QUASISTATIC EVOLUTION OF CAPACITARY MEASURES

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Abstract. We discuss quasistatic evolution processes for capacitary measures and shapes in order to model debonding membranes. Minimizing movements as well as rate independent processes are investigated and some models are described together with a series of open problems.

Keywords: Debonding membranes, minimizing movements, capacitary measures

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1. Introduction

The minimizing movement theory was introduced by De Giorgi in [16] to study the quasistatic evolution for minimizers of time dependent variational problems. The framework was conceived to be very flexible, and can be applied not only to solutions which vary in a vector space of functions, but also to the cases when the variational functionals to be minimized have a domain which is a general topological space or even a space endowed with a convergence structure.

The situation we consider in the present paper is concerned with a model to study the quasistatic evolution of an adhesive membrane D subjected to a debonding force f depending on time. We refer the reader to the model introduced by Andrews and Shillor in [6] (see also [5]) where the main unknown is the fraction of active bonds on the contact surface; the value 0 stands to describe the fully debonded case, while the value 1 stands for the perfect adhesion (see also [21]). An ordinary differential equation governs the debonding field and a partial differential equation the position of the membrane.

Our model uses the general framework of minimizing movements and, although fundamentally different, has several common features with the one of Andrews and Shillor. In the framework we consider, the equilibrium configuration of the membrane is governed by the PDE

$$\begin{cases} -\Delta u + \mu u = f & \text{in } D \\ u \in H_0^1(D) \cap L^2(D, \mu) \end{cases}$$

where the state function u, representing the vertical displacement of the membrane, varies in the Sobolev space $H_0^1(D)$, and the control variable μ, measuring the adhesion of the membrane, is a nonnegative measure. Here $\mu = 0$ is the complete debonding, while $\mu = +\infty$ represents the perfect adhesion. More precisely, μ is assumed to vary in the class $\mathcal{M}_0$ of all nonnegative Borel measures on D, possibly taking the value $+\infty$, which are of capacitary type, that is they vanish on all sets of capacity zero. This class has been studied in detail in the literature (see for instance [10], [8] and references therein), in connection with several variational problems as shape optimization, thin obstacles, Dirichlet forms, etc.

For every $\mu \in \mathcal{M}_0$ we define the energy of μ related to f by

$$E(\mu, f) = \min\Big\{\frac{1}{2}\int_D |\nabla u|^2\,dx + \frac{1}{2}\int_D u^2\,d\mu - \int_D fu\,dx \;:\; u \in H_0^1(D) \cap L^2(D, \mu)\Big\}$$

and this quantity is one of the key ingredients we use to build a functional which governs the quasistatic evolution of the debonding membrane. The other ingredient is what we call a *dissipation distance*, which is a mapping $\mathcal{D} : \mathcal{M}_0 \times \mathcal{M}_0 \to [0, +\infty]$ satisfying suitable properties (see Definition 3.1). Given a debonding force $f(t, x) \geq 0$ we may now define the time discretization scheme of the quasistatic evolution as follows:

- fix a time step $\epsilon > 0$ and consider the discretized time $t_k^\epsilon = \epsilon k$ for $k \in \mathbb{N}$;
- start from an initial configuration $\mu_0 \in \mathcal{M}_0$, so that $\mu_\epsilon(0) = \mu_0$;
- define $\mu_\epsilon(t_k^\epsilon)$ iteratively, by taking $\mu_\epsilon(t_{k+1}^\epsilon)$ as the solution of the minimum problem

$$\min \Big\{ E\big(\mu, f(t_{k+1}^\epsilon)\big) + \mathcal{D}\big(\mu, \mu(t_k^\epsilon)\big) \Big\} \;;$$

- passing to the limit as $\epsilon \to 0$ provides then a mapping $t \mapsto \mu(t)$ that will be called solution of the generalized minimizing movement scheme.

Several cases of dissipation distances will be discussed in the paper; in all of them it seems natural to require that the debonding process is *irreversible*. In other words, if we define $\mu_1 \preceq \mu_2$ whenever $\mu_2(A) \leq \mu_1(A)$ for all quasi-open sets $A \subseteq D$, we require that

$$\mathcal{D}(\mu_1, \mu_2) = +\infty \quad \text{if } \mu_1 \not\preceq \mu_2.$$

In some cases we are simply able to prove the generalized minimizing movement scheme above; in some other the minimizing movement turns out to satisfy additional properties, introduced by Mielke in [20, 19] in the frame of *rate-independent processes*, which give stability of the movement and an energy inequality. This is for instance the case of

$$\mathcal{D}_\gamma(\mu_1, \mu_2) = \begin{cases} \displaystyle\int_D |w_{\mu_1} - w_{\mu_2}|\, dx & \text{if } \mu_1 \preceq \mu_2 \\ +\infty & \text{otherwise,} \end{cases}$$

where w_μ denotes the solution of

$$\begin{cases} -\Delta w_\mu + \mu w_\mu = 1 \\ w_\mu \in H_0^1(D) \cap L^2(D, \mu) \end{cases} \tag{1.1}$$

and $\preceq$ is a natural order on the class of capacitary measures defined in Section 2.

On the contrary, the case of shape flows where the capacitary measures take only the values 0 and $+\infty$ and can be naturally identified to quasi-open domains, seems to require some refinements and does not fit the general scheme of rate independent processes. In this paper we approach this question and we show that a shape flow verifying the Mielke's stability property exists and can be obtained through the minimizing movement method. In some particular cases, we can also show the flow above satisfies the energy inequality. An example of a radially symmetric debonding membrane, where analytic computations can be carried easily, is given in the last section.

The problem we deal in this paper, although devoted to the model of a debonding membrane, fixes also the frame of moving quasi-open sets or, in general, measures by an energetic approach. Shape gradient flows by minimizing movements were already used to model shape evolution as the mean curvature flow: Almgreen, Taylor and Wang [1] and Fonseca and Katsoulakis [17], or Cardaliaguet and Ley [12]. The crack propagation model of Francfort and Marigo (see also Dal Maso and Toader [15]) is another example of moving shapes by an energetic approach. Moving sets without the use of vector field deformations is not a new question in the context of optimal shape design. The level set approach by a Hamilton-Jacobi equation or the weak evolution of sets studied by Zolésio in [23] are examples of local movements allowing topology to change, but not of energetic type.

2. Preliminary tools

Let $D \subseteq \mathbb{R}^N$ be a bounded open set. The capacity of a subset E in D is

$$\operatorname{cap}(E, D) = \inf\Big\{\int_D |\nabla u|^2\, dx \ : \ u \in \mathcal{U}_E\Big\}\ ,$$

where $\mathcal{U}_E$ is the set of all functions u of the Sobolev space $H^1_0(D)$ such that $u \geq 1$ almost everywhere in a neighborhood of E.

If a property $P(x)$ holds for all $x \in E$ except for the elements of a set $Z \subseteq E$ with $\operatorname{cap}(Z) = 0$, we say that $P(x)$ holds *quasi-everywhere* on E (shortly *q.e.* on E). The expression *almost everywhere* (shortly *a.e.*) refers, as usual, to the Lebesgue measure.

A subset A of D is said to be *quasi-open* if for every $\epsilon > 0$ there exists an open subset A_ϵ of D, such that $A \subseteq A_\epsilon$ and $\operatorname{cap}(A_\epsilon \setminus A, D) < \epsilon$.

A function $f : D \to \mathbb{R}$ is said to be *quasi-continuous* (resp. *quasi-lower semicontinuous*) if for every $\epsilon > 0$ there exists a continuous (resp. lower semicontinuous) function f_ϵ : $D \to \mathbb{R}$ such that $\operatorname{cap}(\{f \neq f_\epsilon\}, D) < \epsilon$, where $\{f \neq f_\epsilon\} = \{x \in D : f(x) \neq f_\epsilon(x)\}$. It is well known (see, e.g., Ziemer [22]) that every function u of the Sobolev space $H^1(D)$ has a quasi-continuous representative, which is uniquely defined up to a set of capacity zero. We shall always identify the function u with its quasi-continuous representative, so that a pointwise condition can be imposed on $u(x)$ for q.e. $x \in D$.

We denote by $\mathcal{M}_0$ the class of all nonnegative Borel measures μ on D, possibly $+\infty$ valued, such that $\mu(B) = 0$ for every Borel set $B \subseteq D$ with $\operatorname{cap}(B, D) = 0$.

We stress the fact that the measures $\mu \in \mathcal{M}_0$ do not need to be finite, and may take the value $+\infty$ even on large parts of D. For instance, for every quasi-open set $\Omega \subseteq D$ the measure

$$\infty_{D\setminus\Omega}(A) = \begin{cases} 0 & \text{if } \operatorname{cap}(A \setminus \Omega) = 0 \\ +\infty & \text{otherwise.} \end{cases} \tag{2.1}$$

belongs to $\mathcal{M}_0$. We use the notation $\mathcal{M}_0^f$ to denote the class of all measures of $\mathcal{M}_0$ which are finite.

