenproblem under perturbations of the domain.

Let $\Omega \subseteq \mathbb{R}^N$ be an open bounded domain with Lipschitz boundary. We say that $\sigma > 0$ is an eigenvalue of the Steklov problem if there exists $u \in W^{1,2}(\Omega) \setminus \{0\}$, such that

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = \sigma u & \text{on } \partial\Omega, \end{cases}$$

where ν is the outward normal at the boundary. The equation above is understood in a weak sense, namely

$$\forall \varphi \in W^{1,2}(\Omega) \, : \, \int_\Omega \nabla u \nabla \varphi dx = \sigma \int_{\partial\Omega} u\varphi d\mathcal{H}^{N-1},$$

where $\mathcal{H}^{N-1}$ denotes the $(N-1)$-dimensional Hausdorff measure. As Ω is Lipschitz, there is a sequence of eigenvalues

$$0 = \sigma_0 \le \sigma_1 \le \sigma_2 \ldots \to +\infty,$$

given by the Rayleigh formula

$$\forall k \in \mathbb{N} \, : \, \sigma_k(\Omega) = \min_{S \in \mathcal{S}_{k+1}} \max_{u \in S \setminus \{0\}} \frac{\int_\Omega |\nabla u|^2 dx}{\int_{\partial\Omega} u^2 d\mathcal{H}^{N-1}}, \tag{1.1}$$

where $\mathcal{S}_{k+1}$ denotes the family of all subspaces of dimension $k+1$ in $W^{1,2}(\Omega)$.

It is known that the eigenfunctions belong to L^∞ (see for instance [7, 11] for a suitable variant of Steklov problem). The proof of this fact usually involves the norm of some Sobolev embedding theorem and consequently the Lipschitz character of the domain appears to play an important role in the estimate.

For different eigenvalues of the Laplace operator, the Lipschitz character of the domain does not enter into the L^∞ estimate of the eigenfunctions. This is the case of the Dirichlet eigenvalues ($-\Delta u = \lambda u$ in Ω, $u = 0$ on $\partial\Omega$) for which the following estimate holds (see for instance [8, Example 2.1.8])

$$\|u\|_\infty \le C_N \lambda^{\frac{N}{4}} \|u\|_2,$$

and of the Robin eigenvalues ($-\Delta u = \lambda u$ in Ω, $\frac{\partial u}{\partial \nu} + \beta u = 0$ on $\partial\Omega$) for $\beta > 0$, where (see [14, Proposition 8])

$$\|u\|_\infty \le C_N \Big(\frac{1}{\beta} + \frac{\beta}{\lambda} + \frac{2}{\lambda^{\frac{1}{2}}}\Big)^N \lambda^{\frac{N}{2}} \|u\|_2.$$

Above, and throughout the paper, by C_N we denote a dimensional constant which may change from line to line.

A natural question is to understand at what extent the L^∞-norm of the Steklov eigenfunctions depend on the geometry of the domain. The purpose of this paper is to investigate this question and to give precise L^∞-estimates depending on the norm $C_{trace}(\Omega)$ of the trace operator on the space of functions of bounded variation $BV(\Omega)$ (see Section 2)

$$T : BV(\Omega) \to L^1(\partial\Omega). \tag{1.2}$$

In Theorem 3.1, we show that for an eigenfunction u with eigenvalue σ we have

$$\|u\|_\infty \le C\|u\|_{L^2(\partial\Omega)}, \tag{1.3}$$

where C depends only on $N, C_{trace}(\Omega), |\Omega|$ and σ.

We obtain the previous estimate through a Moser iteration technique: we obtain the crucial increase in summability for the scheme not relying as usual on the Sobolev embedding theorem for $W^{1,2}(\Omega)$, but employing the Sobolev embedding

$$BV(\mathbb{R}^N) \to L^{\frac{N}{N-1}}(\mathbb{R}^N).$$

The key idea is to extend u to zero outside Ω and to interpret the new function $\tilde{u}$ as an element of $BV(\mathbb{R}^N)$: the increase in summability turns out to depend on the total variation of $\tilde{u}$, which is given by

$$|D\tilde{u}|(\mathbb{R}^N) = \int_\Omega |\nabla u| \, dx + \int_{\partial\Omega} |u| \, d\mathcal{H}^{N-1}.$$

The dependence of the L^∞-bound on $C_{trace}(\Omega)$ in (1.3) is connected to the fact that the last term in the right-hand side of the above formula is precisely the L^1 norm of the trace of u on $\partial\Omega$.

Contrary to the Dirichlet and Robin boundary conditions, the eigenvalues *alone* can not control the L^∞-norm of the Steklov eigenvalue, even under a control of volume and perimeter. In Section 6 we report an example of a sequence of domains converging to the unit cube, having a first Steklov eigenvalue constant, converging volumes and perimeters, while the corresponding normalized eigenfunctions have an L^∞-norm which blows up.

A uniform bound for the Steklov eigenfunctions plays a crucial role in the study of the stability of eigenvalues/eigenfunctions under perturbation of the geometric domain. Assume that $(\Omega_n)_{n\in\mathbb{N}}$ is a sequence of bounded open sets in $\mathbb{R}^N$ with Lipschitz boundaries such that $\Omega_n \to \Omega$ strongly in $L^1(\mathbb{R}^N)$, where Ω is also Lipschitz regular. Let $u_k^n \in W^{1,2}(\Omega_n)$ be the k-th Steklov eigenfunction with associated eigenvalue σ_k^n obtained from the scheme (1.1). In the analysis of the stability of the equation

$$\forall \varphi \in W^{1,2}(\Omega_n) : \int_{\Omega_n} \nabla u_k^n \nabla \varphi \, dx = \sigma_k^n \int_{\partial\Omega_n} u_k^n \varphi \, d\mathcal{H}^{N-1}, \tag{1.4}$$

a uniform L^∞-bound for $(u_k^n)_{n\in\mathbb{N}}$ entails that the right hand side of the equation behaves loosely speaking as the perimeter of $\partial\Omega_n$. As a consequence, additional geometric assumptions concerning the behaviors of the perimeters are likely to lead to a stability result.

In Theorem 4.1 we show that equation (1.4) is stable provided that

$$1_{\Omega_n} \to 1_\Omega \quad \text{strongly in } L^1(\mathbb{R}^N), \qquad \mathcal{H}^{N-1}(\partial\Omega_n) \to \mathcal{H}^{N-1}(\partial\Omega)$$

and

$$\sup_n C_{trace}(\Omega_n) < +\infty. \tag{1.5}$$

Thanks to estimate (1.3), this last condition is the key to get that the k-th (normalized) eigenfunctions on Ω_n are uniformly bounded in n, so that the right hand side of (1.4) can be suitably handled (see Lemma 4.4 and Lemma 4.5).

We remark that condition (1.5) can be fulfilled even if the Lipschitz character of the domains is deteriorating, i.e., even under suitable singular perturbations of the geometric boundary. The stability of the spectrum in the context of equi-Lipschitz sets has been investigated in [3].

Finally we notice that the trace constant C_{trace} turns out to be characterized geometrically (see for instance [2], [15, Section 5.10], [5]), and so can be easily estimated in some given geometrical situations. In Section 5 we will recall these characterizations, while in Section 6 we will collect some examples in which (1.5) is satisfied, and a stability result then follows.

The paper is organized as follows. In Section 2 we fix the notation and recall some basic facts concerning functions of bounded variation and sets of finite perimeter employed in the rest of the paper. Section 3 is devoted to the proof of the L^∞-bound for Steklov eigenfunctions, while the main application to the stability of the Steklov spectrum is contained in Section 4. In Section 5 we recall the geometric descriptions of C_{trace}, and in Section 6 we collect some applications of our stability result.

2. Notation and preliminaries

In this section we introduce the basic notation and recall some notions employed in the rest of the paper.

Basic notation. If $E \subseteq \mathbb{R}^N$, we will denote with $|E|$ its N-dimensional Lebesgue measure, and by $\mathcal{H}^{N-1}(E)$ its $(N-1)$-dimensional Hausdorff measure: we refer to [9, Chapter 2] for a precise definition, recalling that for sufficiently regular sets $\mathcal{H}^{N-1}$ coincides with the usual area measure. Moreover, we denote by E^c the complementary set of E, and by 1_E its characteristic function, i.e., $1_E(x) = 1$ if $x \in E$, $1_E(x) = 0$ otherwise. If u is a function defined on E, we will denote with $u1_E$ the extension of u to $\mathbb{R}^N$ which is equal to zero outside E.

If $A \subseteq \mathbb{R}^N$ is open and $1 \le p \le +\infty$, we denote by $L^p(A)$ the usual space of p-summable functions on A with norm indicated by $\|\cdot\|_p$. $W^{1,p}(A)$ will stand for the Sobolev space of functions in $L^p(A)$ whose gradient in the sense of distributions belongs to $L^p(A, \mathbb{R}^N)$. Finally $\mathcal{M}_b(A; \mathbb{R}^N)$ will denote the space of $\mathbb{R}^N$-valued Radon measures on A, which can be identified with the dual of $\mathbb{R}^N$-valued continuous functions on A vanishing at the boundary.

Functions of bounded variation. If $A \subseteq \mathbb{R}^N$ is open, we say that $u \in BV(A)$ if $u \in L^1(A)$ and its derivative in the sense of distributions is a finite Radon measure on A, i.e., $Du \in \mathcal{M}_b(A; \mathbb{R}^N)$. $BV(A)$ is called the space of *functions of bounded variation* on A. $BV(A)$ is a Banach space under the norm $\|u\|_{BV(A)} := \|u\|_{L^1(A)} + \|Du\|_{\mathcal{M}_b(A;\mathbb{R}^d)}$. We call $|Du|(A) := \|Du\|_{\mathcal{M}_b(A;\mathbb{R}^d)}$ the *total variation* of u. We refer the reader to [1] for an exhaustive treatment of the space BV.

If $u \in BV(A)$, then the measure Du can be decomposed canonically (and uniquely) as

$$Du = D^a u + D^j u + D^c u.$$

The measure $D^a u$ is the absolutely continuous part (with respect to the Lebesgue measure) of the derivative: the associated density is denoted by $\nabla u \in L^1(A; \mathbb{R}^N)$. The measure $D^j u$ is the *jump* *part* of the derivative and it turns out that

$$D^j u = (u^+ - u^-) \otimes \nu\, \mathcal{H}^{N-1} \lfloor J_u.$$

Here J_u is the *jump set* of u, ν is the normal to J_u, while $u^\pm$ are the two traces of u on the jump set. Finally $D^c u$ is called the *Cantor part* of the derivative, and it vanishes on sets which are σ-finite with respect to $\mathcal{H}^{N-1}$.

Notice that $W^{1,1}(A) \subseteq BV(A)$: moreover if $u \in W^{1,1}(A)$, $\|u\|_{BV(A)} = \|u\|_{W^{1,1}(A)}$.

We will make use the following standard properties of BV.

