Minimization of the k-th eigenvalue of the Dirichlet Laplacian

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Abstract

For every $k \in \mathbb{N}$ we prove the existence of a quasi-open set minimizing the k-th eigenvalue of the Dirichlet Laplacian among all sets of prescribed Lebesgue measure. Moreover, we prove that every minimizer is bounded and has finite perimeter. The key point is the observation that such quasi-open sets are shape subsolutions for an energy minimizing free boundary problem.

1 Introduction

The question of minimizing the first Dirichlet eigenvalue of the Laplace operator among all open sets of prescribed measure is an old one. It was conjectured by Rayleigh in 1877 that the solution is the ball and was proved in the twenties by Faber and Krahn. It was also noticed that the minimizer for the second eigenvalue is the union of two and equal disjoint balls. For $k \geq 3$, not only the optimal shape is unknown (some conjectures based on numerical evidences are formulated in [12]) but also the existence of an open and smooth optimal shape is an unsolved question. For $k = 3$, the existence of a quasi-open set was proved in [6], while for $k \geq 4$ it was only proved that a minimizer exists provided that for every lower index a bounded minimizer exists.

The key general existence result is due to Buttazzo and Dal Maso [7] where, in particular, it is proved that for every bounded open set $D \subseteq \mathbb{R}^N$ the problem

$$\min\{\lambda_k(A) | A \subseteq D, |A| = c, A \text{ quasi-open}\}$$

has a solution. Here, $\lambda_k(A)$ stands for the k-th eigenvalue of the Dirichlet Laplacian on the set A, multiplicities being counted. To replace D with $\mathbb{R}^N$ is not an trivial matter (see the Note at the end of the paper concerning the recent result of Mazzoleni and Pratelli). The main challenge of [3] was to give a concentration-compactness result for the resolvent operators, which could explain the behaviour of a minimizing sequence of domains in $\mathbb{R}^N$. The missing step in order to prove the existence of a solution for λ_{k+1} was related to the regularity of the optimal shapes for $\lambda_1, .., \lambda_k$.

The main purpose of this paper is to introduce the analysis of the *shape subsolutions* for an energy minimizing free boundary problem as a tool to handle shape optimization problems associated to min-max functionals. As a direct consequence, one can not only prove the existence of a solution for the problem

$$\min\{\lambda_k(A) | A \subseteq \mathbb{R}^N, |A| = c\} \tag{1}$$

in the family of quasi-open sets, but also to extract some qualitative information, e.g. to prove that *every* solution is a bounded set with finite perimeter.

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In the literature, the regularity of the free boundaries is well understood only for energy like minimizers (see [1], [8], [11], [2]). If for $k = 1$ or $k = 2$, problem (1) can be seen as a "classical" free boundary problem and thus regularity can be extracted using the ideas introduced by Alt and Caffarelli [1], for $k \geq 3$, the k-th eigenvalue is a critical point and the regularity of the free boundary problems associated with this kind of functionals requires a new approach.

The key tool we introduce in order to deal with this kind of problems relays on the analysis of the *shape subsolutions* for the energy. We observe that every solution of (1) is a shape subsolution for an energy minimization problem, i.e. there exists $\Lambda > 0$ (small enough) such that

$$\forall \tilde{A} \subseteq A \quad \min_{u \in H_0^1(A)} \frac{1}{2} \int |\nabla u|^2 dx - \int u dx + \Lambda |A| \leq \min_{u \in H_0^1(\tilde{A})} \frac{1}{2} \int |\nabla u|^2 dx - \int u dx + \Lambda |\tilde{A}|. \tag{2}$$

The minimization of the (torsion) energy

$$E(A) = \min_{u \in H_0^1(A)} \frac{1}{2} \int |\nabla u|^2 dx - \int u dx \tag{3}$$

among all (quasi-open) sets A of prescribed measure, possibly satisfying some inclusion constraints, is a classical free boundary problem (see [2]) for which existence of a solution and primary regularity results are quite well understood.