Throughout the paper use the following monotonicity relation on the class $\mathcal{M}_0$ (which is not equivalent to the monotonicity in the sense of measures, see [13]). We say that

$$\mu_1 \preceq \mu_2 \text{ if for every quasi-open set } A \subseteq D \text{ we have } \mu_1(A) \geq \mu_2(A).$$

This order turns out to be equivalent (see [13]) to

$$\forall u \in H^1_0(D) \quad \int_D u^2 d\mu_1 \geq \int_D u^2 d\mu_2.$$

We introduce the space $X_\mu(D)$ as the vector space of all functions $u \in H^1_0(D)$ such that $\int_D u^2\, d\mu < \infty$ and which can be seen as $H^1_0(D) \cap L^2(D, \mu)$; moreover we endow the space $X_\mu(D)$ with the norm

$$\|u\|_{X_\mu(D)} = \Big(\int_D |\nabla u|^2\, dx + \int_D u^2\, d\mu\Big)^{1/2}$$

which comes from the scalar product

$$(u, v)_{X_\mu(D)} = \int_D \nabla u \nabla v\, dx + \int_D uv\, d\mu.$$

It is possible to show (see [10]) that with the scalar product above the space $X_\mu(D)$ becomes a Hilbert space.

Since $X_\mu(D)$ can be embedded into $H^1_0(D)$ by the identity mapping, the dual space $H^{-1}(D)$ of $H^1_0(D)$ can be considered as a subspace of the dual space $X'_\mu(D)$. We then write for $f \in H^{-1}(D)$

$$\langle f, v\rangle_{X'_\mu(D)} = \langle f, v\rangle_{H^{-1}(D)} \qquad \forall v \in X_\mu(D)$$

and so, when $f \in L^2(D)$

$$\langle f, v\rangle_{X'_\mu(D)} = \int_D fv\,dx \qquad \forall v \in X_\mu(D).$$

Consider now a measure $\mu \in \mathcal{M}_0$. By the Riesz representation theorem, for every $f \in X'_\mu(D)$ there exists a unique $u \in X_\mu(D)$ such that

$$(u, v)_{X_\mu(D)} = \langle f, v\rangle_{X'_\mu(D)} \qquad \forall v \in X_\mu(D). \tag{2.2}$$

By the definition of scalar product in $X_\mu(D)$ this turns out to be equivalent to

$$\int_D \nabla u \nabla v\,dx + \int_D uv\,d\mu = \langle f, v\rangle_{X'_\mu(D)} \qquad \forall v \in X_\mu(D) \tag{2.3}$$

that we simply write in the form

$$u \in X_\mu(D), \qquad -\Delta u + \mu u = f \text{ in } X'_\mu(D).$$

The solution of (2.3) will be denoted by $u_{\mu,f}$ and the mapping $f \mapsto u_{\mu,f}$ by R_μ.

Definition 2.1. *We say that a sequence (μ_n) of measures in $\mathcal{M}_0$ γ-converges to a measure $\mu \in \mathcal{M}_0$ if and only if*

$$R_{\mu_n}(f) \to R_\mu(f) \text{ weakly in } H^1_0(D) \qquad \forall f \in H^{-1}(D).$$

It is possible to show (see for instance [8]) that (μ_n) γ-converges to μ if and only if $R_{\mu_n}(1) \to R_\mu(1)$ weakly in $H^1_0(D)$. In the following we denote by w_μ the function $R_\mu(1)$, which is characterized as the unique solution of the PDE

$$\begin{cases} -\Delta w_\mu + \mu w_\mu = 1 \\ w_\mu \in X_\mu(D). \end{cases} \tag{2.4}$$

The γ-convergence of μ_n to μ can be also shown to be equivalent to the Γ-convergence of the energy functionals $u \mapsto \|u\|^2_{X_{\mu_n}(D)}$ to $u \mapsto \|u\|^2_{X_\mu(D)}$ with respect to the $L^2(D)$-strong topology (see for instance [8, 13]).

Moreover, for every $\mu \in \mathcal{M}_0$ we define

$$\operatorname{cap}(\mu, D) = \inf\Big\{\int_D |\nabla u|^2\,dx + \int_D (u-1)^2\,d\mu \ :\ u \in H^1_0(D)\Big\}. \tag{2.5}$$

We refer to [14] for the following result.

Proposition 2.2. *The space $\mathcal{M}_0$, endowed with the topology of γ-convergence, is a compact metric space for the distance*

$$d(\mu_1, \mu_2) = \int_D |w_{\mu_1} - w_{\mu_2}|dx.$$

Moreover, the class of measures of the form $\infty_{D\setminus A}$, with A open (and smooth) subset of D, is dense in $\mathcal{M}_0$.

If we define the energy of μ related to f by

$$E(\mu, f) = \min\Big\{\frac{1}{2}\|u\|^2_{X_\mu(D)} - \langle f, u\rangle \ :\ u \in X_\mu(D)\Big\} \tag{2.6}$$

we have that $u_{\mu,f}$ is the unique minimizer in (2.6), so that

$$E(\mu, f) = \frac{1}{2}\|u_{\mu,f}\|^2_{X_\mu(D)} - \langle f, u_{\mu,f}\rangle \ .$$

For the convenience of the reader we recall the notion of generalized minimizing movements associated to a functional, first introduced in [16]. Consider a topological space $\mathcal{S}$, or more in general a set $\mathcal{S}$ endowed with a convergence structure, and a functional

$$[0,T] \times \mathcal{S} \times \mathcal{S} \ni (t, v, w) \mapsto \mathcal{F}(t, v, w) \in \overline{\mathbb{R}}.$$

For every fixed $\varepsilon > 0$, the Euler scheme of time step ε and initial condition $u_0 \in \mathcal{S}$ consists in constructing a function $u_\varepsilon(t) = w([t/\varepsilon])$, where $[\cdot]$ stands for the integer part function, in the following way:

$$w(0) = u_0, \qquad w(n+1) \in \text{Argmin}\,\{\mathcal{F}((n+1)\varepsilon, \cdot, w(n))\}.$$

Definition 2.3. *We say that $u : [0,T] \to \mathcal{S}$ is a generalized minimizing movement associated to $\mathcal{F}$ with initial condition u_0, and we write $u \in GMM(\mathcal{F}, \mathcal{S}, u_0)$, if there exist a sequence $\varepsilon_n \to 0^+$ such that for any $t \in [0,T]$, $u_{\varepsilon_n}(t) \to u(t)$ in $\mathcal{S}$.*

In our case, we deal with irreversible processes which are modeled by a monotonicity condition, so we endow the set $\mathcal{S}$ with an order relation $\preceq$. More precisely, we assume that

(S1) the convergence in $\mathcal{S}$ is compact, i.e. for every sequence there exists a convergent subsequence;
(S2) the order $\preceq$ is compatible with the convergence, i.e. if $u_n \to u$ and $v_n \to v$ are such that $u_n \preceq v_n$, then $u \preceq v$;
(S3) every nondecreasing function $\psi : \mathbb{R} \to \mathcal{S}$ is sequentially continuous up to countably many points.

We finally denote by $X([0,T], \mathcal{S})$ the class of functions $t \mapsto u(t)$ which are nondecreasing in the sense that

$$t_1 \le t_2 \ \ \Rightarrow \ \ u(t_1) \preceq u(t_2) \ \text{ in } \ \mathcal{S}.$$

Lemma 2.4. *In the framework above, for every $(u_n) \in X([0,T], \mathcal{S})$ there exists a subsequence (u_{n_k}) and a function $u \in X([0,T], \mathcal{S})$ such that*

$$\forall t \in [0,T] \ \ u_{n_k}(t) \longrightarrow u(t) \ \text{ in } \ \mathcal{S}.$$

Proof. The proof follows the scheme of the so called Helly's theorem. Fix a countable and dense subset Q of $[0,T]$. By property (S1) and using the standard diagonal procedure for integer numbers we construct a subsequence (u_{n_k}) such that

$$\forall q \in Q \ \ u_{n_k}(q) \to u(q) \ \text{ in } \ \mathcal{S}.$$

The function $q \mapsto u(q)$ is defined only on Q and is monotone on Q by property (S2). We define $u(t)$ for all $t \notin Q$ by taking the limit of any convergent sequence $u(q_n)$ with $q_n \in Q$, $q_n \uparrow t$. The function $u(t)$ so defined belongs to $X([0,T], \mathcal{S})$ and an easy argument shows that $u_{n_k}(t) \to u(t)$ for every $t \in [0,T]$ which is a point of continuity of $t \mapsto u(t)$. By property (S3), all $t \in [0,T]$, except an at most countable set P, are of this kind. Since P is countable, a further diagonal argument provides a new subsequence $(u_{n'_k})$ such that

$$\forall p \in P \quad u_{n'_k}(p) \to u'(p).$$

Redefining the limit function $u(t)$ on P achieves the proof. $\square$

Let us now precise our abstract framework for irreversible processes. We denote by χ_E the function which takes the value 0 on E and $+\infty$ elsewhere.