Theorem 2.1. *The following items hold true.*

(a) Sobolev embedding. *The space $BV(\mathbb{R}^N)$ is continuously embedded in $L^p(\mathbb{R}^N)$ for every $1 \le p \le \frac{N}{N-1}$. The embedding is compact in $L^p_{loc}(\mathbb{R}^N)$ for every $1 \le p < \frac{N}{N-1}$.*

(b) Lower semicontinuity of the total variation. *If $A \subseteq \mathbb{R}^N$ is open and $(u_n)_{n\in\mathbb{N}}$ is a bounded sequence in $BV(A)$ with $u_n \to u$ strongly in $L^1(A)$, then $u \in BV(A)$ and*

$$|Du|(A) \le \liminf_n |Du_n|(A).$$

We will be concerned with the trace operator in BV. If $\Omega \subseteq \mathbb{R}^N$ is an open bounded domain with Lipschitz boundary, there exists a continuous linear operator

$$T : BV(\Omega) \to L^1(\partial\Omega)$$

such that, denoting $T(u)$ on $\partial\Omega$ still by u, the following integration by parts formula holds true: for every $\varphi \in C^1_c(\mathbb{R}^N)$

$$\int_\Omega u \partial_i \varphi \, dx = \int_{\partial\Omega} u\varphi\nu_i \, d\mathcal{H}^{N-1} - \int_\Omega \varphi \, dD_i u, \tag{2.1}$$

where ν_i denotes the i-th component of the exterior normal ν. We will denote with $C_{trace}(\Omega)$ the norm of T. Thanks to (2.1), T is a lifting to $BV(\Omega)$ of the trace operator on $W^{1,1}(\Omega)$ (with the same norm).

A consequence of (2.1) is the following result which is pivotal to our analysis: if $u \in W^{1,1}(\Omega)$, we have $u1_\Omega \in BV(\mathbb{R}^N)$ with

$$\|u1_\Omega\|_{BV(\mathbb{R}^N)} = \int_\Omega |\nabla u| \, dx + \int_{\partial\Omega} |u| \, d\mathcal{H}^{N-1} + \int_\Omega |u| \, dx.$$

Sets of finite perimeter. Given $E \subseteq \mathbb{R}^N$ measurable and $A \subseteq \mathbb{R}^N$ open, we say that E has finite perimeter in A (or simply has finite perimeter if $A = \mathbb{R}^N$) if

$$Per(E; A) := \sup\left\{ \int_E div(\varphi)\, dx \,:\, \varphi \in C^\infty_c(A; \mathbb{R}^N), \|\varphi\|_\infty \le 1 \right\} < +\infty.$$

If $|E| < +\infty$, then E has finite perimeter if and only if $1_E \in BV(\mathbb{R}^N)$. It turns out that

$$D1_E = \nu_E \mathcal{H}^{N-1} \lfloor \partial^* E, \qquad Per(E; \mathbb{R}^N) = \mathcal{H}^{N-1}(\partial^* E),$$

where $\partial^* E$ is called the *reduced boundary* of E, and ν_E is the associated inner approximate normal (see [1, Section 3.5]). It turns out that $\partial^* E \subseteq \partial E$, but the topological boundary can in in general be much larger than the reduced one.

We will make use of the following isoperimetric inequalities. If $E \subseteq \mathbb{R}^N$ has finite perimeter with $|E| < +\infty$, then

$$|E|^{\frac{N-1}{N}} \le C_N \mathcal{H}^{N-1}(\partial^* E),$$

where C_N is a dimensional constant. If $\Omega \subseteq \mathbb{R}^N$ is a bounded open set with Lipschitz boundary,

$$\min\{|E|^{\frac{N-1}{N}}, |\Omega \setminus E|^{\frac{N-1}{N}}\} \le C\mathcal{H}^{N-1}(\partial^* E \cap \Omega) \tag{2.2}$$

for every $E \subseteq \Omega$ of finite perimeter, where C depends only on Ω.

3. The L^∞-bound for the eigenfunctions

Let $\Omega \subseteq \mathbb{R}^N$ be an open bounded domain with Lipschitz boundary. Recall that $u \in W^{1,2}(\Omega)$ is an eigenfunction for the Steklov problem with associated eigenvalue σ if $u \neq 0$ and

$$\forall \varphi \in W^{1,2}(\Omega) \,:\, \int_\Omega \nabla u \nabla \varphi dx = \sigma \int_{\partial\Omega} u\varphi d\mathcal{H}^{N-1}. \tag{3.1}$$

The following L^∞-estimate for eigenfunctions of the Steklov problem holds true, involving the norm $C_{trace}(\Omega)$ of the trace operator (1.2).

Theorem 3.1 **(**L^∞**-bound for Steklov eigenfunctions).** *Let $\Omega \subseteq \mathbb{R}^N$ be an open bounded domain with Lipschitz boundary. Then for every eigenfunction $u \in W^{1,2}(\Omega)$ of the Steklov problem with associated eigenvalue σ we have $u \in L^\infty(\Omega)$ with*

$$\|u\|_\infty \leq C\sqrt{\|u^2\|_{W^{1,1}(\Omega)}} \leq \tilde{C}\|u\|_{L^2(\partial\Omega)}, \tag{3.2}$$

where C and $\tilde{C}$ depend only on $N, C_{trace}(\Omega), |\Omega|$ and σ.

Proof. We divide the proof in several steps.

Step 1: We claim that if $|u|^\alpha \in W^{1,1}(\Omega)$ with $\alpha \geq 2$, then $|u|^{\alpha\chi} \in W^{1,1}(\Omega)$ with

$$\||u|^{\alpha\chi}\|_{W^{1,1}(\Omega)} \leq C\left(\frac{\alpha\chi}{\sqrt{\alpha-1}} + |\Omega|^{\frac{1}{2N}}\right)\||u|^\alpha\|^\chi_{W^{1,1}(\Omega)}, \tag{3.3}$$

where

$$\chi := \frac{2N-1}{2N-2} = 1 + \frac{1}{2N-2} > 1 \tag{3.4}$$

and $C = C(N, \sigma, C_{trace}(\Omega))$.

For every $M > 0$, consider

$$u_M := (u \wedge M) \vee (-M)$$

and

$$\varphi_M = |u_M|^{\alpha-2}u_M.$$

Since φ_M is the composition of $u_M \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$ with the C^1 function $F(s) := |s|^{\alpha-2}s$, by the chain rule for Sobolev functions we get $\varphi_M \in W^{1,2}(\Omega)$, with

$$\nabla\varphi_M = (\alpha-1)|u_M|^{\alpha-2}\nabla u_M.$$

Testing equation (3.1) with $\varphi = \varphi_M$, since $|u|^\alpha \in W^{1,1}(\Omega)$ and in view of the trace theorem on $BV(\Omega)$ (recall that $W^{1,1}(\Omega) \subseteq BV(\Omega)$ with the same norm) , we get

$$\begin{aligned}\int_\Omega \nabla u(\alpha-1)|u_M|^{\alpha-2}\nabla u_M \,dx &= \sigma\int_{\partial\Omega} u|u_M|^{(\alpha-2)}u_M \,d\mathcal{H}^{N-1}\\ &\leq \sigma\int_{\partial\Omega}|u|^\alpha \,d\mathcal{H}^{N-1}\\ &\leq \sigma C_{trace}(\Omega)\||u|^\alpha\|_{W^{1,1}(\Omega)}.\end{aligned}$$

By the definition of u_M we get

$$\int_\Omega \nabla u(\alpha-1)|u_M|^{\alpha-2}\nabla u_M \,dx = \int_{\{-M<u<M\}}(\alpha-1)|u_M|^{\alpha-2}|\nabla u|^2 dx.$$

Hence, letting $M \to +\infty$, from the Monotone Convergence Theorem, we get

$$\int_\Omega (\alpha-1)|u|^{\alpha-2}|\nabla u|^2 \,dx \leq \sigma C_{trace}(\Omega)\||u|^\alpha\|_{W^{1,1}(\Omega)}. \tag{3.5}$$

Let us prove now that $|u|^{\alpha\chi} \in W^{1,1}(\Omega)$. Notice that, since $u_M \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$, by composition we have

$$|u_M|^{\alpha\chi} \in W^{1,1}(\Omega).$$

Let us compute explicitly the $W^{1,1}$ norm, and then let $M \to +\infty$.

(a) By using Hölder inequality and (3.5) we have

$$
\begin{aligned}
\int_\Omega |\nabla(|u_M|^{\alpha\chi})|\,dx &= \alpha\chi\int_\Omega |u_M|^{\alpha\chi-1}|\nabla u_M|\,dx \\
&= \alpha\chi\int_\Omega |u_M|^{\alpha\chi-\frac{\alpha}{2}}|u_M|^{\frac{\alpha}{2}-1}|\nabla u_M|\,dx \\
&\le \alpha\chi\int_\Omega |u|^{\alpha\chi-\frac{\alpha}{2}}|u|^{\frac{\alpha}{2}-1}|\nabla u|\,dx \\
&\le \frac{\alpha\chi}{\sqrt{\alpha-1}}\sqrt{\int_\Omega |u|^{2\alpha\chi-\alpha}\,dx}\,\sqrt{(\alpha-1)\int_\Omega |u|^{\alpha-2}|\nabla u|^2\,dx} \\
&\le \frac{\alpha\chi}{\sqrt{\alpha-1}}\sqrt{\int_\Omega |u|^{\alpha(2\chi-1)}\,dx}\,\sqrt{\sigma C_{trace}(\Omega)\||u|^\alpha\|_{W^{1,1}(\Omega)}} \\
&\le C_1\frac{\alpha\chi}{\sqrt{\alpha-1}}\sqrt{\int_\Omega |u|^{\frac{\alpha N}{N-1}}\,dx}\,\sqrt{\||u|^\alpha\|_{W^{1,1}(\Omega)}},
\end{aligned}
\tag{3.6}
$$

where in the last inequality we have used the explicit value of χ given in (3.4), while $C_1 = C(\sigma, C_{trace}(\Omega))$.

Let us observe now that $|u|^\alpha \in W^{1,1}(\Omega)$ together with the Lipschitz regularity of Ω yields $|u|^\alpha 1_\Omega \in BV(\mathbb{R}^N)$; hence by Sobolev's embedding theorem (see Theorem 2.1) it follows that $|u|^\alpha 1_\Omega \in L^{\frac{N}{N-1}}(\mathbb{R}^N)$ with

$$
\left(\int_\Omega |u|^{\frac{\alpha N}{N-1}}\,dx\right)^{\frac{N-1}{N}} \le C_N\||u|^\alpha 1_\Omega\|_{BV(\mathbb{R}^N)}\,,
$$

where C_N is a constant depending on N. Since

$$
\begin{aligned}
\||u|^\alpha 1_\Omega\|_{BV(\mathbb{R}^N)} &= \int_\Omega |u|^\alpha\,dx + \int_\Omega \nabla(|u|^\alpha)\,dx + \int_{\partial\Omega} |u|^\alpha\,d\mathcal{H}^{N-1} \\
&\le (1 + C_{trace}(\Omega))\||u|^\alpha\|_{W^{1,1}(\Omega)},
\end{aligned}
$$

we finally obtain

$$
\left(\int_\Omega |u|^{\frac{\alpha N}{N-1}}\,dx\right)^{\frac{N-1}{N}} \le C_2\||u|^\alpha\|_{W^{1,1}(\Omega)}
\tag{3.7}
$$

with $C_2 = C(N, C_{trace}(\Omega))$. Using (3.7) in (3.6) we get

$$
\begin{aligned}
\int_\Omega |\nabla(|u_M|^{\alpha\chi})|\,dx &\le C_1\frac{\alpha\chi}{\sqrt{\alpha-1}}\sqrt{C_2\||u|^\alpha\|^{\frac{N}{N-1}}_{W^{1,1}(\Omega)}}\,\sqrt{\||u|^\alpha\|_{W^{1,1}(\Omega)}} \\
&= C_3\frac{\alpha\chi}{\sqrt{\alpha-1}}\||u|^\alpha\|^{\frac{2N-1}{2N-2}}_{W^{1,1}(\Omega)} = C_3\frac{\alpha\chi}{\sqrt{\alpha-1}}\||u|^\alpha\|^{\chi}_{W^{1,1}(\Omega)},
\end{aligned}
\tag{3.8}
$$

with $C_3 = C(N, \sigma, C_{trace}(\Omega))$.