The main challenge in our case, is that the minimizers of (1) are only subsolutions in the sense of (2), so that full information on the free boundary can not, a priori, be obtained. Roughly speaking, the main point which allows us to pass from (1) to (2) is that the variation of eigenvalues for inner local pertubations of the geometric domain can be controlled by the variation of the torsion energy. This is not anymore the case for outer geometric perturbations. Shortly, if we replace the strong information about the minimality in (1) by the weak information given by (2), we are able to handle the saddle point character of the k-th eigenvalue in the free boundary problem, but we lose all the information related to the outer perturbations. This is a main stop in order to extract more regularity for the solutions of (1), as for example openness of the optimal shapes.

2 Analysis of shape subsolutions

Let A be a measurable set of finite measure. We denote

$$\tilde{H}_0^1(A) := \{u \in H^1(\mathbb{R}^N) : u = 0 \text{ a.e. on } \mathbb{R}^N \setminus A\}.$$

It is well known that there exists a quasi-open set $\omega_A \subseteq A$ such that

$$H_0^1(\omega_A) = \tilde{H}_0^1(A).$$

With this observation, and based on the monotonicity properties of the eigenvalues with respect to inclusions, solving problem (1), in the family of quasi-open sets with the classical definition of the Sobolev space, or in the family of measurable sets associated to $\tilde{H}_0^1$, is equivalent. This remark remains true for a larger class of shape optimization problems associated to decreasing functionals with respect to inclusions. We refer the reader to [7] or [5, Chapter 4] for a collection of results involving the Dirichlet spectrum of the Laplace operator on quasi-open sets and γ-convergence.

As a consequence of the identity above, one can endow the family of measurable sets with a distance issued from the γ-convergence. Precisely, for two measurable sets A and B of finite measure, we introduce

$$d_\gamma(A, B) = \int |u_A - u_B| dx,$$

where $u_A \in H^1_0(A)$ denotes the minimizer in (3) extended by zero to an element of $H^1(\mathbb{R}^N)$.

Definition 2.1 *We say that a measurable set A of finite measure is a local shape subsolution for the energy problem if there exists $\delta > 0$ and $\Lambda > 0$ such that, for every measurable set $\tilde{A}$, $|\tilde{A} \setminus A| = 0$, $d_\gamma(A, \tilde{A}) \le \delta$ we have*

$$E(A) + \Lambda|A| \le E(\tilde{A}) + \Lambda|\tilde{A}|. \tag{4}$$

The locality in the definition above is expressed through the γ-distance.

Here is the main result of this section.

Theorem 2.2 *Assume A is a local shape subsolution for the energy. Then A is bounded, has finite perimeter and its fine interior has the same measure as A (i.e. $|A \setminus \omega_A| = 0$).*

Proof **Fine interior.** Using the subsolution property with $\tilde{A} = \omega_A$ and the fact that $E(A) = E(\omega_A)$ we get $|A| \le |\omega_A|$, so that $|A \setminus \omega_A| = 0$.

Finite perimeter. Let us simply denote $u := u_A$ the unique minimizer of the torsion energy $E(A)$ in (3). In order to prove that A has finite perimeter we consider for every $\varepsilon > 0$, the test function $u_\varepsilon = (u - \varepsilon)^+$ and write from (4)

$$\frac{1}{2}\int |\nabla u|^2 dx - \int u dx + \Lambda|\{u > 0\}| \le \frac{1}{2}\int |\nabla u_\varepsilon|^2 dx - \int u_\varepsilon dx + \Lambda|\{u_\varepsilon > 0\}|.$$

This perturbation is valid, since $\lim_{\varepsilon \to 0} d_\gamma(A, \{u_\varepsilon > 0\}) = 0$.