Theorem 2.5. *Let $\mathcal{S}$ and $\mathcal{F}$ as above and assume that for every $t \in [0,T]$ and every $u \in \mathcal{S}$ the problem*

$$\min\{\mathcal{F}(t,v,u) : v \in \mathcal{S},\ u \preceq v\}$$

has at least one solution. Then for every $u_0 \in S$ there exists a generalized minimizing movement associated to the functional $(t,v,u) \mapsto \mathcal{F}(t,v,u) + \chi_{u \preceq v}$.

Proof. The proof follows straightforwardly by taking an arbitrary sequence $\varepsilon_n \to 0$ and the corresponding u_{ε_n} provided by the Euler scheme. The existence of a generalized minimizing movement follows by Lemma 2.4. The only point to verify is that $u_{\varepsilon_n} \in X([0,T],\mathcal{S})$ which easily follows by the presence of the irreversibility term $\chi_{u \preceq v}$. □

3. A model for adhesive contact membrane

Although all results of this paper hold in any dimension of the space $N \geq 2$, the relevant mechanical case is $N = 2$. Let D be a smooth bounded open set of $\mathbb{R}^2$ and let $f : [0,T] \to L^2(D,\mathbb{R}^+)$ be a given function. We consider D as an elastic membrane in adhesive contact with the plane on its entire surface or on some region. Let $\Omega_0 \subseteq D$ be the region where D is not sticked to the plane. The empirical model can be understood as follows: if the contact is "strong" (or roughly speaking the glue is strong), under the action of a force f, the membrane will take the position given by u on Ω_0, remaining sticked on $D \setminus \Omega_0$. At the initial time $t = 0$, the function u solves the membrane equation on Ω_0

$$\begin{cases} -\Delta u = f(0,\cdot) \text{ in } \Omega_0 \\ u \in H^1_0(\Omega_0). \end{cases} \tag{3.1}$$

Roughly speaking, in presence of a "strong" adhesive contact, a "weak" force f does not have any effect on the sticked part $D \setminus \Omega_0$.

Our purpose is to consider situations where, by breaking the adhesive contact, the unsticked region may increase in time. The debonding process minimizes a balance of energies consisting of the internal energy of the membrane and the debonding energy, which in our formulation is called dissipation distance. An evolution of the shape which represents the unsticked part of the membrane is an evolution of the form $t \mapsto \Omega_t$ such that at every time t equation (3.1) is satisfied on Ω_t with the force $f(t,\cdot)$.

However, depending on the dissipation energy under consideration, it may happen that a relaxation process occurs and a mixture of glue and free material appears. This is a reason for which the model we propose uses capacitary measures. In other words, depending on the dissipation energy we consider (i.e. the energy consumed to debond the membrane), one can obtain a domain solution of the form $t \mapsto \Omega_t$ or a relaxed solution of the form $t \mapsto \mu_t$.

The only model we found in the literature, adopting a somehow similar point of view, is due to Andrew and Shillor [6] (see also [5] for a refinement). We do not explain it here in details, but we point out its main features.

The main unknown in the Andrews-Shillor model is the bonding field β representing the fraction of active bonds on the contact surface, and has values between 0 and 1. Instead of the evolution of the surface as previously emphasized, it is the bonding field which evolves in time by solving the equation

$$\beta_t(t,x) = -c_1 u^2(t,x)\beta(t,x)$$

while u solves on the region $\{u(t,\cdot) > 0\}$ the partial differential equation

$$-\Delta u(t,x) + c_2 u(t,x)\beta^2(t,x) = f(t,x). \tag{3.2}$$

We notice that a bonding field equal to 1 on a time interval $[0, t_0)$ on a closed set K implies from the first equation that $\beta_t = 0$ on $[0, t_0)$ and consequently that $u(t, x) = 0$ for $t \in [0, t_0)$ and $x \in K$. This does imply that the membrane is sticked on K, as one should expect! In this model β belongs to $W^{1,\infty}([0, T]; L^\infty)$, hence its time derivative is defined only almost everywhere. This means that membranes sticked on a set of zero Lebesgue measure (like a segment in two dimensions) cannot be considered, since β does not act pointwise on it. In our model, we overcome this fact, since we can consider membranes which are sticked on sets of zero Lebesgue measure (but of positive capacity).

Moreover, we notice the following technical point. For a large constant c_2, on a region where β is close to 1, the membrane should behave in a way close to a sticked membrane. Nevertheless, a sticked membrane is modeled by the infinite value of the measure acting on the sticked region! This technical gap between $+\infty$ and the maximal value of the measure modeling the glue, which is $c_2 dx$ in (3.2), can be also overcome by using capacitary measures.

3.1. **The minimizing movement model.** We fix a smooth bounded open set D of $\mathbb{R}^2$ and a function $f : [0, T] \to L^2(D, \mathbb{R}^+)$. Let $\mu_0 \in \mathcal{M}_0$ be the initial state of the membrane.

Definition 3.1. *A dissipation distance on $\mathcal{M}_0$ is a mapping $\mathcal{D} : \mathcal{M}_0 \times \mathcal{M}_0 \to [0, +\infty]$ satisfying the following conditions:*

(i) $\mathcal{D}(\mu, \mu) = 0$ *for every* $\mu \in \mathcal{M}_0$*;*

(ii) $\mathcal{D}(\mu_1, \mu_3) \le \mathcal{D}(\mu_1, \mu_2) + \mathcal{D}(\mu_2, \mu_3)$ *for every* $\mu_1, \mu_2, \mu_3 \in \mathcal{M}_0$.

We say that the dissipation distance $\mathcal{D}$ is irreversible if

$$\mu_2 \not\preceq \mu_1 \quad \Rightarrow \mathcal{D}(\mu_1, \mu_2) = +\infty.$$

Example 3.2. Here are two examples of irreversible dissipation distances on $\mathcal{M}_0$:

- if w_μ is the function defined in (2.4), we set

$$\mathcal{D}_\gamma(\mu_1, \mu_2) = \int_D |w_{\mu_1} - w_{\mu_2}|\, dx + \chi_{\mu_2 \preceq \mu_1} \ ;$$

- if $\mathrm{cap}(\mu, D)$ is the quantity defined in (2.5), we set

$$\mathcal{D}_{\mathrm{cap}}(\mu_1, \mu_2) = -\mathrm{cap}(\mu_1, D) + \mathrm{cap}(\mu_2, D) + \chi_{\mu_2 \preceq \mu_1} \ .$$

We notice that both the quantities $\int_D |w_{\mu_1} - w_{\mu_2}|\, dx$ and $-\mathrm{cap}(\mu_1, D) + \mathrm{cap}(\mu_2, D)$ are γ-continuous in each variable.

Given an irreversible dissipation distance $\mathcal{D}$ on $\mathcal{M}_0$ we shall study the generalized minimizing movement associated to the functional $\mathcal{F} : [0, T] \times \mathcal{M}_0 \times \mathcal{M}_0 \to \overline{\mathbb{R}}$ defined by

$$\mathcal{F}(t, \mu_1, \mu_2) = E(\mu_1, f(t)) + \mathcal{D}(\mu_1, \mu_2), \tag{3.3}$$

where E is the energy defined in (2.6).

3.2. **The rate independent model.** In this section, we formalize a rate independent model for the evolution of the debonding membrane. We recall the general frame due to Mielke [20] (see also Mainik and Mielke [19]).

A pair $(u, \mu) : [0, T] \to H^1_0(D) \times \mathcal{M}_0$ is called a solution of the rate-independent problem associated with the energy E and the dissipation distance $\mathcal{D}$ if the following relations hold.

(S) *Stability*: For all $t \in [0, T]$ and all $\tilde{\mu} \in \mathcal{M}_0$ we have

$$E(\mu(t), f(t)) \le E(\tilde{\mu}, f(t)) + \mathcal{D}(\tilde{\mu}, \mu(t)) \ ; \tag{3.4}$$

(E) *Energy inequality*: For all $s,t\in[0,T]$, with $s<t$ we have

$$E(\mu(t),f(t))+Diss_{\mathcal{D}}(\mu,[s,t])\le E(\mu(s),f(s))+\int_s^t\langle\partial_f E(\mu(\tau),f(\tau)),\dot f(\tau)\rangle\,d\tau \qquad (3.5)$$

where

$$Diss_{\mathcal{D}}(\mu,[s,t])=\sup_{N\in\mathbb{N},\ s=t_0<\dots<t_N=t}\sum_{j=1}^N\mathcal{D}(\mu(t_{j-1}),\mu(t_j))\ ,$$

and

$$\langle\partial_f E(\mu(\tau),f(\tau)),\dot f(\tau)\rangle=-\int_D u_{\mu(\tau),f(\tau)}\dot f(t)dx,$$

being $u_{\mu,f}$ the solution of (2.3).