(b) By using Holder's inequality with exponents

$$
p = \frac{N}{(N-1)\chi} = \frac{2N}{2N-1} \qquad \text{and} \qquad p' = 2N,
$$

and in view of (3.7) we get

$$\begin{aligned}\int_\Omega |u_M|^{\alpha\chi}\,dx &\le \left(\int_\Omega |u_M|^{\alpha\chi p}\,dx\right)^{\frac{1}{p}} |\Omega|^{\frac{1}{p'}}\\ &\le \left(\int_\Omega |u|^{\alpha\chi p}\,dx\right)^{\frac{1}{p}} |\Omega|^{\frac{1}{p'}}\\ &= \left(\int_\Omega |u|^{\frac{\alpha N}{N-1}}\,dx\right)^{\frac{2N-1}{2N}} |\Omega|^{\frac{1}{2N}}\\ &\le \left(\left(C_2\||u|^\alpha\|_{W^{1,1}(\Omega)}\right)^{\frac{N}{N-1}}\right)^{\frac{2N-1}{2N}} |\Omega|^{\frac{1}{2N}}\\ &= C_4\||u|^\alpha\|_{W^{1,1}(\Omega)}^{\frac{2N-1}{2N-2}}|\Omega|^{\frac{1}{2N}}\\ &= C_4\||u|^\alpha\|_{W^{1,1}(\Omega)}^{\chi}|\Omega|^{\frac{1}{2N}},\end{aligned} \tag{3.9}$$

with $C_4 = C(N, C_{trace}(\Omega))$.

Gathering (3.8) and (3.9) it follows

$$\||u_M|^{\alpha\chi}\|_{W^{1,1}(\Omega)} \le C_5\left(\frac{\alpha\chi}{\sqrt{\alpha-1}} + |\Omega|^{\frac{1}{2N}}\right)\||u|^\alpha\|_{W^{1,1}(\Omega)}^{\chi},$$

with $C_5 = \max\{C_3, C_4\} = C(N, \sigma, C_{trace}(\Omega))$.

Letting $M \to +\infty$ and using the Dominated Convergence Theorem, it follows that $|u|^{\alpha\chi} \in W^{1,1}(\Omega)$ with

$$\||u|^{\alpha\chi}\|_{W^{1,1}(\Omega)} \le C_5\left(\frac{\alpha\chi}{\sqrt{\alpha-1}} + |\Omega|^{\frac{1}{2N}}\right)\||u|^\alpha\|_{W^{1,1}(\Omega)}^{\chi}. \tag{3.10}$$

This concludes the proof of Step 1 choosing $C = C_5$ in (3.3).

Step 2: Let us prove that $u \in L^\infty(\Omega)$ with

$$\|u\|_\infty \le C\sqrt{\|u^2\|_{W^{1,1}(\Omega)}}, \tag{3.11}$$

where $C = C(N, \sigma, C_{trace}(\Omega), |\Omega|)$. We employ a Moser-iteration technique.

Let $\alpha := 2\chi^m$ with $m \in \mathbb{N}$ in (3.10). Since

$$\frac{2\chi^{m+1}}{\sqrt{2\chi^m - 1}} \le 4\chi^{\frac{m}{2}+1},$$

we may write (recall that $\chi > 1$)

$$\begin{aligned}\||u|^{2\chi^{m+1}}\|_{W^{1,1}(\Omega)} &\le C_5\left(\frac{2\chi^{m+1}}{\sqrt{2\chi^m-1}} + |\Omega|^{\frac{1}{2N}}\right)\||u|^{2\chi^m}\|_{W^{1,1}(\Omega)}^{\chi}\\ &\le C_5\left(4\chi^{\frac{m}{2}+1} + |\Omega|^{\frac{1}{2N}}\right)\||u|^{2\chi^m}\|_{W^{1,1}(\Omega)}^{\chi}\\ &= C_6\chi^{\frac{m}{2}+1}\||u|^{2\chi^m}\|_{W^{1,1}(\Omega)}^{\chi},\end{aligned} \tag{3.12}$$

where

$$C_6 := C_5(4 + |\Omega|^{\frac{1}{2N}}) = C(N, \sigma, C_{trace}(\Omega), |\Omega|).$$

Taking into account (3.12) we thus obtain

$$\||u|^{2\chi^{m+1}}\|_{W^{1,1}(\Omega)}^{\chi^{-m-1}} \le (C_6)^{\chi^{-m-1}}\left(\chi^{\frac{m}{2}+1}\right)^{\chi^{-m-1}}\||u|^{2\chi^m}\|_{W^{1,1}(\Omega)}^{\chi^{-m}}.$$

Iterating again estimate (3.12) in the right-hand side of the above inequality we get

$$\||u|^{2\chi^{m+1}}\|_{W^{1,1}(\Omega)}^{\chi^{-m-1}} \le (C_6)^{\lambda}\chi^{\beta}\|u^2\|_{W^{1,1}(\Omega)} \le C\|u^2\|_{W^{1,1}(\Omega)}, \tag{3.13}$$

where

$$\lambda := \sum_{j=1}^{m+1} \chi^{-j}, \qquad \beta := \sum_{j=1}^{m} \left(\frac{j}{2}+1\right) \chi^{-j-1} \qquad \text{and} \qquad C = C(N,\sigma,C_{trace}(\Omega),|\Omega|).$$

Hence, the trace theorem in BV and (3.13) yield

$$\left(\int_{\partial\Omega} |u|^{2\chi^{m+1}}\, d\mathcal{H}^{N-1}\right)^{\chi^{-m-1}} \leq C_7 \|u^2\|_{W^{1,1}(\Omega)}$$

with $C_7 = C(N,\sigma,C_{trace}(\Omega),|\Omega|)$. Letting $m \to +\infty$, we therefore obtain

$$\|u^2\|_{L^\infty(\partial\Omega)} \leq C_7 \|u^2\|_{W^{1,1}(\Omega)},$$

from which

$$\|u\|_{L^\infty(\partial\Omega)} \leq \sqrt{C_7}\sqrt{\|u^2\|_{W^{1,1}(\Omega)}}.$$

Being u harmonic, (3.11) follows from the maximum principle, so that Step 2 is concluded.

Step 3: Conclusion. Let us prove that

$$\|u^2\|_{W^{1,1}(\Omega)} \leq \tilde{C} \|u\|^2_{L^2(\partial\Omega)} \tag{3.14}$$

with $\tilde{C} = \tilde{C}(N,\sigma,|\Omega|)$. Then estimate (3.2) follows immediately from Step 2.

We first observe that testing equation (3.1) with u we get

$$\int_\Omega |\nabla u|^2\, dx = \sigma \int_{\partial\Omega} u^2\, d\mathcal{H}^{N-1}. \tag{3.15}$$

Hence

$$\int_\Omega \nabla(u^2)\, dx = \int_\Omega 2u\nabla u\, dx \leq \int_\Omega u^2\, dx + \int_\Omega |\nabla u|^2\, dx = \int_\Omega u^2\, dx + \sigma \int_{\partial\Omega} u^2\, d\mathcal{H}^{N-1}$$

from which we infer

$$\|u^2\|_{W^{1,1}(\Omega)} \leq 2\int_\Omega u^2\, dx + \sigma \int_{\partial\Omega} u^2\, d\mathcal{H}^{N-1}. \tag{3.16}$$

From $u^2 \in W^{1,1}(\Omega)$ and the Lipschitz regularity of Ω we get $u^2 1_\Omega \in BV(\mathbb{R}^N)$. Using Hölder's inequality, the Sobolev embedding $BV(\mathbb{R}^N)$ into $L^{\frac{N}{N-1}}(\mathbb{R}^N)$, and Young's inequality we get

$$\begin{aligned}\int_\Omega u^2\, dx &\leq \left(\int_\Omega u^{\frac{2N}{N-1}}\, dx\right)^{\frac{N-1}{N}} |\Omega|^{\frac{1}{N}}\\ &\leq C_N |D(u^2 1_\Omega)|(\mathbb{R}^N)|\Omega|^{\frac{1}{N}}\\ &= C_N\left(\int_\Omega 2u|\nabla u|\, dx + \int_{\partial\Omega} u^2\, d\mathcal{H}^{N-1}\right)|\Omega|^{\frac{1}{N}}\\ &\leq C_N\left(\varepsilon\int_\Omega u^2\, dx + \frac{1}{\varepsilon}\int_\Omega |\nabla u|^2\, dx + \int_{\partial\Omega} u^2\, d\mathcal{H}^{N-1}\right)|\Omega|^{\frac{1}{N}}.\end{aligned}$$

Choosing ε such that $C_N\varepsilon|\Omega|^{\frac{1}{N}} = \frac{1}{2}$ we can absorb the first integral in the right-hand side of the above inequality into the left-hand side and using (3.15) we infer

$$\int_\Omega u^2\, dx \leq C(N,\sigma,|\Omega|)\int_{\partial\Omega} u^2\, d\mathcal{H}^{N-1}. \tag{3.17}$$

Inequality (3.14) follows now from (3.16) and (3.17). □

4. The stability result

As mentioned in the Introduction, our main application of Theorem 3.1 is to an issue of stability of the Steklov spectrum under the variation of the domain. The following result holds true.

Theorem 4.1 (A stability result for the Steklov spectrum). *Let $(\Omega_n)_{n\in\mathbb{N}}$ be a sequence of bounded Lipschitz domains such that*

$$1_{\Omega_n} \to 1_\Omega \qquad \text{strongly in } L^1(\mathbb{R}^N), \tag{4.1}$$

$$\mathcal{H}^{N-1}(\partial\Omega_n) \to \mathcal{H}^{N-1}(\partial\Omega) \tag{4.2}$$

and

$$\sup_n C_{trace}(\Omega_n) < +\infty, \tag{4.3}$$

where $\Omega \subseteq \mathbb{R}^N$ is also a bounded Lipschitz domain, and $C_{trace}(\Omega_n)$ is the norm of the trace operator (1.2)*. Then the Steklov spectrum is stable, i.e., for every $k \geq 0$*

$$\sigma_k(\Omega_n) \to \sigma_k(\Omega). \tag{4.4}$$

Moreover, up to a subsequence,

$$u_k^n 1_{\Omega_n} \to u_k 1_\Omega \qquad \text{strongly in } L^2(\mathbb{R}^N) \tag{4.5}$$

and

$$\nabla u_k^n 1_{\Omega_n} \to \nabla u_k 1_\Omega \qquad \text{strongly in } L^2(\mathbb{R}^N;\mathbb{R}^N), \tag{4.6}$$

where $u_k^n \in W^{1,2}(\Omega_n)$ and $u_k \in W^{1,2}(\Omega)$ are normalized eigenfunctions corresponding to $\sigma_k(\Omega_n)$ and $\sigma_k(\Omega)$ respectively.