Consequently

$$\frac{1}{2}\int_{0 \le u \le \varepsilon} |\nabla u|^2 dx + \Lambda|\{0 \le u \le \varepsilon\}| \le \int (u - u_\varepsilon) dx = \int_{0 \le u \le \varepsilon} u + \varepsilon|\{u > \varepsilon\}| \le \varepsilon|A|.$$

Using the Cauchy-Schwarz inequality we have

$$(\int_{0 \le u \le \varepsilon} |\nabla u| dx)^2 \le \int_{0 \le u \le \varepsilon} |\nabla u|^2 dx |\{0 \le u \le \varepsilon\}| \le \frac{2}{\Lambda}\varepsilon^2|A|^2,$$

so that

$$\int_{0 \le u \le \varepsilon} |\nabla u| dx \le \sqrt{2/\Lambda}\varepsilon|A|.$$

Using the co-area formula, we find $\delta_n > 0, \delta_n \to 0$ such that

$$\mathcal{H}^{N-1}(\partial^*\{u > \delta_n\}) \le \sqrt{2/\Lambda}|A|.$$

Passing to the limit, we get

$$\mathcal{H}^{N-1}(\partial^*\{u > 0\}) \le \sqrt{2/\Lambda}|A|.$$

Since $\omega_A = \{u > 0\}$ and since $\omega_A = A$ a.e., we get that the perimeter of A is less than $\sqrt{2/\Lambda}|A|$.

Boundedness. The proof of the boundedness involves an estimate of Alt-Caffarelli type (see [1]) adapted to the energy minimization problem. First we notice that

$$\lim_{r \to 0} \sup_{x \in \mathbb{R}^N} d_\gamma(A, A \setminus \overline{B}_r(x)) \le \limsup_{r \to 0} (|u_A|^2_\infty + |u_A|_\infty) \operatorname{cap}(B_r) = 0,$$

so that for r smaller than some $r_0 = r_0(|A|, \delta)$ and for every $x \in \mathbb{R}^N$ we have $d_\gamma(A, A \setminus \overline{B}_r(x)) < \delta$.

We give the following.

Lemma 2.3 *Let $\tau \in (0,1)$. There exists $r_0 > 0, C_0 > 0$ such that for every $x_0 \in \mathbb{R}^N$ and $r \in (0, r_0)$*

$$\sup_{x \in B_{\sqrt{\tau} r}(x_0)} u(x) \leq C_0 r \Longrightarrow u = 0 \text{ on } B_{\tau r}(x_0). \tag{5}$$

Proof [of the Lemma] The proof follows a perturbation argument similar to [1, Lemma 3.4] adapted to the energy E. In the following computations we assume $x_0 = 0$ in order to simplify notations. For $\varepsilon > 0$, we introduce the following function:

$$\begin{cases} -\Delta v_r = 1 & \text{in } B_{\sqrt{\tau} r} \setminus \overline{B}_{\tau r} \\ v_r = 0 & \text{on } \partial B_{\tau r} \\ v_r = \varepsilon & \text{on } \partial B_{\sqrt{\tau} r}, \end{cases}$$

Rescaling v_r by $\psi_r(x) = v_r(rx)$, we get

$$\begin{cases} -\Delta \psi_r = r^2 & \text{in } B_{\sqrt{\tau}} \setminus \overline{B}_{\tau} \\ \psi_r = 0 & \text{on } \partial B_{\tau} \\ \psi_r = \varepsilon & \text{on } \partial B_{\sqrt{\tau}}, \end{cases}$$

Let $\varepsilon = \sup_{x \in B_{\sqrt{\tau} r}} u(x)$ (which will be controlled by $C_0 r$). Since $u \in L^\infty(\mathbb{R}^N)$, with norm controlled by $|A|$ (as a consequence of the isoperimetric inequalitiy for the L^∞ norm of the torsion function), we have $\varepsilon < +\infty$.

By the construction of v_r we have that $u \wedge v_r \in H_0^1(A \setminus \overline{B}_{\tau r})$, so we can use it as a test function. We have

$$\frac{1}{2}\int_{B_{\sqrt{\tau} r}} |\nabla u|^2 dx - \int_{B_{\sqrt{\tau} r}} u dx + \Lambda |B_{\tau r} \cap A| \leq \frac{1}{2}\int_{B_{\sqrt{\tau} r} \setminus \overline{B}_{\tau r}} |\nabla (u \wedge v_r)|^2 dx - \int_{B_{\sqrt{\tau} r} \setminus \overline{B}_{\tau r}} u \wedge v_r dx.$$