Notice that $Diss_{\mathcal{D}}(\mu,[s,t])$ represents a kind of length of the curve $\tau\mapsto\mu(\tau)$, $\tau\in[s,t]$, in the space $\mathcal{M}_0$ endowed with the pseudo distance $\mathcal{D}$.

3.3. **The main results.** Here are the main results of the paper.

Theorem 3.3. *Let $T>0$ and let $f:[0,T]\to L^2(D,\mathbb{R}^+)$. Let $\mathcal{D}$ be an irreversible dissipation distance which is γ-l.s.c. in the first variable, and let $\mu_0\in\mathcal{M}_0$ be an initial condition. Then there exists a generalized minimizing movement $\mu\in GMM(\mathcal{F},\mathcal{M}_0,\mu_0)$ associated to $\mathcal{F}$ and to the initial condition μ_0, where $\mathcal{F}$ is defined by* (3.3).

Theorem 3.4. *Assume that $f\in W^{1,\infty}\big([0,T];L^2(D)\big)$, $\mu_0\in\mathcal{M}_0^f$ and $\mathcal{D}=\mathcal{D}_\gamma$. Then there exists a solution of the rate-independent problem* (3.4)-(3.5).

We notice that the topology of the space $\mathcal{S}=\mathcal{M}_0$ plays a crucial role in the definition of the minimizing movements. On the other hand, the stability and energy inequalities in the rate independent definition do not require any topological structure. The topology on $\mathcal{S}$ is just a tool to construct a convenient minimizing movement $u(t)$, and we will prove that its lower semicontinuous envelope satisfies the qualitative properties of the rate independent model.

4. Proof of the results

This section is devoted to the proof of Theorems 3.3 and 3.4.

Proof. of Theorem 3.3. All we have to prove is that Theorem 2.5 applies. We take $\mathcal{S}=\mathcal{M}_0$ endowed with the topology of the γ-convergence which is known to be metric and compact, which gives condition (S1).

The order on $\mathcal{S}$ is taken to be $\preceq$ as defined in Section 2. Let us prove condition (S2). Let μ_n and ν_n be two sequences in $\mathcal{M}_0$ γ-converging to μ and ν, respectively, and assume that $\nu_n\preceq\mu_n$. For any $u\in H_0^1(D)$, by the γ-convergence we may deduce (see for instance [8, Chapter 4]) the existence of a sequence u_n converging to u weakly in $H_0^1(D)$ such that

$$\int_D|\nabla u|^2dx+\int_D u^2d\nu=\lim_{n\to\infty}\int_D|\nabla u_n|^2dx+\int_D u_n^2d\nu_n.$$

Using the fact that $\nu_n\preceq\mu_n$ and one of the properties of the γ-convergence (see [8, Chapter 4]), we obtain

$$\begin{aligned}\int_D|\nabla u|^2dx+\int_D u^2d\mu&\le\liminf_{n\to\infty}\int_D|\nabla u_n|^2dx+\int_D u_n^2d\mu_n\\&\le\lim_{n\to\infty}\int_D|\nabla u_n|^2dx+\int_D u_n^2d\nu_n\\&=\int_D|\nabla u|^2dx+\int_D u^2d\nu,\end{aligned}$$

which gives

$$\int_D u^2 d\mu \le \int_D u^2 d\nu.$$

As stated in Section 2, this is equivalent to $\nu \preceq \mu$.

Concerning condition (S3), we notice that the continuity points of every nondecreasing mapping $\mathbb{R} \ni t \mapsto \mu(t) \in \mathcal{M}_0$ coincide with the continuity points of the mapping

$$\mathbb{R} \ni t \mapsto \int_D w_{\mu(t)} dx \in \mathbb{R},$$

which is nonincreasing from $\mathbb{R}$ to $\mathbb{R}$ and hence with at most countably many discontinuity points.

The conditions of Theorem 2.5 on the functional $\mathcal{F}$ are easily verified. □

Remark 4.1. Notice that the irreversibility condition on the dissipation distance $\mathcal{D}$ is expressed in terms of the order $\preceq$ in $\mathcal{M}_0$. This is because a measure μ takes into account the quantity of active bonds which decreases with the time. However, the conclusion of Theorem 3.3 still remains valid if the irreversibility condition is expressed in terms of another (unphysical) order defined by

$$\mu \preceq \nu, \quad \Longleftrightarrow \quad \int_D w_\mu d\mu \le \int_D w_\nu d\nu.$$

Proposition 4.2. *The dissipation distances $\mathcal{D}_\gamma$ and $\mathcal{D}_{\text{cap}}$ introduced in Example 3.2 are γ-l.s.c. in the first variable.*

Proof. Since the quantities $\int_D |w_{\mu_1} - w_{\mu_2}|\, dx$ and $-\text{cap}(\mu_1, D) + \text{cap}(\mu_2, D)$ are γ-continuous in each variable, the only thing to be proved is that the irreversibility term $\chi_{\mu_2 \preceq \mu_1}$ is γ-l.s.c. with respect to μ_1. This is a consequence of property (S2) previously proved. □

Proof. of Theorem 3.4. We start by the following preliminary technical compactness result.

Lemma 4.3. *Let $\mu \in \mathcal{M}_0^f$. Then, the injection*

$$H_0^1(D) \cap L^\infty(D, \mu) \hookrightarrow L^2(D, \mu)$$

is compact.

Proof. Let (u_n) be a sequence converging weakly to u in $H_0^1(D)$. Assume first that

$$|u_n| \le M \quad \text{a.e. in } D$$

hence q.e. (hence μ-a.e.).

Our purpose is to prove that $u_n \to u$ (for a subsequence) in $L^2(D, \mu)$. We shall prove first a local version of this convergence, precisely the convergence on a quasi-open set on which w is bounded from below by a positive constant.

We set $w = w_\mu$; then $w > 0$ a.e. in D since μ is finite. We prove that

$$u_n \wedge (\tilde{M} w - \varepsilon)^+ \overset{L^2(D,\mu)}{\longrightarrow} u \wedge (\tilde{M} w - \varepsilon)^+$$

for every $\varepsilon > 0$ and $\tilde{M} > 0$.

For simplicity set $u_n^\varepsilon = u_n \wedge (\tilde{M} w - \varepsilon)^+$ and $u^\varepsilon = u \wedge (\tilde{M} w - \varepsilon)^+$. Then, $\{w > \varepsilon/\tilde{M}\}$ is a quasi-open set and $u_n^\varepsilon, u^\varepsilon \in H_0^1(\{w > \varepsilon/\tilde{M}\})$.

Let us observe that $\nu = 1 + \Delta w$ is a positive measure belonging to $H^{-1}(D)$. Since $-\Delta w + \mu w = 1$ we have

$$\mu(A) = \int_A \frac{1}{w}\, d\nu \qquad \forall A \subseteq D$$

so that

$$\int_D (u_n^\varepsilon)^2\, d\mu = \int_D (u_n^\varepsilon)^2 \frac{1}{w}\, d\nu = \langle 1+\Delta w, (u_n^\varepsilon)^2 \frac{1}{w}\rangle_{H^{-1}(D)\times H_0^1(D)}.$$

Since supp $u_n^\varepsilon \subseteq \{w > \varepsilon/\tilde{M}\}$, then $(u_n^\varepsilon)^2 \frac{1}{w} \in L^\infty(D)$. Hence

$$\int_D (u_n^\varepsilon)^2\, d\mu = \int_D (u_n^\varepsilon)^2 \frac{1}{w}\, dx - 2\int_D \frac{\nabla w}{w} u_n^\varepsilon \nabla u_n^\varepsilon\, dx + \int_D \nabla w \frac{\nabla w}{w^2}(u_n^\varepsilon)^2\, dx.$$

Using the compact embedding $H_0^1(D) \hookrightarrow L^2(D)$, the right-hand side of the equality above converges to

$$\int_D (u^\varepsilon)^2 \frac{1}{w}\, dx - 2\int_D \frac{\nabla w}{w} u^\varepsilon \nabla u^\varepsilon\, dx + \int_D \nabla w \frac{\nabla w}{w^2}(u^\varepsilon)^2\, dx = \int_D (u^\varepsilon)^2\, d\mu.$$

We prove now the weak convergence in $L^2(D,\mu)$, in the regions where w is bounded from below by a positive constant. Since $u_n^\varepsilon \in H_0^1(\{w > \frac{\varepsilon}{\tilde{M}}\}) \cap L^\infty(D,\mu)$, it is enough to consider only test functions $\psi \in H_0^1(\{w > \frac{\varepsilon}{\tilde{M}}\}) \cap L^\infty(D,\mu)$. We want to prove that

$$\int_D u_n^\varepsilon \psi d\mu \to \int_D u^\varepsilon \psi d\mu.$$

Indeed, this can be done as above and comes from the fact that

$$\langle 1+\Delta w, \frac{u_n^\varepsilon \psi}{w}\rangle_{H^{-1}(D)\times H_0^1(D)} \to \langle 1+\Delta w, \frac{u^\varepsilon \psi}{w}\rangle_{H^{-1}(D)\times H_0^1(D)}.$$

Therefore $u_n^\varepsilon \to u^\varepsilon$ in $L^2(D,\mu)$.