Remark 4.2. The key assumption (4.3) concerns the uniform bound on the trace constants C_{trace} of the converging domains: we refer the reader to Section 5 for a review of the geometric characterization of C_{trace}, and to Section 6 for some applications to singular perturbations of the domain. Assumption (4.2) is crucial for the convergence of the spectrum even in the case of equilipschitz domains (for which the control (4.3) comes from free), as [3, Example 3.6] shows.

Remark 4.3. The hypothesis above on the Lipschitz regularity is not requesting uniform constants. Moreover, the fact that the domains are assumed to be Lipschitz is not crucial for the stability issue, being merely a classical setting in which the Steklov problem is usually considered, and its spectrum is known to consist of eigenvalues. One can replace this hypothesis, by asking Ω_n, Ω to be bounded open sets, with a topological boundary of finite Hausdorff measure such that

$$\mathcal{H}^{N-1}(\partial\Omega_n \setminus \partial^*\Omega_n) = \mathcal{H}^{N-1}(\partial\Omega \setminus \partial^*\Omega) = 0,$$
$$T_n : W^{1,2}(\Omega_n) \to L^2(\partial\Omega_n),\ T : W^{1,2}(\Omega) \to L^2(\partial\Omega) \text{ compact.}$$

Above, since the boundary are not assumed to be Lipschitz, the operators T and T_n are defined using the BV trace. In this case, the Steklov problem still has a spectrum consisting on eigenvalues, which can be obtained by the usual min-max formula.

4.1. Some technical lemmas.

We need some preliminary lemmas.

Lemma 4.4. *Let $(\Omega_n)_{n\in\mathbb{N}}$ be a sequence of bounded Lipschitz domains such that*

$$1_{\Omega_n} \to 1_\Omega \qquad \text{strongly in } L^1(\mathbb{R}^N),$$

where $\Omega \subseteq \mathbb{R}^N$ is also a bounded Lipschitz domain.

Let $u_n \in W^{1,2}(\Omega_n)$ with

$$u_n 1_{\Omega_n} \to u 1_\Omega \qquad \text{strongly in } L^2(\mathbb{R}^N),$$

and

$$\|\nabla u_n\|_{L^2(\Omega_n;\mathbb{R}^N)} \leq C$$

for some $u \in W^{1,2}(\Omega)$ and $C \geq 0$ independent of n. Then

$$\liminf_n \int_{\partial\Omega_n} |u_n|\, d\mathcal{H}^{N-1} \geq \int_{\partial\Omega} |u|\, d\mathcal{H}^{N-1}.$$

Proof. By considering positive and negative parts, it is not restrictive to assume $u_n, u \geq 0$. Let us consider

$$v_n := u_n 1_{\Omega_n} \in BV(\mathbb{R}^N).$$

Since

$$v_n \to u 1_\Omega \qquad \text{strongly in } L^2(\mathbb{R}^N),$$

by the lower semicontinuity of the total variation for BV functions (see Theorem 2.1) we infer that for every open set $A \subseteq \mathbb{R}^N$

$$\int_{A\cap\Omega} |\nabla u|\,dx + \int_{A\cap\partial\Omega} u\,d\mathcal{H}^{N-1} \leq \liminf_n \left[\int_{A\cap\Omega_n} |\nabla u_n|\,dx + \int_{A\cap\partial\Omega_n} u_n\,d\mathcal{H}^{N-1}\right].$$

Let $\partial\Omega \subseteq A$. We deduce

$$\int_{\partial\Omega} u\,d\mathcal{H}^{N-1} \leq \liminf_n \left[\int_{A\cap\Omega_n} |\nabla u_n|\,dx + \int_{A\cap\partial\Omega_n} u_n\,d\mathcal{H}^{N-1}\right]$$
$$\leq \liminf_n \left[\|\nabla u_n\|_{L^2(\Omega_n;\mathbb{R}^N)}|A|^{1/2} + \int_{\partial\Omega_n} u_n\,d\mathcal{H}^{N-1}\right] \leq C|A|^{1/2} + \liminf_n \int_{\partial\Omega_n} u_n\,d\mathcal{H}^{N-1}.$$

If we let A shrink to $\partial\Omega$, the conclusion follows. □

Lemma 4.5. *Let $(\Omega_n)_{n\in\mathbb{N}}$ be a sequence of bounded Lipschitz domains such that*

$$1_{\Omega_n} \to 1_\Omega \qquad \textit{strongly in } L^1(\mathbb{R}^N), \tag{4.7}$$

and

$$\mathcal{H}^{N-1}(\partial\Omega_n) \to \mathcal{H}^{N-1}(\partial\Omega), \tag{4.8}$$

where $\Omega \subseteq \mathbb{R}^N$ is also a bounded Lipschitz domain. Let $u_n \in W^{1,2}(\Omega_n)$ be such that

$$\|\nabla u_n\|_{L^2(\Omega_n;\mathbb{R}^N)} + \|u_n\|_\infty \leq C \tag{4.9}$$

for some C independent of n.

Then there exists $u \in W^{1,2}(\Omega)$ such that up to a subsequence

$$u_n 1_{\Omega_n} \to u 1_\Omega \qquad \textit{strongly in } L^2(\mathbb{R}^N), \tag{4.10}$$

$$\nabla u_n 1_{\Omega_n} \rightharpoonup \nabla u 1_\Omega \qquad \textit{weakly in } L^2(\mathbb{R}^N;\mathbb{R}^N), \tag{4.11}$$

and

$$\int_{\partial\Omega_n} u_n\,d\mathcal{H}^{N-1} \to \int_{\partial\Omega} u\,d\mathcal{H}^{N-1}. \tag{4.12}$$

Proof. Let us divide the proof in two steps.

Step 1. Let us prove (4.10) and (4.11). Consider

$$v_n := u_n 1_{\Omega_n} \in BV(\mathbb{R}^N).$$

Notice that

$$\|v_n\|_{BV(\mathbb{R}^N)} = \int_{\Omega_n} |\nabla u_n|\,dx + \int_{\partial\Omega_n} |u_n|\,d\mathcal{H}^{N-1} + \int_{\Omega_n} |u_n|\,dx$$
$$\leq \|\nabla u_n\|_{L^2(\Omega_n;\mathbb{R}^N)}|\Omega_n|^{1/2} + \|u_n\|_\infty \mathcal{H}^{N-1}(\partial\Omega_n) + \|u_n\|_\infty|\Omega_n| \leq C_1,$$

for some $C_1 \geq 0$ independent of n.

Thanks to Theorem 2.1, we find a function $v \in BV(\mathbb{R}^N)$ such that, up to a subsequence,

$$v_n \to v \qquad \text{pointwise a.e. in } \mathbb{R}^N. \tag{4.13}$$

Up to a further subsequence we may also assume

$$\nabla u_n 1_{\Omega_n} \rightharpoonup f \qquad \text{weakly in } L^2(\mathbb{R}^N;\mathbb{R}^N) \tag{4.14}$$

for some $f \in L^2(\mathbb{R}^N;\mathbb{R}^N)$.

We claim that

$$v = 0 \qquad \text{a.e. on } \mathbb{R}^N \setminus \Omega \tag{4.15}$$

with

$$Dv = f\,dx \qquad \text{on } \Omega. \tag{4.16}$$

From (4.15) and (4.16) we infer

$$v = u 1_\Omega$$

with $u \in W^{1,2}(\Omega)$ such that $f = \nabla u$ on Ω. Then (4.11) is a consequence of (4.14), while (4.10) follows from (4.13), (4.7) and the uniform L^∞-bound given by (4.9) thanks to the Dominated Convergence Theorem.

In order to conclude the step, let us prove claims (4.15) and (4.16). Claim (4.15) is a consequence of (4.7) and of (4.13). Let us pass to claim (4.16). The lower semicontinuity of the perimeter together with (4.8) yields that

$$\mu_n := \mathcal{H}^{N-1} \llcorner \partial\Omega_n \overset{*}{\rightharpoonup} \mu := \mathcal{H}^{N-1} \llcorner \partial\Omega \qquad \text{weakly}^* \text{ in the sense of measures on } \mathbb{R}^N. \tag{4.17}$$

Let

$$Dv_n = \nabla u_n 1_{\Omega_n}\,dx + u_n \nu_n \mathcal{H}^{N-1} \llcorner \partial\Omega_n =: \lambda_n + \eta_n,$$

where ν_n denotes the inner normal to the boundary. Clearly thanks to (4.14) we have

$$\lambda_n \overset{*}{\rightharpoonup} f\,dx \qquad \text{weakly}^* \text{ in the sense of measures.}$$

Notice that in view of (4.8) and (4.9) we have

$$|\eta_n|(\mathbb{R}^N) = \int_{\partial\Omega_n} |u_n|\,d\mathcal{H}^{N-1} \le \|u_n\|_\infty \mathcal{H}^{N-1}(\partial\Omega_n) \le C_2$$

with C_2 independent of n. Up to a further subsequence we may assume

$$\eta_n \overset{*}{\rightharpoonup} \eta \qquad \text{weakly}^* \text{ in the sense of measures on } \mathbb{R}^N.$$

Since

$$|\eta_n| \le \|u_n\|_\infty \mathcal{H}^{N-1} \llcorner \partial\Omega_n \le C\mu_n,$$

thanks to (4.17) we infer

$$|\eta| \le C\mathcal{H}^{N-1} \llcorner \partial\Omega,$$

so that η turns out to be supported on $\partial\Omega$. Since $Dv = f\,dx + \eta$, claim (4.16) follows.

Step 2. Let us prove (4.12). Let C be the constant appearing in (4.9), so that $\|u_n\|_\infty \le C$. Applying Lemma 4.4 to the positive function $C + u_n$ we obtain

$$\liminf_n \int_{\partial\Omega_n} (C + u_n)\,d\mathcal{H}^{N-1} \ge \int_{\partial\Omega} (C + u)\,d\mathcal{H}^{N-1},$$

so that

$$C \lim_n \mathcal{H}^{N-1}(\partial\Omega_n) + \liminf_n \int_{\partial\Omega_n} u_n\,d\mathcal{H}^{N-1} \ge C\mathcal{H}^{N-1}(\partial\Omega) + \int_{\partial\Omega} u\,d\mathcal{H}^{N-1}.$$

In view of the convergence (4.8) of the perimeters we infer

$$\liminf_n \int_{\partial\Omega_n} u_n\,d\mathcal{H}^{N-1} \ge \int_{\partial\Omega} u\,d\mathcal{H}^{N-1}. \tag{4.18}$$

Applying again Lemma 4.4 to the positive function $C - u_n$ we obtain

$$C \lim_n \mathcal{H}^{N-1}(\partial\Omega_n) - \limsup_n \int_{\partial\Omega_n} u_n\,d\mathcal{H}^{N-1} \ge C\mathcal{H}^{N-1}(\partial\Omega) - \int_{\partial\Omega} u\,d\mathcal{H}^{N-1},$$

which yields similarly

$$\limsup_n \int_{\partial\Omega_n} u_n\,d\mathcal{H}^{N-1} \le \int_{\partial\Omega} u\,d\mathcal{H}^{N-1}. \tag{4.19}$$

The convergence (4.12) follows gathering (4.18) and (4.19). □

Remark 4.6. Under the assumptions of Lemma 4.5, we also have

$$\lim_n \int_{\partial\Omega_n} u_n^2 \, d\mathcal{H}^{N-1} = \int_\Omega u^2 \, d\mathcal{H}^{N-1}.$$

Indeed it is sufficient to apply the result to the functions $v_n := u_n^2$ for which

$$\int_{\Omega_n} |\nabla v_n|^2 \, dx + \|v_n\|_\infty = 4 \int_{\Omega_n} u_n^2 |\nabla u_n|^2 \, dx + \|u_n\|_\infty^2 \leq C,$$

where C is independent of n, and one has

$$v_n 1_{\Omega_n} \to u^2 1_\Omega \qquad \text{strongly in } L^2(\mathbb{R}^N).$$

4.2. **Proof of Theorem 4.1.** We proceed in several steps.