Using the Cauchy-Schwarz inequality and the equation satsified by v_r

$$\frac{1}{2}\int_{B_{\tau r}} |\nabla u|^2 dx - \int_{B_{\tau r}} u dx + \Lambda |B_{\tau r} \cap A| \leq \frac{1}{2}\int_{(B_{\sqrt{\tau} r} \setminus \overline{B}_{\tau r}) \cap \{u > v\}} |\nabla v_r|^2 - |\nabla u|^2 dx + \int_{B_{\sqrt{\tau} r} \setminus \overline{B}_{\tau r}} (u - v)^+ dx$$

$$\leq \int_{(B_{\sqrt{\tau} r} \setminus \overline{B}_{\tau r}) \cap \{u > v\}} |\nabla v_r|^2 - \nabla u \nabla v_r dx + \int_{B_{\sqrt{\tau} r} \setminus \overline{B}_{\tau r}} (u - v)^+ dx$$

$$= -\int_{B_{\sqrt{\tau} r} \setminus \overline{B}_{\tau r}} \nabla (u - v_r)^+ \nabla v dx + \int_{B_{\sqrt{\tau} r} \setminus \overline{B}_{\tau r}} (u - v_r)^+ dx$$

$$= -\int_{\partial B_{\tau r}} \frac{\partial v_r}{\partial n} (u - v_r)^+ d\mathcal{H}^{N-1} = |v_r'(\tau r)| \int_{\partial B_{\tau r}} (u - v_r)^+ d\mathcal{H}^{N-1} = |v_r'(\tau r)| \int_{\partial B_{\tau r}} u d\mathcal{H}^{N-1}.$$

Finally

$$\frac{1}{2}\int_{B_{\tau r}} |\nabla u|^2 dx - \int_{B_{\tau r}} u dx + \Lambda |B_{\tau r} \cap A| \leq |v_r'(\tau r)| \int_{\partial B_{\tau r}} u d\mathcal{H}^{N-1}. \tag{6}$$

Let us assume $\varepsilon \leq \frac{\Lambda}{2}$, which is controlled by the choice of C_0 and r_0. Then (6) gives

$$\frac{1}{2}\int_{B_{\tau r}} |\nabla u|^2 dx + \frac{1}{2}\Lambda |B_{\tau r} \cap A| \leq |v_r'(\tau r)| \int_{\partial B_{\tau r}} u d\mathcal{H}^{N-1}. \tag{7}$$

On the other hand we have by rescaling the boundary trace theorem in $W^{1,1}(B_1)$

$$\int_{\partial B_{\tau r}} u d\mathcal{H}^{N-1} \leq C(N)\Big[\frac{1}{\tau r}\int_{B_{\tau r}} u dx + \int_{B_{\tau r}} |\nabla u| dx\Big].$$

So using hypothesis (5) we get

$$\int_{\partial B_{\tau r}} u d\mathcal{H}^{N-1} \leq C(N)\Big[\frac{1}{\tau r} C_0 r |B_{\tau r}\cap A| + \int_{B_{\tau r}} |\nabla u| dx\Big]$$

$$= C(N)\Big[\frac{1}{\tau} C_0 |B_{\tau r}\cap A| + \int_{B_{\tau r}} |\nabla u| dx\Big]$$

$$\leq C(N)\Big[(\frac{1}{\tau} C_0 + \frac{1}{2})|B_{\tau r}\cap A| + \frac{1}{2}\int_{B_{\tau r}} |\nabla u|^2 dx\Big].$$

Combining

$$\int_{\partial B_{\tau r}} u d\mathcal{H}^{N-1} \leq C(N)\Big[(\frac{1}{\tau} C_0 + \frac{1}{2})|B_{\tau r}\cap A| + \frac{1}{2}\int_{B_{\tau r}} |\nabla u|^2 dx\Big]$$

with (7) we get

$$\min\{\frac{1}{2}, \Lambda\} \leq |v_r'(\tau r)| C(N)((\frac{1}{\tau} C_0 + \frac{1}{2}) + \Lambda),$$

as soon as $\int_{B_{\tau r}} |\nabla u|^2 dx + |B_{\tau r}\cap A| > 0$. This leads to a contradiction, as soon as $|v_r'(\tau r)|$ is small enough. But $|v_r'(\tau r)| = \frac{\psi_r'(\tau)}{r} \to 0$ when r and C_0 are vanishing, hence for some r_0, C_0 small enough and all $r \leq r_0$ we get a contradiction. Finally we should have $\int_{B_{\tau r}} |\nabla u|^2 dx + |B_{\tau r}\cap A| = 0$, which gets the conclusion of the Lemma. □