We have shown that:

$$\forall \tilde{M} > 0\,,\ \forall \epsilon > 0\,,\ u_n \wedge (\tilde{M}w - \varepsilon)^+ \to u \wedge (\tilde{M}w-\varepsilon)^+ \text{ in } L^2(D,\mu).$$

Let us fix $\delta > 0$ and denote by K_δ the set $\{w > \delta\}$. Then

$$1_{K_\delta} u_n \to 1_{K_\delta} u \text{ in } L^2(D,\mu).$$

Indeed, we can find $\tilde{M} > 0$ such that:

$$(\tilde{M}w - \varepsilon) \geq \tilde{M}\delta - \varepsilon \geq M \geq u \text{ a.e. in } K_\delta.$$

So $u_n \wedge (\tilde{M}w-\varepsilon)^+ = u_n$ a.e. in K_δ. Since μ is finite and $(u_n)_n$ is bounded in $L^\infty(D,\mu)$, we get

$$1_{\{w>0\}} u_n \xrightarrow{L^2(D,\mu)} 1_{\{w>0\}} u.$$

Since cap$(\{w=0\}) = 0$ we have $\mu(\{w=0\}) = 0$, hence $u_n \xrightarrow{L^2(D,\mu)} u$.

Assume now that $|u_n| \leq M$ μ-a.e. and that $u_n \rightharpoonup u$ weakly in $H_0^1(D)$. For every $M' \geq M$, we have that $(u_n \wedge M') \vee (-M') \to (u \wedge M') \vee (-M')$ strongly in $L^2(D,\mu)$. But $(u_n \wedge M') \vee (-M') = u_n$ μ-a.e., hence $u_n \to (u \wedge M') \vee (-M')$ strongly in $L^2(D,\mu)$. Making $M' \to \infty$, we get $u_n \to u$ strongly in $L^2(D,\mu)$. □

Remark 4.4. Thanks to the fact that finite measures are regular, we notice that for measures in $\mathcal{M}_0^f$ the order relation $\nu \preceq \mu$ defined in Section 2, is equivalent to the standard order on measures $\mu \leq \nu$, that is

for every Borel subset $E \subseteq D$, $\mu(E) \leq \nu(E)$.

Lemma 4.5. *Let μ_0 and μ_n be measures of $\mathcal{M}_0^f$ satisfying $\mu_n \leq \mu_0$ for every $n \in \mathbb{N}$. Then, the following assertions are equivalent:*

i) *μ_n γ-converges to μ;*

ii) *$\mu_n \rightharpoonup \mu$ for the weak* convergence of measures.*

Proof. Note that from the Radon-Nikodym theorem we can write

$$\mu_n = f_n \cdot \mu_0$$

with $0 \leq f_n \leq 1$ μ_0-a.e. Consequently, $\mu_n \rightharpoonup \mu$ weakly* is equivalent to $f_n \rightharpoonup f$ weakly in $L^2(D, \mu_0)$. Since both the γ-convergence and the weak* convergence are compact, in order to achieve the proof it is enough to show only one of the implications.

We will prove ii) $\Rightarrow$ i). In view of the remark above, if is enough to show that if $f_n \rightharpoonup f$ weakly in $L^2(D, \mu_0)$ then $\mu_n \to \mu$ in the γ-convergence. To prove the γ-convergence of the measures, we show the Γ-convergence of the corresponding energy functionals (see for instance [8, 13]).

$\Gamma - \limsup$ **inequality**: Let us consider first $u \in H_0^1(D) \cap L^\infty(D, \mu_0)$. Then

$$\limsup_{n\to\infty} \Big[\int_D |\nabla u|^2 \, dx + \int_D f_n u^2 \, d\mu_0 \Big] \leq \int_D |\nabla u|^2 \, dx + \int_D f u^2 \, d\mu_0. \tag{4.1}$$

Indeed, since u is bounded μ_0-a.e., the weak $L^2(D, \mu_0)$-convergence of f_n gives

$$\lim_{n\to\infty} \int_D f_n u^2 d\mu_0 = \int_D f u^2 d\mu_0. \tag{4.2}$$

For an arbitrary $u \in H_0^1(D) \cap L^2(D, \mu_0)$, the $\Gamma - \limsup$ condition comes by a diagonal procedure. Setting $u_k = (u \wedge k) \vee (-k)$ we have

$$u_k \to u \qquad \text{strongly in } H_0^1(D).$$

Since u is quasi-continuous, then $u_k \in H_0^1(D) \cap L^\infty(D, \mu_0)$ and so (4.2) holds. Letting $k \to \infty$ we get the $\Gamma - \limsup$ inequality (4.1).

$\Gamma - \liminf$ **inequality**: Let $u_n \rightharpoonup u$ weakly in $H_0^1(D)$; we have to prove that

$$\int_D |\nabla u|^2 \, dx + \int_D f u^2 \, d\mu_0 \leq \liminf_{n\to\infty} \Big[\int_D |\nabla u_n|^2 \, dx + \int_D f_n u_n^2 \, d\mu_0 \Big]. \tag{4.3}$$

Setting $u_{n,k} = (u_n \wedge k) \vee (-k)$ we have

$$u_{n,k} \rightharpoonup u_k \qquad \text{weakly in } H_0^1(D).$$

¿From the compact injection proved in Lemma 4.3 we get

$$u_{n,k} \to u_k \qquad \text{strongly in } L^2(D, \mu_0),$$

hence (for a subsequence) μ_0-a.e. Consequently

$$\int_D u_{n,k}^2 f_n \, d\mu_0 \to \int_D u_k^2 f \, d\mu_0,$$

so that

$$\begin{aligned} \liminf_{n\to\infty} & \int_D |\nabla u_n|^2 \, dx + \int_D u_n^2 f_n \, d\mu_0 \\ & \geq \liminf_{n\to\infty} \int_D |\nabla u_{n,k}|^2 \, dx + \int_D u_{n,k}^2 f_n \, d\mu_0 \\ & \geq \int_D |\nabla u_k|^2 \, dx + \int_D u_k^2 f \, d\mu_0. \end{aligned}$$

Finally, the $\Gamma - \liminf$ inequality (4.3) comes by letting $k \to \infty$. □

The key tool for providing stability of the rate independent movement is the following result.

Lemma 4.6. *Let $\mu, \tilde{\mu} \in \mathcal{M}_0^f$ with $\tilde{\mu} \le \mu \le \mu_0$, and assume that $\mu_n \to \mu$ in the γ-convergence. Then there exists a subsequence of (μ_n) and a sequence of measures $(\tilde{\mu}_{n_k})$ such that*

$$\tilde{\mu}_{n_k} \to \tilde{\mu} \text{ in the } \gamma\text{-convergence, and } \tilde{\mu}_{n_k} \le \mu_{n_k}.$$

Proof. In order to construct the sequence $(\tilde{\mu}_{n_k})$, the main idea is to approach the density of $\tilde{\mu}$ and to use Lemma 4.5. From the Radon-Nykodim theorem we may assume that $d\tilde{\mu} = \tilde{f} d\mu_0$, $d\mu = f d\mu_0$ and $d\mu_n = f_n d\mu_0$. Since $\mu_n \to \mu$ in the γ-convergence, following Lemma 4.5 we have that $\mu_n \rightharpoonup \mu$ weakly*. Moreover, since $0 \le f_n, f \le 1$ μ_0-a.e., we get that $f_n \rightharpoonup f$ weakly* in $L^\infty(D, \mu_0)$ which, since μ_0 is finite, is equivalent to $f_n \rightharpoonup f$ weakly in $L^2(D, \mu_0)$.