Step 1: Upper semicontinuity for the eigenvalues. Let us prove that for every $k \geq 1$

$$\limsup_n \sigma_k(\Omega_n) \leq \sigma_k(\Omega). \tag{4.20}$$

We will use the Courant-Fisher representation

$$\sigma_k(A) = \min_{V \in \mathcal{V}_{k+1}} \max_{v \in V \setminus \{0\}} \frac{\int_A |\nabla v|^2 \, dx}{\int_{\partial A} v^2 \, d\mathcal{H}^{N-1}},$$

where $A \subseteq \mathbb{R}^N$ is an open bounded domain with a Lipschitz boundary, and $\mathcal{V}_{k+1}$ denotes the family of the subspaces of $W^{1,2}(A)$ with dimension $k+1$. The minimum is achieved on the subspace generated by the eigenfunctions v_j associated to $\sigma_j(A)$ with $j = 0, 1, \dots, k$, where we may assume $v_0 = 1$.

Let $V \in \mathcal{V}_{k+1}(\Omega)$ be the subspace generated by the first $(k+1)$ eigenfunctions $u_0, u_1, \dots, u_k$ of Ω, that is

$$V = span\{u_0, \dots, u_k\}.$$

Since Ω has a Lipschitz boundary, we may assume (by regularity given by Theorem 3.1 and by regular extension) that $u_i \in W^{1,2}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. We can then consider the restrictions of u_i to Ω_n and the associated generated vector space V_n which has dimension $k+1$ if n is large enough. Let

$$w_n := \lambda_0^n u_0 + \cdots + \lambda_k^n u_k \in V_n$$

be such that

$$\frac{\int_{\Omega_n} |\nabla w_n|^2 \, dx}{\int_{\partial\Omega_n} w_n^2 \, d\mathcal{H}^{N-1}} = \max_{w \in V_n \setminus \{0\}} \frac{\int_{\Omega_n} |\nabla w|^2 \, dx}{\int_{\partial\Omega_n} w^2 \, d\mathcal{H}^{N-1}}.$$

We may assume $\sum_i (\lambda_i^n)^2 = 1$ with $\lambda_i^n \to \lambda_i$ for $i = 0, \dots, k$. Let $w := \sum_i \lambda_i u_i \in V$. Then the convergence (4.1) of the domains entails

$$\lim_n \int_{\Omega_n} |\nabla w_n|^2 \, dx = \int_\Omega |\nabla w|^2 \, dx.$$

On the other hand, in view of Lemma 4.5 and Remark 4.6 we deduce

$$\lim_n \int_{\partial\Omega_n} w_n^2 \, d\mathcal{H}^{N-1} = \int_{\partial\Omega} w^2 \, d\mathcal{H}^{N-1}.$$

We thus conclude

$$\begin{aligned}\limsup_n \sigma_k(\Omega_n) &\leq \limsup_n \max_{w \in V_n \setminus \{0\}} \frac{\int_{\Omega_n} |\nabla w|^2 \, dx}{\int_{\partial\Omega_n} w^2 \, d\mathcal{H}^{N-1}} = \limsup_n \frac{\int_{\Omega_n} |\nabla w_n|^2 \, dx}{\int_{\partial\Omega_n} w_n^2 \, d\mathcal{H}^{N-1}} \\ &= \frac{\int_\Omega |\nabla w|^2 \, dx}{\int_{\partial\Omega} w^2 \, d\mathcal{H}^{N-1}} \leq \max_{v \in V \setminus \{0\}} \frac{\int_\Omega |\nabla v|^2 \, dx}{\int_{\partial\Omega} v^2 \, d\mathcal{H}^{N-1}} = \sigma_k(\Omega),\end{aligned}$$

so that (4.20) follows.

Step 2: The convergence result for σ_1. Let u_1^n be a normalized eigenfunction for $\sigma_1(\Omega_n)$, that is such that

$$\int_{\partial\Omega_n} u_1^n \, d\mathcal{H}^{N-1} = 0, \quad \int_{\partial\Omega_n} (u_1^n)^2 \, d\mathcal{H}^{N-1} = 1 \quad \text{and} \quad \int_{\Omega_n} |\nabla u_1^n|^2 \, dx = \sigma_1(\Omega_n).$$

Thanks to Step 1, since $\limsup_n \sigma_1(\Omega_n) \le \sigma_1(\Omega)$, we have

$$\int_{\Omega_n} |\nabla u_1^n|^2 \, dx \le C$$

for some C independent of n. Moreover thanks to the uniform bound on $C_{trace}(\Omega_n)$ given by (4.3) and to Theorem 3.1 we may assume that

$$\|u_1^n\|_\infty \le M$$

for some M independent of n.

In view of Lemma 4.5 and Remark 4.6 we infer that up to a subsequence

$$u_1^n 1_{\Omega_n} \to u 1_\Omega \qquad \text{strongly in } L^2(\mathbb{R}^N) \tag{4.21}$$

for some $u \in W^{1,2}(\Omega)$ with

$$\nabla u_1^n 1_{\Omega_n} \rightharpoonup \nabla u 1_\Omega \qquad \text{weakly in } L^2(\mathbb{R}^N;\mathbb{R}^N), \tag{4.22}$$

$$\int_{\partial\Omega} u \, d\mathcal{H}^{N-1} = \lim_n \int_{\partial\Omega_n} u_1^n \, d\mathcal{H}^{N-1} = 0 \tag{4.23}$$

and

$$\int_{\partial\Omega} u^2 \, d\mathcal{H}^{N-1} = \lim_n \int_{\partial\Omega_n} (u_1^n)^2 \, d\mathcal{H}^{N-1} = 1. \tag{4.24}$$

Taking into account Step 1 we deduce

$$\sigma_1(\Omega) \ge \limsup_n \sigma_1(\Omega_n) \ge \liminf_n \sigma_1(\Omega_n) = \liminf_n \int_{\Omega_n} |\nabla u_1^n|^2 \, dx \ge \int_\Omega |\nabla u|^2 \, dx \ge \sigma_1(\Omega),$$

the last inequality coming from the fact that u is admissible for the computation of $\sigma_1(\Omega)$ in view of (4.23) and (4.24). We infer that u is an eigenfunction for $\sigma_1(\Omega)$ and that (4.4) holds true for $k = 1$. Moreover the convergences (4.5) and (4.6) follow from (4.21), (4.22) and the relation

$$\lim_n \int_{\Omega_n} |\nabla u_1^n|^2 \, dx = \lim_n \sigma_1(\Omega_n) = \sigma_1(\Omega) = \int_\Omega |\nabla u|^2 \, dx.$$

Step 3: Convergence of the higher order eigenvalues. Proceeding by induction, thanks to Step 1 and a diagonal argument, it is sufficient to show that if the result is true for the eigenvalues of order $k \le h$, then it is true also for that of order $k = h + 1$.

Let u_{h+1}^n be a normalized eigenfunction for $\sigma_{h+1}(\Omega_n)$, that is such that

$$\int_{\partial\Omega_n} u_{h+1}^n \, d\mathcal{H}^{N-1} = 0, \quad \int_{\partial\Omega_n} (u_{h+1}^n)^2 \, d\mathcal{H}^{N-1} = 1, \qquad \int_{\Omega_n} |\nabla u_{h+1}^n|^2 \, dx = \sigma_{h+1}(\Omega_n),$$

and

$$\int_{\partial\Omega_n} u_{h+1}^n u_j^n \, d\mathcal{H}^{N-1} = 0 \qquad \text{for every } j = 1, \dots, h.$$

Recall that thanks to (4.3) and to Theorem 3.1 we have

$$\|u_j^n\|_\infty \le M$$

for every $j = 1, \dots, h+1$, where M is independent of n.

In view of Step 1 we have

$$\int_{\Omega_n} |\nabla u_{h+1}^n|^2 \, dx = \sigma_{h+1}(\Omega_n) \le C$$

for some C independent of n.

Thanks to Lemma 4.5 and Remark 4.6 we infer that up to a subsequence

$$u^n_{h+1}1_{\Omega_n} \to u1_\Omega \qquad \text{strongly in } L^2(\mathbb{R}^N) \tag{4.25}$$

for some $u \in W^{1,2}(\Omega)$ with

$$\nabla u^n_{h+1}1_{\Omega_n} \rightharpoonup \nabla u1_\Omega \qquad \text{weakly in } L^2(\mathbb{R}^N;\mathbb{R}^N), \tag{4.26}$$

$$\int_{\partial\Omega} u\, d\mathcal{H}^{N-1} = \lim_n \int_{\partial\Omega_n} u^n_{h+1}\, d\mathcal{H}^{N-1} = 0 \tag{4.27}$$

and

$$\int_{\partial\Omega} u^2\, d\mathcal{H}^{N-1} = \lim_n \int_{\partial\Omega_n} (u^n_{h+1})^2\, d\mathcal{H}^{N-1} = 1. \tag{4.28}$$

Moreover we claim that

$$\int_{\partial\Omega} u_j u\, d\mathcal{H}^{N-1} = 0 \qquad \text{for } j = 1, \dots, h. \tag{4.29}$$

Since

$$\sigma_{h+1}(\Omega) = \min\Bigg\{\int_\Omega |\nabla v|^2\,dx \,:\, \int_{\partial\Omega} v\, d\mathcal{H}^{N-1} = 0, \int_{\partial\Omega} v^2\, d\mathcal{H}^{N-1} = 1, \\ \int_{\partial\Omega} u_j v\, d\mathcal{H}^{N-1} = 0 \text{ for } j = 1, \dots, h\Bigg\},$$

taking into account Step 1 we deduce

$$\sigma_{h+1}(\Omega) \geq \limsup_n \sigma_{h+1}(\Omega_n) \geq \liminf_n \sigma_{h+1}(\Omega_n) \\ = \liminf_n \int_{\Omega_n} |\nabla u^n_{h+1}|^2\,dx \geq \int_\Omega |\nabla u|^2\,dx \geq \sigma_{h+1}(\Omega),$$

the last inequality coming from the fact that u is admissible for the computation of $\sigma_{h+1}(\Omega)$ in view of (4.27), (4.28) and claim (4.29). We infer that (4.4) holds true for $k = h + 1$, and that $u \in W^{1,2}(\Omega)$ is an eigenfunction associated to $\sigma_{h+1}(\Omega)$. The convergences (4.5) and (4.6) follow from (4.25), (4.26) and the relation

$$\lim_n \int_{\Omega_n} |\nabla u^n_{h+1}|^2\,dx = \lim_n \sigma_{h+1}(\Omega_n) = \sigma_{h+1}(\Omega) = \int_\Omega |\nabla u|^2\,dx.$$

In order to conclude the proof, we need to show that claim (4.29) holds true. This is a straightforward consequence of Lemma 4.5 applied to the functions $v_n := u^n_j u^n_{h+1}$ for which we have, in view of the convergence result for the eigenvalues of order $j \leq h$,

$$\int_{\Omega_n} |\nabla v_n|^2\,dx + \|v_n\|_\infty \leq 2\int_{\Omega_n} \left[(u^n_j)^2|\nabla u^n_{h+1}|^2 + (u^n_{h+1})^2|\nabla u^n_j|^2\right]\,dx + \|u^n_j\|_\infty\|u^n_{h+1}\|_\infty \\ \leq 2M^2\int_{\Omega_n} \left[|\nabla u^n_{h+1}|^2 + |\nabla u^n_j|^2\right]\,dx + M^2 \leq C$$

for some C independent of n, while (by the Dominated Convergence Theorem)

$$v_n 1_{\Omega_n} \to u_j u_{h+1} 1_\Omega \qquad \text{strongly in } L^2(\mathbb{R}^N).$$

The proof is thus concluded.