Proof of Theorem 2.2 (continuation). Assume for contradiction that the set A is unbounded. There exists a sequence of points $x_n \in A$, such that $|x_n| \to +\infty$ and the distance between any two of them is greater than $3r_0$. We can also assume that $u(x_n) \neq 0$ in the measure theoretic sense.

By the choice of x_n, using Lemma 2.3 we know that $\sup_{B_{\tau r_0}(x_n)} u \geq C_0 r_0 \wedge \frac{1}{2}$. So there exists $y_n \in B_{\tau r_0}(x_n)$ such that $u(y_n) \geq C_0 r_0 \wedge \frac{1}{2}$. Since the function

$$u(x) + \frac{|x-y_n|^2}{2N}$$

is subharmonic in $\mathbb{R}^N$, we get for every $r > 0$ that

$$C_0 r_0 \wedge \frac{1}{2} \leq u(y_n) \leq C(N)\frac{1}{r^N}\int_{B_r(y_n)} \Big[u(x) + \frac{|x-y_n|^2}{2N}\Big] dx.$$

For a small r there exists a constant C independent of n such that

$$\int_{B_r(y_n)} u(x) dx \geq C r^N,$$

which gives that $\int u(x) dx = +\infty$, which is a contradiction with $u \in L^1(A)$. □

Remark 2.4 The construction of the constants C_0 and r_0 in Lemma 2.3 leads to a control of the diameter of A in terms of $|A|$, Λ and δ.

3 Minimization of the k-th eigenvalue of the Dirichlet Laplacian

Let $A \subseteq \mathbb{R}^N$ be a quasi-open set of finite measure. Then the injection $H_0^1(A) \subseteq L^2(A)$ is compact and the spectrum of the Dirichlet Laplacian consists only on eigenvalues which can be denoted (counting the multiplicity)

$$0 < \lambda_1(A) \le \lambda_2(A) \le \dots \quad \le \lambda_k(A) \le \dots \;\to +\infty.$$

For every $k \in \mathbb{N}$, there exists $u_k \in H_0^1(A) \setminus \{0\}$ such that $-\Delta u_k = \lambda(A) u_k$, this equality being understood in the sense

$$\forall \varphi \in H_0^1(A), \quad \int_A \nabla u_k \nabla \varphi dx = \lambda_k(A) \int_A u_k \varphi dx.$$

We consider the following minimization problem

$$A \mapsto \lambda_k(A) + |A| \tag{8}$$

in the family of all quasi-open sets of $\mathbb{R}^N$. We notice that problems (1) and (8) are equivalent, and have the same solutions up to a homothety. In a first step, we assume that A is a solution to problem (8); the existence will be proved in a second step by an induction argument over k. We shall prove first only that, if it exists, A is also a shape subsolution of the energy problem, so that Theorem 2.2 applies.

Theorem 3.1 *Assume A is a solution of problem* (8)*. Then A is a local shape subsolution for the energy problem.*

Proof This is a direct consequence of [4, Theorem 3.4] (see also Lemma 4.1 in the Appendix below) which asserts that there exists a constant $c_k(A)$ depending on A and k such that

$$|\frac{1}{\lambda_k(\tilde{A})} - \frac{1}{\lambda_k(A)}| \le c_k(A) d_\gamma(\tilde{A}, A). \tag{9}$$