We will construct a sequence of functions $(\tilde{f}_{n_k})$ such that $0 \le \tilde{f}_{n_k} \le f_{n_k}$ μ_0-a.e. and $\tilde{f}_{n_k} \rightharpoonup \tilde{f}$ weakly in $L^2(D, \mu_0)$. Using Lemma 4.5 and the fact that μ_0 is finite, we conclude the proof.

In order to simplify the notation, we assume that $D \subseteq (0,1) \times (0,1)$ and $\phi_k^{i,j}$ is the characteristic function of

$$[\frac{i}{2^k}, \frac{i+1}{2^k}) \times [\frac{j}{2^k}, \frac{j+1}{2^k}),$$

for $0 \le i, j \le 2^k - 1$. Note that if $(i,j) \neq (i',j')$ then $\phi_k^{i,j} \neq \phi_k^{i',j'}$ μ_0-a.e. and that the union of all supports of $(\phi_k^{i,j})_{k,i,j}$ is a generating family of Borel sets.

For $k \in \mathbb{N}^*$ and for every $0 \le i, j \le 2^k - 1$ we have that

$$\int_D f_n \phi_k^{i,j} \, d\mu_0 \to \int_D f \phi_k^{i,j} \, d\mu_0 \quad \text{as } n \to \infty.$$

Since $\int_D f \phi_k^{i,j} \, d\mu_0 \ge \int_D \tilde{f} \phi_k^{i,j} \, d\mu_0$, there exists a sequence of constants $(c_{n,k}^{i,j})_n$ belonging to $[0,1]$ such that

$$\int_D (f_n \wedge c_{n,k}^{i,j}) \phi_k^{i,j} \, d\mu_0 \to \int_D \tilde{f} \phi_k^{i,j} \, d\mu_0 \quad \text{as } n \to \infty.$$

Let $\varepsilon_k > 0$ with $\varepsilon_k \downarrow 0$. We fix $n_0 = 1$ and n_k such that $n_k > n_{k-1}$ and for every $0 \le i, j \le 2^k - 1$

$$\Big| \int_D (f_{n_k} \wedge c_{n,k}^{i,j}) \phi_k^{i,j} \, d\mu_0 - \int_D \tilde{f} \phi_k^{i,j} \, d\mu_0 \Big| \le \frac{\varepsilon_k}{2^k}.$$

We conclude by observing that the sequence $(\tilde{f}_{n_k})$ built as $\tilde{f}_{n_k} = \sum_{i,j} (f_{n_k} \wedge c_{n,k}^{i,j}) \phi_k^{i,j}$ satisfies all the requirements. □

Construction of the solution. Being in a particular case of Theorem 3.3, the candidate of the solution to the rate independent model is the lower semicontinuous envelope of the minimizing movement constructed in the proof of Theorem 3.3. We have to verify the stability property (3.4) and the energy inequality (3.5).

The stability property. Let $\mu(t)$ be a solution of the generalized minimizing movement model constructed as in the previous section with $Q = \{mT2^{-n} \ : \ m, n \in \mathbb{N}, \ 0 \le m \le 2^n\}$ and $\varepsilon_n = T2^{-n}$.

We denote by $\overline{\mu}(t)$ the lower semicontinuous envelope of μ with respect to t, i.e.

$$\overline{\mu}(t) = \gamma - \lim_{s \uparrow t} \mu(s) = \sup_{s<t} \mu(s),$$

where the last supremum is intended in the sense of the order $\preceq$ intoduced on $\mathcal{S} = \mathcal{M}_0$.

In order to prove that the mapping $[0,T] \ni t \mapsto \overline{\mu}(t)$ satisfies the stability property, we fix $t \in [0,T]$ and consider $\nu \in \mathcal{M}_0^f$ such that $\overline{\mu}(t) \preceq \nu$. If this last inequality does not hold, then $\mathcal{D}_\gamma(\nu, \overline{\mu}(t)) = \infty$ and (3.4) holds trivially.

We want to prove the stability inequality

$$E(\overline{\mu}(t), f(t)) \leq E(\nu, f(t)) + \int_D |w_{\overline{\mu}(t)} - w_\nu| \, dx. \tag{4.4}$$

Let $t_n \in Q$, $t_n \uparrow t$ and a subsequence of (ε_n) (still denoted by (ε_n)), such that

$$\overline{\mu}(t) = \gamma - \lim_{n\to\infty} \mu_{\varepsilon_n}(t_n).$$

Since $\mu_{\varepsilon_n}(t_n)$ is obtained through the minimizing movement procedure, we have

$$\begin{aligned} E(\mu_{\varepsilon_n}(t_n), f(t_n)) \quad &+ \int_D |w_{\mu_{\varepsilon_n}(t_n)} - w_{\mu_{\varepsilon_n}(t_n-\varepsilon_n)}| dx \\ &\leq E(\theta, f(t_n)) + \int_D |w_\theta - w_{\mu_{\varepsilon_n}(t_n-\varepsilon_n)}| \, dx \end{aligned} \tag{4.5}$$

for every $\mu_{\varepsilon_n}(t_n - \varepsilon_n) \preceq \theta$.

By Lemma 4.6 there exists a subsequence (ε_n) and measures $\nu_n \in \mathcal{M}_0$ such that

$$\nu_n \to \nu \text{ in the } \gamma\text{-convergence and } \nu_n \leq \mu_{\varepsilon_n}(t_n).$$

Notice that $\mu_{\varepsilon_n}(t_n - \varepsilon_n) \preceq \mu_{\varepsilon_n}(t_n) \preceq \nu_n$ so that we can use ν_n as test measure in (4.5) which gives

$$\begin{aligned} E(\mu_{\varepsilon_n}(t_n), f(t_n)) \quad &+ \int_D |w_{\mu_{\varepsilon_n}(t_n)} - w_{\mu_{\varepsilon_n}(t_n-\varepsilon_n)}| dx \\ &\leq E(\nu_n, f(t_n)) + \int_D |w_{\nu_n} - w_{\mu_{\varepsilon_n}(t_n-\varepsilon_n)}| \, dx, \end{aligned} \tag{4.6}$$

that is

$$E(\mu_{\varepsilon_n}(t_n), f(t_n)) + \int_D w_{\mu_{\varepsilon_n}(t_n)} dx \leq E(\nu_n, f(t_n)) + \int_D w_{\nu_n} \, dx. \tag{4.7}$$

Passing to the limit as $n \to \infty$ we obtain the stability inequality (4.4).

The energy inequality. We start to prove that for every $\mu \in \mathcal{M}_0^f$

$$\langle \partial_f E(\mu, f(t)), \dot{f}(t) \rangle = \partial_t E(\mu, f(t)).$$

Let us first prove that the right hand side of this equality is well defined. Denote

$$e(u, t) = \int_D |\nabla u|^2 \, dx + \int_D u^2 \, d\mu - \int_D f(t) u dx$$

so that

$$E(\mu, f(t)) = \min \left\{ e(u, t) \ : \ u \in H_0^1(D) \cap L^2(\mu) \right\} = e(u_{\mu, f(t)}, f(t)).$$

Let $t \in [0, T]$ and $h \in \mathbb{R}$ be such that $t + h \in [0, T]$. We have:

$$\begin{aligned} e(u_{\mu, f(t+h)}, f(t+h)) - e(u_{\mu, f(t+h)}, f(t)) &\leq E(\mu, f(t+h)) - E(\mu, f(t)) \\ &\leq e(u_{\mu, f(t)}, f(t+h)) - e(u_{\mu, f(t)}, f(t)). \end{aligned}$$

Replacing e by its expression we obtain:

$$\begin{aligned} -\int_D (f(t+h) - f(t)) u_{\mu, f(t+h)} \, dx &\leq E(\mu, f(t+h)) - E(\mu, f(t)) \\ &\leq -\int_D (f(t+h) - f(t)) u_{\mu, f(t)} \, dx. \end{aligned}$$

Dividing by h and passing to the limit as $h \to 0$, we obtain that $\partial_t E(\mu, f(t))$ exists and

$$\langle \partial_f E(\mu, f(t)), \dot{f}(t) \rangle = \partial_t E(\mu, f(t)) = -\int_D u_{\mu, f(t)} \dot{f}(t) \, dx. \tag{4.8}$$

Let us now prove the energy inequality (3.5). We fix first $s, t \in [0,T]$ with $0 \le s < t \le T$. As previously, let $s_n, t_n \in Q$, $s_n \uparrow s, t_n \uparrow t$ and

$$\overline{\mu}(s) = \gamma - \lim_{n\to\infty} \mu_{\varepsilon_n}(s_n), \;\; \overline{\mu}(t) = \gamma - \lim_{n\to\infty} \mu_{\varepsilon_n}(t_n).$$