Remark 4.7 (Stability of the spectrum and uniform L^∞-bound for eigenfunctions). In the proof above, the uniform bound (4.3) on $C_{trace}(\Omega_n)$ is employed to infer the uniform L^∞-bound for normalized Steklov eigenfunctions

$$\sup_n \|u^n_k\|_{L^\infty(\Omega_n)} < +\infty. \tag{4.30}$$

An inspection of the proof shows that the bound (4.30) (in addition to (4.1) and (4.2)) is sufficient to guarantee the stability of the Steklov spectrum. Since the constant trace C_{trace} admits geometric characterizations, as we report in Section 5, checking the uniform bound (4.3) is in principle easier than verifying (4.30).

5. Geometric control of the BV-trace

In this section we recall some results concerning the geometric description of the norm $C_{trace}(\Omega)$ of the trace operator

$$T : BV(\Omega) \longrightarrow L^1(\partial\Omega),$$

where $\Omega \subseteq \mathbb{R}^N$ is an open bounded domain with a sufficiently smooth boundary.

There are several geometric descriptions of $C_{trace}(\Omega)$, all of being based on the same principle. The difference is only coming at the passage from localised versions around the boundary to global inequalities. We refer the reader to [2] (see [15, Section 5.10] as well) and to the more recent paper [5]. The constant C_{trace} is finite even on domains which are not Lipschitz regular. This may occur even in domains with inner peaks or in domains with cracks. We report below a list of results in this sense.

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded open set such that $\mathcal{H}^{N-1}(\partial\Omega) < +\infty$. Following [2], $C_{trace}(\Omega)$ is finite provided that $Q(\Omega) < +\infty$, where

$$Q(\Omega) := \sup_{x\in\partial\Omega} \limsup_{r\to 0} \sup\left\{ \frac{\mathcal{H}^{N-1}(\partial^* E \cap \partial^*\Omega)}{\mathcal{H}^{N-1}(\partial^* E \cap \Omega)} : E \subseteq \Omega \cap B_r(x), Per(E,\Omega) < +\infty \right\}. \tag{5.1}$$

If $Q(\Omega) < +\infty$, for every $\varepsilon > 0$ there exists a constant $C(\Omega, \varepsilon)$ such that

$$\forall u \in BV(\Omega) \,:\, \int_{\partial^*\Omega} |u| d\mathcal{H}^{N-1} \le (Q(\Omega) + \varepsilon) \int_\Omega |Du| + C(\Omega,\varepsilon) \int_\Omega |u| dx. \tag{5.2}$$

The constant $C(\Omega, \varepsilon)$ turns out to depend upon a partition of unity $\{\varphi_i\}_{i=1,\dots,k}$ of $\partial\Omega$ subordinated to a finite family of balls $\{B_{r_j}(x_j)\}_{j=1,\dots,k}$ with $x_j \in \partial\Omega$ and such that

$$\sup\left\{ \frac{\mathcal{H}^{N-1}(\partial^* E \cap \partial^*\Omega)}{\mathcal{H}^{N-1}(\partial^* E \cap \Omega)} : E \subseteq \Omega \cap B_{r_j}(x_j), Per(E,\Omega) < +\infty \right\} < Q(\Omega) + \varepsilon.$$

More precisely, an inspection of the proof of [2, Theorem 4] shows that we may choose

$$C(\Omega,\varepsilon) = (Q(\Omega) + \varepsilon) \max \sum_{i=1}^{k} |\nabla\varphi_i|. \tag{5.3}$$

Note that the trace of $u \in BV(\Omega)$ is defined only on the reduced boundary of Ω. In general, the reduced boundary $\partial^*\Omega$ is a subset of the topological boundary $\partial\Omega$ and their difference may be of strictly positive $\mathcal{H}^{N-1}$-measure. This is the case, for instance, when Ω has an inner crack. A rather similar interpretation for domains which satisfy moreover $\mathcal{H}^{N-1}(\partial\Omega \setminus \partial^*\Omega) = 0$ is given in [15, Theorem 5.10.7].

Concerning the uniform bound (4.3) in our main stability result, the following result holds true.

Proposition 5.1. *Let $D \subseteq \mathbb{R}^N$ be bounded, and let $(\Omega_n)_{n\in\mathbb{N}}$ be a sequence of open bounded domains with $\mathcal{H}^{N-1}(\partial\Omega_n) < +\infty$ and $\Omega_n \subseteq D$. Assume that there exist $\bar{Q}, \bar{r} > 0$ such that for every $x \in \partial\Omega_n$*

$$\sup\left\{ \frac{\mathcal{H}^{N-1}(\partial^* E \cap \partial^*\Omega_n)}{\mathcal{H}^{N-1}(\partial^* E \cap \Omega_n)} : E \subseteq \Omega_n \cap B_{\bar{r}}(x), Per(E,\Omega_n) < +\infty \right\} \le \bar{Q}. \tag{5.4}$$

Then the constants $C_{trace}(\Omega_n)$ are uniformly bounded.

Proof. Up to a subsequence, we may assume that $\partial\Omega_n \to K$ in the Hausdorff metric for some compact set $K \subseteq \mathbb{R}^N$. Let us consider a partition of unity $\{\varphi_i\}_{i=1,\dots,k}$ of an open neighborhood of K subordinated to the family of balls $\{B_{\bar{r}/2}(x_i)\}_{i=1,\dots,k}$, with $x_i \in K$. Let $x_i^n \in \partial\Omega_n$ be such that $x_i^n \to x_i$. Then for n large we have that $\{\varphi_i\}_{i=1,\dots,k}$ is also a partition of unity of a neighborhood of $\partial\Omega_n$ subordinated to the family of balls $\{B_{\bar{r}}(x_i^n)\}_{i=1,\dots,k}$. In view of (5.1), (5.2), (5.3), and taking into account (5.4), the conclusion follows. ☐

A more direct way to avoid the presence of the unknown constant $C(\Omega_n, \varepsilon)$ is to use the following results from [5]. Assume Ω is a bounded open set with finite perimeter, such that $\mathcal{H}^{N-1}(\partial\Omega \setminus \partial^*\Omega) = 0$ and such that there exists a constant $C > 0$ with the property that for $E \subseteq \Omega$ of finite perimeter

$$\min\{\mathcal{H}^{N-1}(\partial^* E \cap \partial\Omega), \mathcal{H}^{N-1}(\partial\Omega \setminus \partial^* E)\} \le C\mathcal{H}^{N-1}(\partial^* E \cap \Omega).$$

One introduces the constants

$$C_{mv}(\Omega) = \sup_{E\subseteq\Omega} \frac{|E|\mathcal{H}^{N-1}(\partial\Omega \setminus \partial^* E) + |\Omega \setminus E|\mathcal{H}^{N-1}(\partial^* E \cap \partial\Omega)}{|\Omega|\mathcal{H}^{N-1}(\partial^* E \cap \Omega)}$$

and

$$C_{med}(\Omega) = \sup_{E\subseteq\Omega, |E|\le \frac{|\Omega|}{2}} \frac{\mathcal{H}^{N-1}(\partial^* E \cap \partial\Omega)}{\mathcal{H}^{N-1}(\partial^* E \cap \Omega)}.$$

Then, for every $u \in BV(\Omega)$ we have

$$\|u - u_{mv}\|_{L^1(\partial\Omega)} \le C_{mv}(\Omega)|Du|(\Omega),$$

$$\|u - u_{med}\|_{L^1(\partial\Omega)} \le C_{med}(\Omega)|Du|(\Omega),$$

where

$$u_{mv} := \frac{1}{|\Omega|}\int_\Omega u dx \qquad \text{and} \qquad u_{med} := \inf\left\{t \in \mathbb{R} : |\{u > t\}| \le \frac{1}{2}|\Omega|\right\}$$

(see [12, Theorem 9.6.4 and Theorem 9.5.2] and [4, Theorem 1.1]).

Immediately, one gets the trace inequalities

$$\|u\|_{L^1(\partial\Omega)} \le C_{mv}(\Omega)|Du|(\Omega) + \frac{\mathcal{H}^{N-1}(\partial\Omega)}{|\Omega|}\|u\|_{L^1(\Omega)},$$

and

$$\|u\|_{L^1(\partial\Omega)} \le C_{med}(\Omega)|Du|(\Omega) + 2\frac{\mathcal{H}^{N-1}(\partial\Omega)}{|\Omega|}\|u\|_{L^1(\Omega)}.$$

Consequently, in order to apply Theorem 4.1, the knowledge of the values $C_{mv}(\Omega_n)$ or $C_{med}(\Omega_n)$ are enough to get full control of $C_{trace}(\Omega_n)$, since the isoperimetric ratios in the right-hand side of the above inequalities are already uniformly bounded, by hypothesis.

6. Further remarks and examples

In this section, we give several examples of stability of Steklov eigenvalues for some specific domain variation and an example showing that the L^∞-norm of the eigenfunctions can not be controlled by the corresponding eigenvalue *alone*, as it is the case of Dirichlet and Robin eigenvalues.

Example 1. (Vanishing homothetic holes) In this example, we just give an interpretation of the asymptotic result of Nazarov [13] from the perspective of our approach. Assume Ω and ω are two open, bounded, Lipschitz sets containing the origin. One defines $\Omega_\varepsilon := \Omega \setminus \varepsilon\overline{\omega}$, for $\varepsilon > 0$ small. Nazarov gave the precise asymptotic behaviour of $\sigma_k(\Omega_\varepsilon)$, when $\varepsilon \to 0^+$. A simpler question is just to understand that the spectrum behaves continuously with respect to this singular perturbation, namely to prove that

$$\sigma_k(\Omega_\varepsilon) \to \sigma_k(\Omega). \tag{6.1}$$

In view of Theorem 4.1, we need simply to check that the trace constant of Ω_ε remains bounded as ε vanishes: we will make use of the Anzellotti-Giaquinta characterization described in Section 5.