Since $E(\tilde{A}) - E(A) = \frac{1}{2} d_\gamma(\tilde{A}, A)$, for δ small enough, such that $\delta \le \frac{1}{4c_k(A)} \frac{1}{\lambda_k(A)}$ and for every $\tilde{A}$ such that $\tilde{A} \subseteq A$ and $d_\gamma(\tilde{A}, A) \le \delta$ we get that

$$\lambda_k(\tilde{A}) - \lambda_k(A) \le c'_k(A)(E(\tilde{A}) - E(A)) \tag{10}$$

where $c'_k(A)$ depends on δ, k and A. Combining the optimality of A expressed in (8) by $|A| - |\tilde{A}| \le \lambda_k(\tilde{A}) - \lambda_k(A)$ together with (10) we obtain that A is a local shape subsolution for the energy problem, with constant $\Lambda = \frac{1}{c'_k(A)}$. □

We recall the following result from [6, Theorem 3.5] which is based on a concentration - compactness argument for the resolvent operators (see [3, Theorem 2.2]).

Lemma 3.2 *Assume that a bounded minimizer exists for problem* (1) *for $k = 1, .., N$. Then, at least one minimizer exists for $k = N + 1$.*

We can now formulate the existence result and give a piece of qualitative information on the minimizers.

Theorem 3.3 *For every $k \in \mathbb{N}$, problem* (1) *has at least one solution. Moreover, every solution is bounded and has finite perimeter.*

Proof The fact that every solution, if it exists, is bounded and has finite perimeter is a consequence of Theorems 2.2 and 3.1.

For $k = 1, 2$ the (unique) solution is a ball and two equal balls, respectively. The existence of a solution for an arbitrary k, follows from the previous lemma provided that a bounded solution exists for $j = 1, 2, .., k-1$. Since this is true for $k = 1, 2$, an induction argument based on Theorems 2.2 and 3.1 concludes the proof.

□

4 Appendix

For the convenience of the reader, we recall the proof of inequality (9) given in [4, Theorem 3.4], rephrased here in the context of quasi-open sets.

Lemma 4.1 *Let $A \subseteq \mathbb{R}^N$ be a quasi-open set of finite measure. For every $k \in \mathbb{N}$, there exists a constant $c_k(A)$ depending only on A such that for every $j \le k$ and for every quasi-open set $B \subseteq A$ we have*

$$\Big|\frac{1}{\lambda_j(A)} - \frac{1}{\lambda_j(B)}\Big| \le c_k(A) d_\gamma(A, B). \tag{11}$$

Proof Let us fix $k \in \mathbb{N}$. We consider $V_k \subseteq L^2(\mathbb{R}^N)$ the linear space generated by the first k eigenfunctions of the Dirichlet Laplacian on A. The space V_k is a finite dimensional subspace of $H^1_0(A)$. We denote by $R_A : L^2(A) \to L^2(A)$ the resolvent operator of the Dirichlet-Laplacian $R_A = (-\Delta)^{-1}$, defined by $R_A(f) = u_{A,f}$, where $u_{A,f} \in H^1_0(A)$ satisfies $-\Delta u_{A,f} = f$, in the sense that

$$\forall \varphi \in H^1_0(A), \quad \int_A \nabla u_{A,f} \nabla \varphi dx = \int_A f \varphi dx.$$

We set $P_k : L^2(A) \to V_k$ is the L^2-orthogonal projector on V_k and introduce the finite rank, positive, self adjoint operators

$$T^A_k = P_k \circ R_A \circ P_k,$$
$$T^B_k = P_k \circ R_B \circ P_k.$$

Denoting $\mu_j(T^A_k)$, $\mu_j(T^B_k)$ the j-th eigenvalues of the operators T^A_k, T^B_k respectively (multiplicites being counted), we have

$$\forall j = 1, ..k, \;\; \mu_j(T^B_k) \le \frac{1}{\lambda_j(B)}, \;\; \mu_j(T^A_k) = \frac{1}{\lambda_j(A)} \tag{12}$$

Indeed, for every $j = 1, .., k$ we have (see for instance [10, Corollaries 3 and 4, pages 1089-1090])

$$\mu_j(T^B_k) = \mu_j(P_k \circ R_B \circ P_k) \le \mu_1(P_k)^2 \frac{1}{\lambda_j(B)}.$$

Since $\mu_1(P_k) = |P_k|_{\mathcal{L}(L^2(A))} = 1$ the first inequality in (12) comes immediately.