Between two consecutive points of the form $s_n + k\varepsilon_n$ and $s_n + (k+1)\varepsilon_n$, the minimizing movement inequality gives

$$\begin{aligned} E(\mu_{\varepsilon_n}(s_n+(k+1)\varepsilon_n), f(s_n+(k+1)\varepsilon_n)) \;\; &+ \int_D |w_{\mu_{s_n+(k+1)\varepsilon_n}} - w_{\mu_{s_n+k\varepsilon_n}}| dx \\ &\le E(\mu_{\varepsilon_n}(s_n+k\varepsilon_n), f(s_n+(k+1)\varepsilon_n)). \end{aligned} \tag{4.9}$$

Using (4.8) gives

$$\begin{aligned} &E(\mu_{\varepsilon_n}(s_n+k\varepsilon_n), f(s_n+(k+1)\varepsilon_n)) \\ &\quad = E(\mu_{\varepsilon_n}(s_n+k\varepsilon_n), f(s_n+k\varepsilon_n)) \;\; - \int_{s_n+k\varepsilon_n}^{s_n+(k+1)\varepsilon_n} \int_D u_{\mu_{\varepsilon_n}(s_n+k\varepsilon_n), f(\tau)} \dot f(\tau)\, d\tau, \end{aligned}$$

and summing up from $k=0$ to $(t_n - s_n)/\varepsilon_n - 1$ we get

$$\begin{aligned} &E(\mu_{\varepsilon_n(t_n)}, f(t_n)) - E(\mu_{\varepsilon_n(s_n)}, f(s_n)) + \int_D |w_{\mu_{\varepsilon_n(t_n)}} - w_{\mu_{\varepsilon_n(s_n)}}|\, dx \\ &\quad \le - \sum_{k=0}^{(t_n-s_n)/\varepsilon_n - 1} \int_{s_n+k\varepsilon_n}^{s_n+(k+1)\varepsilon_n} \int_D u_{\mu_{\varepsilon_n}(s_n+k\varepsilon_n), f(\tau)} \dot f(\tau)\, d\tau. \end{aligned}$$

Passing to the limit as $n \to \infty$, we obtain (3.4) as desired. □

Proposition 4.7. *Under the assumptions of Theorem 3.4, we have that the function*

$$t \mapsto E(\overline{\mu}(t), f(t)) + \int_D w_{\overline{\mu}(t)}\, dx$$

is Lipschitz continuous on $[0,T]$.

Proof. For a generic μ, the equality

$$E(\mu, f(t)) - E(\mu, f(s)) = -\int_s^t \int_D u_{\mu, f(\tau)} \dot f(\tau)\, d\tau$$

holds. Writing the previous equality for $\mu = \overline{\mu}(t)$ we get

$$E(\overline{\mu}(t), f(t)) - E(\overline{\mu}(t), f(s)) = -\int_s^t \int_D u_{\overline{\mu}(t), f(\tau)} \dot f(\tau)\, d\tau. \tag{4.10}$$

Since $\overline{\mu}(s) \preceq \overline{\mu}(t)$ we have from the stability property

$$E(\overline{\mu}(s), f(s)) \le E(\overline{\mu}(t), f(s)) + \int_D (w_{\overline{\mu}(t)} - w_{\overline{\mu}(s)})\, dx. \tag{4.11}$$

Inequalities (4.10) and (4.11) give

$$\begin{aligned} &E(\overline{\mu}(t), f(t)) - E(\overline{\mu}(s), f(s)) + \int_D (w_{\overline{\mu}(t)} - w_{\overline{\mu}(s)})\, dx \\ &\quad \ge -\int_s^t \int_D u_{\overline{\mu}(t), f(\tau)} \dot f(\tau)\, d\tau. \end{aligned} \tag{4.12}$$

Since the L^2-norm of the function $u_{\overline{\mu}(\tau), f(t)}$ is uniformly bounded in τ and t by a constant C, we have by the inequality (4.12) and the energy inequality (3.5):

$$\left| E(\overline{\mu}(t), f(t)) + \int_D w_{\overline{\mu}(t)}\, dx - \left(E(\overline{\mu}(s), f(s)) + \int_D w_{\overline{\mu}(s)}\, dx \right) \right| \le C \int_s^t \left\| \dot f(\tau) \right\|_{L^2(D)} d\tau$$

which concludes the proof. ☐

5. Further remarks

The choice of the dissipation distance is related to the mechanical model and has a key influence on the relaxation process of the moving shapes/measures. In the sequel we give an example related to the delamination model, where no relaxation occurs, that is for every t the measure $\mu(t)$ is of the form $\infty_{D\setminus\Omega(t)}$ for a suitable shape flow $t \mapsto \Omega(t)$.

In particular, we will prove that for every nonnegative debonding force $f(t)$, a shape flow $t \mapsto \Omega(t)$ satisfying the generalized minimizing movement scheme exists. Moreover, its lower semicontinuous envelope $\overline{\Omega}(t)$ is stable in the sense of Mielke [20, 19], similarly to (3.4). Since the class of domains is not compact for the γ-convergence (in fact it is dense in $\mathcal{M}_0$ as shown in [14]) the proof of stability and the energy inequality cannot be obtained in the same way as for measures. Also, the abstract frameworks proposed in [20, 19] seem not adapted to our situation. This is why we are able to obtain only the stability condition for a general debonding force.

5.1. Dissipation proportional to the surface measure.

A delamination model suggesting that the dissipation distance is related to the surface measure is investigated in [18]. The purpose of this section is to prove that no relaxation process occurs in this case. Precisely, if the inital state is a quasi-open set (i.e. not an arbitrary measure) then for the same energy as previously and without any additional hypothesis on the debonding force, the solution consists only on shapes.

Let $\mathcal{A}$ be the family of quasi-open subsets of a bounded design region D. A quasi-open set A stands for the region where the the membrane is not sticked. There is a natural identification between the quasi-open set A and the measure $\infty_{D\setminus A}$. We may endow the family $\mathcal{A}$ with the $w\gamma$-convergence (see [8, 9] for details): we say that $A_n \xrightarrow{w\gamma} A$ if $w_{A_n} \longrightarrow w$, weakly in $H^1_0(D)$ and $A = \{w > 0\}$. Clearly, this convergence is compact and weaker than the γ-convergence.

Let us define a new dissipation distance by setting

$$\mathcal{D}_m(A_1, A_0) = |A_1 \setminus A_0| + \chi_{A_0 \subseteq A_1}.$$

Here, $\chi_{A_0\subseteq A_1} = 0$ if $A_0 \subseteq A_1$ q.e., and $+\infty$ if not. We consider the functional

$$\mathcal{F}(t, A_1, A_2) = E(A_1, f(t)) + \mathcal{D}_m(A_1, A_2), \tag{5.1}$$

where

$$E(A, f) = \min\Big\{\frac{1}{2}\int_D |\nabla u|^2\,dx - \int_D uf\,dx \ :\ u \in H^1_0(A)\Big\}. \tag{5.2}$$

Notice that, even if the $w\gamma$-convergence is compact, the energy (5.2) is not $w\gamma$-continuous and the dissipation distance $\mathcal{D}_m$ is not pairwise $w\gamma$-lower semicontinuous. The interplay between the $w\gamma$-convergence and the dissipation distance also fails because one can easily find domains $A_n \subseteq A$ such that $\mathcal{D}_m(A, A_n) \to 0$ and A_n does not $w\gamma$-converge to A.

Theorem 5.1. *Let $T > 0$ and let $f : [0,T] \to L^2(D, \mathbb{R}^+)\big)$. Let $A_0 \in \mathcal{A}$ be an initial quasi-open set and let $\mathcal{A}$ be endowed with the $w\gamma$-convergence. Then there exists a generalized minimizing movement $A \in GMM(\mathcal{F}, \mathcal{A}, A_0)$ associated to $\mathcal{F}$ and to the initial condition A_0.*

Proof. As for Theorem 3.3, given $B \in \mathcal{A}$ we first want to prove the existence of a solution for the minimizing problem:

$$\min\Big\{E\big(A, f\big) + \mathcal{D}_m\big(A, B\big) \ :\ A \in \mathcal{A}\Big\}. \tag{5.3}$$

This is immediate because of the compactness of the $w\gamma$-convergence on $\mathcal{A}$ and of the lower semicontinuity of both the Dirichlet energy and of the Lebesgue measure with respect to the $w\gamma$-convergence.