Without loosing generality, assume that $diam(\omega) = 1$ and that $d(0, \partial\Omega) = 3$, and let $0 < \varepsilon \le 1$. Let us consider $x_\varepsilon \in \partial(\Omega \setminus \varepsilon\overline{\omega})$ and a measurable set E with finite perimeter contained in $(\Omega \setminus \varepsilon\overline{\omega}) \cap B_1(x_\varepsilon)$. Clearly, the set E can not touch simultaneously $\partial\Omega$ and $\partial(\varepsilon\overline{\omega})$. If it touches only $\partial\Omega$ then the ratio

$$\frac{\mathcal{H}^{N-1}(\partial^* E \cap \partial\Omega_\varepsilon)}{\mathcal{H}^{N-1}(\partial^* E \cap \Omega_\varepsilon)} = \frac{\mathcal{H}^{N-1}(\partial^* E \cap \partial\Omega)}{\mathcal{H}^{N-1}(\partial^* E \cap \Omega)}$$

is uniformly bounded, as Ω is Lipschitz. If it touches $\partial(\varepsilon\overline{\omega})$ we observe first that setting

$$E_\varepsilon := E \cap B_{2\varepsilon}(x_\varepsilon),$$

taking into account that $\varepsilon\omega \subset B_{2\varepsilon}(x_\varepsilon)$ we have

$$\mathcal{H}^{N-1}(\partial^* E_\varepsilon \cap \Omega_\varepsilon) \le \mathcal{H}^{N-1}(\partial^* E \cap \Omega_\varepsilon)$$

as the projection onto the sphere $\partial B_{2\varepsilon}(x_\varepsilon)$ is a contraction on $\mathbb{R}^N \setminus B_{2\varepsilon}(x_\varepsilon)$. Consequently

$$\frac{\mathcal{H}^{N-1}(\partial^* E \cap \partial\Omega_\varepsilon)}{\mathcal{H}^{N-1}(\partial^* E \cap \Omega_\varepsilon)} \le \frac{\mathcal{H}^{N-1}(\partial^* E_\varepsilon \cap \partial\Omega_\varepsilon)}{\mathcal{H}^{N-1}(\partial^* E_\varepsilon \cap \Omega_\varepsilon)} = \frac{\mathcal{H}^{N-1}\left(\partial^*(\frac{1}{\varepsilon}E_\varepsilon) \cap \partial\omega\right)}{\mathcal{H}^{N-1}\left(\partial^*(\frac{1}{\varepsilon}E_\varepsilon) \cap (B_2(\frac{1}{\varepsilon}x_\varepsilon) \setminus \overline{\omega})\right)},$$

where $\frac{1}{\varepsilon}E_\varepsilon$ is the homothety of E_ε of center the origin and scale $\frac{1}{\varepsilon}$. Since ω is Lipschitz, the ratio is uniformly bounded from above, independently on E_ε.

We can thus apply Proposition 5.1 and conclude that the constants $C_{trace}(\Omega_\varepsilon)$ are uniformly bounded: from Theorem 4.1 we thus infer that (6.1) holds true.

Example 2. (Vanishing multiple random convex holes) In what follows C_N will denote a dimensional constant which may change from line to line. Let Ω be a bounded open set with Lipschitz boundary. For every $n \in \mathbb{N}$, we consider $K_1^n, \dots, K_n^n$ pairwise disjoint, nonempty closed convex subsets of Ω and denote

$$\Omega_n := \Omega \setminus (K_1^n \cup \dots K_n^n).$$

Let r_n be the maximal diameter of $(K_i^n)_i$ and d_n the minimal distance between any couple from $\{K_1^n, \dots, K_n^n, \partial\Omega\}$.

If

$$n^{\frac{1}{N-1}} r_n = o(d_n), \tag{6.2}$$

we prove that for every $k \in \mathbb{N}$

$$\sigma_k(\Omega_n) \to \sigma_k(\Omega). \tag{6.3}$$

We start with some estimates on r_n and d_n. First, as the measure of Ω is fixed, and any convex set has a tubular neighbourhood of size $\frac{d_n}{2}$ which does not intersect any other convex, we get that there exists a dimensional constant C_N such that

$$d_n^N n \le C_N |\Omega|,$$

so that in particular $d_n \to 0$. Since K_i^n is convex, this implies that

$$\mathcal{H}^{N-1}(\cup_{i=1}^n \partial K_i^n) \le C_N n r_n^{N-1} = o(d_n^{N-1}) \to 0, \tag{6.4}$$

where C_N is another dimensional constant.

From the previous estimates, we see that the convergence of the spectrum (6.3) is a consequence of Theorem 4.1 provided that we get a uniform estimate for the trace constant of the domain Ω_n.

In order to estimate the trace constant, we follow the Anzellotti-Giaquinta characterization described in Section 5. Let us choose a point $x_n \in \partial\Omega_n$ and estimate the ratio

$$\frac{\mathcal{H}^{N-1}(\partial^* E \cap \partial\Omega_n)}{\mathcal{H}^{N-1}(\partial^* E \cap \Omega_n)} \tag{6.5}$$

for a set E lying in $B_{\bar{r}}(x_n) \cap \Omega_n$ with $\bar{r}$ sufficiently small. The key idea is to prove that we can decouple the estimate around each set K_i^n and the boundary $\partial\Omega$.

Since Ω is Lipschitz, we know that there exists a constant $M > 0$ such that

$$\mathcal{H}^{N-1}(\partial^* E \cap \partial\Omega) \le M \mathcal{H}^{N-1}(\partial^* E \cap \Omega). \tag{6.6}$$

Indeed we can choose $\bar{r}$ in such a way that $|E| \le 1$ and $|E| \le |\Omega \setminus E|$. Then (6.6) follows from the trace theorem in BV (applied to 1_E) and the relative isoperimetric inequality on Ω (see (2.2)).

Assume that

$$\mathcal{H}^{N-1}(\partial^* E \cap (\cup_i \partial K_i^n)) \le \frac{1}{2}\mathcal{H}^{N-1}(\partial^* E \cap \Omega). \tag{6.7}$$

We may write

$$\begin{aligned}\mathcal{H}^{N-1}(\partial^* E \cap \partial\Omega_n) &= \mathcal{H}^{N-1}(\partial^* E \cap \partial\Omega) + \mathcal{H}^{N-1}(\partial^* E \cap (\cup_i \partial K_i^n)) \\ &\le \mathcal{H}^{N-1}(\partial^* E \cap \partial\Omega) + \frac{1}{2}\mathcal{H}^{N-1}(\partial^* E \cap \Omega),\end{aligned}$$

while

$$\begin{aligned}\mathcal{H}^{N-1}(\partial^* E \cap \Omega) &= \mathcal{H}^{N-1}(\partial^* E \cap \Omega_n) + \mathcal{H}^{N-1}(\partial^* E \cap (\cup_i \partial K_i^n)) \\ &\le \mathcal{H}^{N-1}(\partial^* E \cap \Omega_n) + \frac{1}{2}\mathcal{H}^{N-1}(\partial^* E \cap \Omega),\end{aligned}$$

so that $\mathcal{H}^{N-1}(\partial^* E \cap \Omega) \le 2\mathcal{H}^{N-1}(\partial^* E \cap \Omega_n)$. Taking into account (6.6) we conclude that

$$\frac{\mathcal{H}^{N-1}(\partial^* E \cap \partial\Omega_n)}{\mathcal{H}^{N-1}(\partial^* E \cap \Omega_n)} \le 2\frac{\mathcal{H}^{N-1}(\partial^* E \cap \partial\Omega) + \frac{1}{2}\mathcal{H}^{N-1}(\partial^* E \cap \Omega)}{\mathcal{H}^{N-1}(\partial^* E \cap \Omega)} \le 2M + 1.$$

Let us assume that (6.7) does not hold: using again (6.6) we may write

$$\mathcal{H}^{N-1}(\partial^* E \cap (\cup_i \partial K_i^n)) > \frac{1}{2}\mathcal{H}^{N-1}(\partial^* E \cap \Omega) \ge \frac{1}{2M}\mathcal{H}^{N-1}(\partial^* E \cap \partial\Omega). \tag{6.8}$$

Let us denote $d_{\partial\Omega_n} : \Omega_n \to [0, diam(\Omega)]$ the distance function to $\partial\Omega_n$ and let us set

$$E_t := E \cap \{x \in \Omega_n : d_{\partial\Omega_n}(x) > t\} \qquad \text{and} \qquad F_t := E \setminus \{x \in \Omega_n : d_{\partial\Omega_n}(x) > t\}.$$

Assume that for some $t \in (0, \frac{d_n}{3})$ (recall that d_n is the minimal distance), we have that

$$\mathcal{H}^{N-1}(\partial^* F_t \cap \Omega_n) \le \mathcal{H}^{N-1}(\partial^* E \cap \Omega_n). \tag{6.9}$$

Since

$$\frac{\mathcal{H}^{N-1}(\partial^* E \cap \partial\Omega_n)}{\mathcal{H}^{N-1}(\partial^* E \cap \Omega_n)} \le \frac{\mathcal{H}^{N-1}(\partial^* F_t \cap \partial\Omega_n)}{\mathcal{H}^{N-1}(\partial^* F_t \cap \Omega_n)}$$

a uniform estimate for (6.5) follows since the set $F_t \cap \Omega_n$ can be decomposed around each K_i^n and $\partial\Omega$. Around $\partial\Omega$ the inequality holds with constant M and around each K_i^n the inequality holds with constant 1, the norm of the projection on the convex sets.

Assuming that for every $t \in (0, \frac{d_n}{3})$, (6.9) fails, we get that for almost every $t \in (0, \frac{d_n}{3})$

$$\mathcal{H}^{N-1}(\partial^* E_t \cap \partial^*\{d_{\partial\Omega_n} > t\}) > \mathcal{H}^{N-1}(\partial^* E_t \setminus \partial^*\{d_{\partial\Omega_n} > t\}).$$

Setting

$$\sigma(t) := \mathcal{H}^{N-1}(\partial^* E_t \cap \partial^*\{d_{\partial\Omega_n} > t\})) \qquad \text{and} \qquad v(t) := |E_t|,$$

we get from the isoperimetric inequality

$$2\sigma(t) \ge C_N v(t)^{\frac{N-1}{N}}.$$

Since $v'(t) = -\sigma(t)$ (from the coarea formula), integrating over $(0, \frac{d}{3})$ we get that

$$2N|E|^{\frac{1}{N}} \ge C_N \frac{d_n}{3}.$$

On the other hand, we may write using the isoperimetric inequality, (6.6), (6.8) and (6.4)

$$\begin{aligned}|E|^{\frac{N-1}{N}} &\le C_N \mathcal{H}^{N-1}(\partial^* E) \\ &= C_N \left(\mathcal{H}^{N-1}(\partial^* E \cap \partial\Omega) + \mathcal{H}^{N-1}(\partial^* E \cap \Omega)\right) \le C_N(M+1)\mathcal{H}^{N-1}(\partial^* E \cap \Omega) \\ &\le 2C_N(M+1)\mathcal{H}^{N-1}(\partial^* E \cap \cup \partial K_i^n) \le 2C_N(M+1)\mathcal{H}^{N-1}(\cup \partial K_i^n) \le 2C_N(M+1) n r_n^{N-1}.\end{aligned}$$

We conclude that

$$n r_n^{N-1} \ge C d_n^{N-1},$$

in contradiction with our assumption (6.2).

By Proposition 5.1 we infer that the trace constants $C_{trace}(\Omega_n)$ are uniformly bounded, so that the convergence of the spectrum (6.3) follows.

Example 3. (Merging domains) The question is to see what happens to the Steklov spectrum in case of two disconnecting smooth domains. We refer to the paper of Girouard and Polterovich [10] for the case of two discs with vanishing intersection and explain it in our framework.