For proving the second inequality in (12) we notice in the same way that

$$\mu_j(T^A_k) \le \frac{1}{\lambda_j(A)}. \tag{13}$$

Moreover, for every $j = 1, .., k$ if u_j is the j-th eigenfunction of R_A associated with $\lambda_j(A)$ we have that

$$T_k^A u_j = P_k \circ R_A \circ P_k u_j = \mu_j(A) u_j, \tag{14}$$

since $P_k u_j = u_j$. Combining (13) and (14) we get the second inequality in (12).

Consequently, we have for every $j = 1, .., k$

$$0 \le \frac{1}{\lambda_j(A)} - \frac{1}{\lambda_j(B)} \le \mu_j(T_k^A) - \mu_j(T_k^B)$$

$$\le |T_k^A - T_k^B|_{\mathcal{L}(L^2(A))} = |P_k \circ R_A \circ P_k - P_k \circ R_B \circ P_k|_{\mathcal{L}(L^2(A))}.$$

But

$$|P_k \circ R_A \circ P_k - P_k \circ R_B \circ P_k|_{\mathcal{L}(L^2(A))} = \sup_{|u|_{L^2(A)} \le 1} \langle (P_k \circ R_A \circ P_k - P_k \circ R_B \circ P_k)u, u \rangle_{L^2(A) \times L^2(A)}$$

$$= \sup_{|u|_{L^2(A)} \le 1} \langle (R_A - R_B) P_k u, P_k u \rangle_{L^2(A) \times L^2(A)}.$$

Let us notice that $Range(P_k) \subseteq L^\infty(A)$ and moreover

$$P_k : L^2(A) \to L^\infty(A)$$

is bounded. Indeed, let $u \in L^2(A), |u|_{L^2(A)} \le 1$ and $P_k u = \alpha_1 u_1 + ... + \alpha_k u_k$. Here, the eigenfunctions $u_1, .., u_k$ of the Dirichlet Laplacian on A are supposed to be L^2-normalised. Since $|P_k u|_{L^2(A)} \le 1$ we get $\sum_{j=1}^k \alpha_j^2 \le 1$, hence $|\alpha_j| \le 1$ for every $j = 1, .., k$. From [9, Example 2.1.8] (which holds as well for quasi-open sets of finite measure as a consequence of the density of smooth open sets in the family of quasi-open sets, for the γ-distance) we have that

$$|u_j|_{L^\infty(A)} \le C \lambda_j(A)^{N/4},$$

where the constant C depends only on the dimension of the space. Finally, we observe that

$$|P_k u|_{L^\infty(A)} \le C \sum_{j=1}^k |\alpha_j| \lambda_j(A)^{N/4} := C_k(A).$$

We have

$$\langle (R_A - R_B) P_k u, P_k u \rangle_{L^2(A) \times L^2(A)} \le \int_A |R_A(P_k u) - R_B(P_k u)||P_k u| dx$$

$$\le C_k(A) \int_A |R_A(P_k u) - R_B(P_k u)| dx \le 2C_k(A) \int_A R_A(|P_k u|) - R_B(|P_k u|) dx$$

$$\le 2C_k(A)^2 \int_A R_A(1) - R_B(1) dx = 2C_k(A)^2 d_\gamma(A, B).$$

The last inequality is a consequence of the weak maximum principle. □

Note: The results of the paper were presented by the author during the ANR GAOS meeting in Chambéry, June 2011. The author was informed before the submission of this paper about the result in [13] which states that if Ω is a quasi-open set of finite measure and $k \in \mathbb{N}$, there exists a quasi-open set $\tilde{\Omega}$ such that $|\Omega| = |\tilde{\Omega}|$, $\lambda_j(\tilde{\Omega}) \le \lambda_j(\Omega)$ for every $j = 1, .., k$ and $diam(\tilde{\Omega}) \le C(k, |\Omega|, N)$. This argument, combined with the result of Buttazzo-Dal Maso leads to the existence of (bounded) solutions for a general class of monotonous functionals of eigenvalues, so that the existence part of the result in Theorem 3.3 could also be obtained using [13].

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