In a second step, we construct, as in the proof of Theorem 3.3, the discrete solution $A_\varepsilon(t)$ of the Euler scheme and pass to the limit as $\varepsilon \to 0$, so obtaining a minimizing movement $A(t)$. As in Section 4, the irreversibility assumption in the dissipation distance plays a key role in the proof. We notice that monotonicity is preserved through the $w\gamma$-convergence. □

Theorem 5.2. *Assume that $f \in W^{1,\infty}\big([0,T];L^2(D)\big)$. Then the lower semicontinuous envelope $\overline{A}$ of the GMM-solution (u, A) above, satisfies the stability property*

$$E(\overline{A}(t), f(t)) \leq E(B, f(t)) + \mathcal{D}_m(B, \overline{A}(t)), \ \ \forall t \in [0,T], \ \forall B \supset \overline{A}(t) \ q.e. \tag{5.4}$$

As in Section 4, the main difficulty to prove the stability property in the sense of relation (3.4) is the analogous of Lemma 4.6 for quasi-open sets.

Lemma 5.3. *Let (A_n) be a sequence in $\mathcal{A}$ such that $A_n \to A$ in the $w\gamma$-convergence. Given $\tilde{A} \in \mathcal{A}$ such that $A \subseteq \tilde{A}$ q.e., there exists a sequence $(\tilde{A}_{n_k})$ in $\mathcal{A}$ and a subsequence (A_{n_k}) such that*

i) $A_{n_k} \subseteq \tilde{A}_{n_k}$ *q.e.;*
ii) $\tilde{A}_{n_k} \to \tilde{A}$ *in the γ-convergence;*
iii) $\limsup_{k\to\infty} |\tilde{A}_{n_k} \setminus A_{n_k}| \leq |\tilde{A} \setminus A|$.

Proof. For the proof of points i) and ii) we refer the reader to [11] (see also [9]). In order to prove the upper semicontinuity property iii), we follow the construction given in [11] and notice that $\tilde{A}_{n_k}$ can be written in the form

$$\tilde{A}_{n_k} = A_{n_k} \cup \{w_{\tilde{A}} > \varepsilon_k\},$$

where $\varepsilon_k \downarrow 0$ and ε_k is chosen in relation with n_k. From the $w\gamma$-convergence and the compact injection $H_0^1(D) \hookrightarrow L^2(D)$ we have that $w_{A_{n_k}} \to w$ in $L^2(D)$ and

$$\liminf_{k\to\infty} 1_{\{w_{A_{n_k}} > \delta\}} \geq 1_{\{w > \delta\}} \quad \text{a.e.}$$

We have

$$\begin{aligned}\tilde{A}_{n_k} \setminus A_{n_k} &\subseteq (\{w_{\tilde{A}} > \varepsilon_k\} \cup A_{n_k}) \setminus A_{n_k} \subseteq \{w_{\tilde{A}} > \varepsilon_k\} \setminus A_{n_k} \\ &\subseteq \{w_{\tilde{A}} > \varepsilon_k\} \setminus \{w_{A_{n_k}} > \delta\} \subseteq \tilde{A} \setminus \{w_{A_{n_k}} > \delta\}\end{aligned}$$

so that passing to the limit as $k \to \infty$, we obtain

$$\limsup_{k\to\infty} |\tilde{A}_k \setminus A_{n_k}| \leq |\tilde{A} \setminus \{w > \delta\}|.$$

Making $\delta \to 0$, the proof is achieved. □

Proof. of Theorem 5.2 We follow step by step the proof of Theorem 3.4, the set $\tilde{A}_n$ constructed by Lemma 5.3 playing the same role as the measure $\tilde{\mu}_n$ of Lemma 4.6. The only point which needs some attention is concerned with the passage to the limit in inequality

$$E(A_{n_k}(t_k), f(t_k)) \leq E(\tilde{A}_{n_k}, f(t_k)) + \mathcal{D}_m(\tilde{A}_{n_k}, A_{n_k}(t_k))$$

where we use the lower semicontinuity of the energy with respect to the $w\gamma$-convergence on the left hand side, the convergence of the energy with respect to the γ-convergence and the upper semi-continuity result of Lemma 5.3 for the dissipation distance on the right hand side. □

Remark 5.4. We do not know if the shape flow $\overline{A}(t)$ satisfies the energy inequality similar to (3.5)

$$E(\overline{A}(t), f(t)) + Diss_{\mathcal{D}_m}(\overline{A}, [s,t]) \leq E(\overline{A}(s), f(s)) + \int_s^t \langle \partial_f E(\overline{A}(\tau), f(\tau)), \dot{f}(\tau)\rangle\, d\tau. \quad (5.5)$$

In Mainik-Mielke [19] a weaker form of the energy inequality is proposed by requiring (5.5) only for $s = 0$. Using shape analysis techniques (see [8]), one could prove that the GMM flow $\overline{A}$ satisfies this weaker form of energy inequality as soon as the $w\gamma$-convergence coincides with the γ-convergence, which is known to be true under some geometrical a priori constraints on the admissible domains, e.g. convexity, exterior cone condition, flat cone condition, uniform Wiener regularity, uniformly bounded number of holes in dimension 2. However, in these cases, if the constraints are imposed *a priori*, it is not clear if the stability property (5.4) holds.

On the other hand, it is possible to prove the rate independence property in the weaker sense of [19] (i.e. stability and weak form of energy inequality) in the following situations:

- the debonding force satisfies $\dot{f}(t) \leq 0$ for every $t \in [0,T]$;
- the dimension of the space is one;
- the dimension of the space is two and A_0 and $f(t)$ are symmetric with respect to a straight line in the sense of Steiner;
- in any dimension of the space if A_0 and $f(t)$ are symmetric in the sense of Schwarz with respect to a point.

Example 5.5. In this example we discuss the case of a two dimensional radially symmetric membrane where analytic computations can be carried easily. Let $D = B(0,1) \subseteq \mathbb{R}^2$ and $A_0 = \emptyset$, that is the membrane is initially fully glued. The force we consider is

$$f : [0,+\infty) \times B(0,1) \to [0,+\infty), \quad f(t,x) = t \cdot 1_{B(0,1/2)}(x).$$

Before computing the minimizing movement, we notice that any time discrete solution (and hence the GMM solution) consists only on balls. This is a consequence of the structure of D and of the fact that the force f is symmetric in the sense of Schwarz. Since the family of balls is compact in the γ-convergence and the Lebesgue measure is continuous in this family, the lower semicontinuous regularization of the GMM is a rate independent movement.

Denoting by a the radius (depending on t) of the debonded part of the membrane and setting $H(r) = \int_0^r s1_{[0,1/2]}(s)ds$, one gets that

$$E\big(B(0,a), f(t)\big) = -\pi t^2 \int_0^a rH^2(r)dr.$$

In order to compute the rate independent movement, one has to find the solution of

$$\min_{a\in[0,1]} E(B(0,a), f(t)) + \pi a^2.$$

Simple computations lead to the following result

- for $0 \leq t \leq 4\sqrt{2}$ the functional above is increasing in a, hence the minimum is attained at $a = 0$ (the membrane remains glued);
- for $4\sqrt{2} \leq t \leq 4\sqrt{2}e^{1/4}$, the functional is not increasing in a, but the minimum is still attained at $a = 0$ (the membrane remains glued);
- for $\sqrt{2}e^{1/4} < t < 8\sqrt{2}$ the minimum is attained at $a = \frac{t}{8\sqrt{2}}$, hence there is a sudden debonding at $t = \sqrt{2}e^{1/4}$ followed by a continuous debonding up to $t = 8\sqrt{2}$;
- for $t \geq 8\sqrt{2}$ the membrane is fully debonded.

Open problem. We remark that in dimension 2, as a consequence of the Alt-Caffarelli result [2], for a smooth force f every domain $A_\varepsilon(t_n)$ in the Euler scheme is of class C^∞ on the free parts $\partial A_\varepsilon(t_n) \setminus \partial A_\varepsilon(t_n - \varepsilon)$. We do not know whether A_t is smooth provided that A_0 and f are smooth.

5.2. **Dissipation related to surface tension.** If instead of arbitrary quasi-open sets, one considers the family of convex subsets of D, endowed with the topology of the $w\gamma$-convergence (which in this case is equivalent to the Hausdorff complementary metric [8]) a natural dissipation distance is

$$\mathcal{D}_s(A_1, A_0) = |\mathcal{H}^{N-1}(\partial A_1) \setminus H^{N-1}(\partial A_0)| + \chi_{A_0 \subseteq A_1}.$$

At a discrete time step, one has to solve the minimization problem

$$\min \Big\{ E\big(A, f\big) + \mathcal{D}_s\big(A, A_0\big)\big) : A \text{ convex subset of } D \Big\}. \tag{5.6}$$

The question of obtaining for this problem a minimizing movement solution and a related stability and energy properties is a naural issue.

Open problem. Following [7], in any dimension of the space, every domain $A_\varepsilon(t_n)$ in the Euler scheme for problem (5.6) is of class C^1 on the free parts, provided that f is uniformly bounded. Again, we do not know if $A(t)$ is smooth provided that both A_0 and f are smooth.

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