Let $x_\varepsilon = (1-\varepsilon, 0, \dots, 0), y_\varepsilon = (-1+\varepsilon, 0, \dots, 0) \in \mathbb{R}^N$ and assume

$$\Omega_\varepsilon := B_1(x_\varepsilon) \cup B_1(y_\varepsilon) \qquad \text{and} \qquad \Omega := B_1(x_0) \cup B_1(y_0),$$

where $x_0 := (1, 0, \dots, 0)$ and $y_0 := (-1, 0, \dots, 0)$. Although Ω has a cuspidal point, the spectrum of the Steklov problem is still consisting of eigenvalues, as the trace operator $T : W^{1,2}(\Omega) \mapsto L^2(\partial\Omega)$ is compact. We claim that the Steklov eigenvalues of Ω_ε converge to those of Ω . Indeed, following Remark 4.3 to cope with the non regularity of Ω, one has only to evaluate $C_{trace}(\Omega_\varepsilon)$ and to prove that it is uniformly bounded. For this, take a function $u \in BV(\Omega_\varepsilon)$ and write the trace inequality in $B_1(x_\varepsilon), B_1(y_\varepsilon)$ for $u|_{B_1(x_\varepsilon)}, u|_{B_1(y_\varepsilon)}$, respectively. Adding the two inequalities, we get that $C_{trace}(\Omega_\varepsilon)$ is not larger than the double of $C_{trace}(B_1)$.

Example 4. (Inward cusps) This example is devoted to domains with inward cusps (inner peaks).

Let Ω be a smooth domain of $\mathbb{R}^2$, outside an inward cusp. Assume that the cusp lies at the origin, and that in a neighbourhood U the domain is locally the subgraph of a continuous function $f : (-\delta, \delta) \to \mathbb{R}^+$, such that

$$f(0) = 0, \quad f \in C^1((-\delta, 0) \cup (0, \delta)), \quad \lim_{t \to 0^-} f'(t) = -\infty, \quad \text{and} \quad \lim_{t \to 0^+} f'(t) = +\infty.$$

Assume that $f_\varepsilon : [-\varepsilon, \varepsilon] \to \mathbb{R}^+$ is a C^1 perturbation of f on $[-\varepsilon, \varepsilon]$ such that $f_\varepsilon(\pm\varepsilon) = f(\pm\varepsilon)$, $f'_\varepsilon(\pm\varepsilon) = f'(\pm\varepsilon)$ and f_ε is convex on $(-\varepsilon, \varepsilon)$. We notice first that $W^{1,2}(\Omega)$ has a compact trace in $L^2(\partial\Omega)$ so the Steklov spectrum consists of eigenvalues. The question is whether the Steklov spectrum is stable.

In view of Theorem 4.1 and of Remark 4.3, one has to evaluate the constants $C_{trace}(\Omega_\varepsilon)$, where Ω_ε is the subgraph of f_ε. We follow, as in the previous examples, the Anzellotti-Giaquinta characterization described in Section 5. It is sufficient to localize only at the origin. Taking a set $E \subseteq \Omega_\varepsilon \cap B_r(0)$ with $B_r(0) \subseteq U$, and decomposing E in

$$E = [E \cap (\Omega_\varepsilon \setminus \Omega)] \cup [E \cap \Omega \cap \{x_1 \geq 0\}] \cup [E \cap \Omega \cap \{x_1 \leq 0\}],$$

one gets uniform bounds for the ratio

$$\frac{\mathcal{H}^{N-1}(\partial^* E \cap \partial\Omega_\varepsilon)}{\mathcal{H}^{N-1}(\partial^* E \cap \Omega_\varepsilon)}$$

from the analysis of each piece, since $\Omega \cap \{x_1 > 0\}$ and $\Omega \cap \{x_1 < 0\}$ are Lipschitz regular, while $\Omega_\varepsilon \setminus \Omega$ has a convex exposure to the projection operator onto the boundary of Ω_ε.

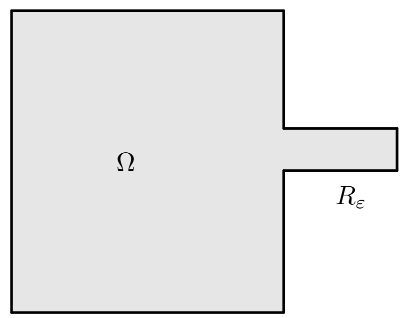
Example 5. (The L^∞-norm of Steklov eigenfunctions are not controlled by the eigenvalues alone) We report below an example of a sequence of Lipschitz domains converging to a cube, satisfying the first two requirements of Theorem 4.1, while the associated σ_k is constant but not converge to the k-th eigenvalue of the limiting domain. Thanks to Remark 4.7, this shows that the L^∞-norms of the (normalized) Steklov eigenfunctions blow up, even if volume and perimeter are converging and σ_k is constant. This behavior is clearly different from what happens in the case of Dirichlet and Robin boundary conditions.

We consider for simplicity the two dimensional case: our example is an adaptation to the case of Steklov conditions of a classical example of Courant and Hilbert [6, Page 420] (see also the examples of Girouard and Polterovich [10]). Let us consider for $\varepsilon > 0$

$$\Omega :=]-1, 1[^2 \qquad \text{and} \qquad R_\varepsilon := [1, 1+\varepsilon[\times]0, \varepsilon^3[,$$

and let us set

$$\Omega_\varepsilon := \Omega \cup R_\varepsilon.$$



As $\varepsilon \to 0$ we have

$$1_{\Omega_\varepsilon} \to 1_\Omega \qquad \text{strongly in } L^1(\mathbb{R}^2) \qquad \text{and} \qquad \mathcal{H}^1(\partial\Omega_\varepsilon) \to \mathcal{H}^1(\Omega).$$

It is easy to see that for every $k \geq 1$ we have $\sigma_k(\Omega_\varepsilon) \to 0 \neq \sigma_k(\Omega)$. Let us indeed consider the piecewise linear function $\phi : \mathbb{R} \to \mathbb{R}$ given by

$$\phi(s) := \begin{cases} s & \text{if } 0 \leq s \leq \frac{1}{4} \\ \frac{1}{2} - s & \text{if } \frac{1}{4} \leq s \leq \frac{3}{4} \\ s - 1 & \text{if } \frac{3}{4} \leq s \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

with $\int_0^1 \phi(s)\, ds = 0$. If we consider the admissible test function for the computation of $\sigma_1(\Omega_\varepsilon)$ given by

$$\varphi_\varepsilon(x) := \phi\left(\frac{x_1 - 1}{\varepsilon}\right),$$

we get (the support is contained in R_ε)

$$\sigma_1(\Omega_\varepsilon) \leq \frac{\varepsilon^3 \int_0^\varepsilon \frac{1}{\varepsilon^2} \phi'(s/\varepsilon)^2\, ds}{2\int_0^\varepsilon \phi(s/\varepsilon)^2\, ds} = \frac{\varepsilon}{2} \frac{\int_0^1 (\phi')^2\, ds}{\int_0^1 \phi^2\, ds} \to 0 \neq \sigma_1(\Omega).$$

For $k > 1$, it is sufficient to consider the admissible k-uple of functions $\{\varphi_\varepsilon^{k,1}, \dots, \varphi_\varepsilon^{k,k}\}$ with disjoint supports contained in R_ε given by

$$\varphi_\varepsilon^{k,i}(x) := \phi\left(\frac{-i\varepsilon + k(x_1 - 1)}{\varepsilon}\right) \qquad i = 1, \dots, k,$$

and the same computation leads again to

$$\sigma_k(\Omega_\varepsilon) \leq \frac{\varepsilon}{2} \frac{\int_0^1 (\phi')^2\, ds}{\int_0^1 \phi^2\, ds} \to 0 \neq \sigma_k(\Omega). \tag{6.10}$$

Since the conclusion of Theorem 4.1 is violated, this means that the trace constants $C_{trace}(\Omega_\varepsilon)$ are not uniformly bounded. We can directly check this fact by considering the function φ_ε above for which

$$\|\varphi_\varepsilon\|_{L^1(\partial\Omega_\varepsilon)} = \frac{\varepsilon}{8} \qquad \text{and} \qquad \|\varphi_\varepsilon\|_{BV(\Omega_\varepsilon)} = \|\varphi_\varepsilon\|_{W^{1,1}(\Omega_\varepsilon)} = \varepsilon^3 + \frac{\varepsilon^4}{8}.$$

Notice that if we replace the rectangle R_ε with the rectangle

$$R_\varepsilon^t := [1, 1 + t\varepsilon[\times]0, \varepsilon^3[, \qquad t \in [0,1]$$

and set $\Omega_\varepsilon^t := Q \cup R_\varepsilon^t$, we have that

$$t \in [0,1] \mapsto \Omega_\varepsilon^t$$

is a deformation from the square Q (obtained for $t = 0$) and the domain Ω_ε (obtained for $t = 1$). Along this deformation, in which ε is fixed, we have that the trace constants $C_{trace}(\Omega_\varepsilon^t)$ are uniformly bounded since we have

$$C_{trace}(\Omega_\varepsilon^t) \le C_{trace}(\hat{R}_\varepsilon^t) + C_{trace}(A_1) + C_{trace}(A_2)$$

where

$$\hat{R}_\varepsilon^t :=]-1, 1 + \varepsilon t[\times]0, \varepsilon^3[, \qquad A_1 :=]-1, 1[\times]\varepsilon^3, 1[, \qquad A_2 :=]-1, 1[\times]-1, 0[.$$

Notice that $C_{trace}(\hat{R}_\varepsilon^t)$ can be estimated uniformly in t since by means of a well controlled dilation we can transform the rectangle $\hat{R}_\varepsilon^t$ into the rectangle $]-1, 1[\times]0, \varepsilon^3[$.

In view of Theorem 4.1 and of (6.10), we conclude that for every $k \ge 1$ there exists $\varepsilon_0 > 0$ such that for every $\varepsilon < \varepsilon_0$ we can select $t = t_\varepsilon$ such that

$$\sigma_k(\Omega_\varepsilon^{t_\varepsilon}) = \frac{1}{2}\sigma_k(\Omega).$$

Again as $\varepsilon \to 0$ we have

$$1_{\Omega_\varepsilon^{t_\varepsilon}} \to 1_\Omega \qquad \text{strongly in } L^1(\mathbb{R}^2) \qquad \text{and} \qquad \mathcal{H}^1(\partial\Omega_\varepsilon^{t_\varepsilon}) \to \mathcal{H}^1(\Omega),$$

with

$$\sigma_k(\Omega_\varepsilon^{t_\varepsilon}) \text{ constant but } \sigma_k(\Omega_\varepsilon^{t_\varepsilon}) \ne \sigma_k(\Omega).$$

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(Dorin Bucur) Univ. Grenoble Alpes, Univ. Savoie Mont Blanc, CNRS, LAMA 73000 Chambéry, France
E-mail address, D. Bucur: dorin.bucur@univ-savoie.fr

(Alessandro Giacomini) DICATAM, Sezione di Matematica, Università degli Studi di Brescia, Via Branze 43, 25123 Brescia, Italy
E-mail address, A. Giacomini: alessandro.giacomini@unibs.it

(Paola Trebeschi) DICATAM, Sezione di Matematica, Università degli Studi di Brescia, Via Branze 43, 25123 Brescia, Italy
E-mail address, P. Trebeschi: paola.trebeschi@unibs.it