Boundary behavior of viscous fluids: Influence of wall roughness and friction-driven boundary conditions

Dorin Bucur [*] Eduard Feireisl[†] Šárka Nečasová[‡]

Laboratoire de Mathématiques, CNRS UMR 5127 , Université de Savoie
Campus Scientifique, 73376 Le-Bourget-Du-Lac, France

Mathematical Institute of the Academy of Sciences of the Czech Republic
Žitná 25, 115 67 Praha 1, Czech Republic

Abstract

We consider a family of solutions to the evolutionary Navier-Stokes system supplemented with the complete slip boundary conditions on domains with rough boundaries. We give a complete description of the asymptotic limit by means of $\Gamma-$convergence arguments, and identify a general class of boundary conditions.

1 Introduction

The influence of wall roughness on the slip behavior of a viscous fluid has been discussed in several recent studies (see Priezjev et al. [23], [24]). In the case of *impermeable boundary*, the commonly accepted hypothesis reads

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0, \tag{1}$$

where $\mathbf{u}$ denotes the velocity of the fluid and $\mathbf{n}$ is the outer normal vector on the boundary of a spatial domain $\Omega \subset R^N$, $N = 2, 3$, occupied by the fluid.

The behavior of the tangential component $[\mathbf{u}]_{tan}$ is a more delicate issue. For many years, the *no-slip boundary condition*

$$\mathbf{u} = [\mathbf{u}]_{tan}|_{\partial\Omega} = 0 \tag{2}$$

has been the most widely used given its success in reproducing the standard velocity profiles for incompressible viscous fluids. Although the no-slip hypothesis seems to be in a

[*]The work of D.B. was supported by the Nečas Center for Mathematical Modelling LC06052 financed by MSMT, and partially by CNRS - AVČR project

[†]The work of E.F. was supported by Grant 201/08/0315 of GA ČR in the framework of the general research programme of the Academy of Sciences of the Czech Republic, Institutional Research Plan AV0Z10190503

[‡]The work of Š.N. supported by Grant IAA100190804 of GA AVČR in the framework of the general research programme of the Academy of Sciences of the Czech Republic, Institutional Research Plan AV0Z10190503,and partially by CNRS - AVČR project

good agreement with experiments, it leads to certain rather surprising conclusions, the most striking one being the absence of collisions of rigid objects immersed in a linearly viscous fluid (see Hesla [16], Hillairet [17]).

In contrast with (2), the so-called *Navier's boundary conditions*

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0, \ [S\mathbf{n}]_{tan} + \beta \mathbf{u}|_{\partial\Omega} = 0, \tag{3}$$

where S is the viscous stress tensor, offer more freedom and are likely to provide a physically acceptable solution at least to some of the paradoxical phenomena resulting from the no-slip boundary condition (see, for instance, Málek and Rajagopal [20], Moffat [21]).

There have been several attempts in the literature to provide a rigorous justification of the no-slip boundary conditions based on the idea that the physical boundary is never smooth but contains small asperities that drive the fluid to rest under the mere impermeability hypothesis (1). Results of this type have been shown by Casado-Diaz et al. [9] in the case of periodically distributed asperities and later extended to the general case in [7]. Conversely, Jäger and Mikelic [18], and Basson and Gérard-Varet [4] identified the Navier's slip boundary condition (3) as a suitable approximation of the behavior of a viscous fluid out of a boundary layer created by the *no-slip* condition imposed on a rough boundary. It is worth noting that these results are not contradictory but reflect two conceptually different approaches in mathematical modeling of viscous fluids. Recent developments in macrofluidic and nanofluidic technologies have renewed interest in the slip behavior that may become significant in the small spatial scales even for a relatively small Reynolds number (cf. Priezjev and Troian [24]). We notice that microscopic asperities may significantly change the solution of the equation with Navier boundary conditions,while the same asperities have only a mild effect on the solution of the same equation with no slip boundary conditions. This phenomenon may play a significant role in the control of the solution with respect to the shape of the domain, as the drag minimization problem, where large shape deformations can be associated to microscopic rugosity.

In the present study, we identify a general class of the so-called *friction-driven boundary conditions* enforced by the geometrical properties of the boundary. To fix ideas, consider a ball $D \subset R^N$, $N = 2, 3$, together with a family of domains $\Omega_\varepsilon \subseteq D$ satisfying the uniform cone condition (e.g. [15, Definition 2.4.1]). More specifically, given $\pi/2 > \omega > 0$, $h > 0$, let

$$C(x, \omega, h, \xi) = \{y \in R^N : \|y - x\| \leq h, (y - x, \xi) > \cos(\omega)\|y - x\|\}$$

be the cone with vertex at x, aperture 2ω, height h, and orientation given by a unit vector ξ. The uniform cone condition requires the existence of a fixed $\omega > 0$, $h > 0$ such that for any $\varepsilon > 0$, $x_0 \in \partial\Omega_\varepsilon$, there exists a unit vector $\xi_{x_0} \in R^N$ such that

$$C(x, \omega, h, \xi_{x_0}) \subseteq \Omega_\varepsilon \text{ whenever } x \in B(x_0, \omega) \cap \partial\overline{\Omega}_\varepsilon.$$

In addition, we assume that the family $\{\Omega_\varepsilon\}_{\varepsilon>0}$ converges to Ω in the sense that

$$1_{\Omega_\varepsilon} \to 1_\Omega \text{ in } L^1(D) \text{ as } \varepsilon \to 0$$

where 1_Ω is the characteristic function of Ω. As a direct consequence of the uniform cone condition, the sequence of domains converges also in the Hausdorff complementary topology, specifically,

$$d(\cdot, R^N \setminus \Omega_\varepsilon) \to d(\cdot, R^N \setminus \Omega) \text{ uniformly on } D,$$

where $d(\cdot, F)$ denotes the distance function to the set F.

We consider the *Navier-Stokes system* in the form

$$\partial_t \mathbf{u}_\varepsilon + \mathrm{div}(\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) + \nabla p_\varepsilon = \nu \mathrm{div}\mathbf{D}[\mathbf{u}_\varepsilon] + \mathbf{g}_\varepsilon,\ \nu > 0, \tag{4}$$

$$\mathrm{div}\mathbf{u}_\varepsilon = 0 \tag{5}$$

in $(0, T) \times \Omega_\varepsilon$, supplemented with the initial condition

$$\mathbf{u}_\varepsilon(0, \cdot) = \mathbf{u}_{0,\varepsilon} \text{ in } \Omega_\varepsilon, \tag{6}$$

and the *complete slip boundary conditions*

$$\mathbf{u}_\varepsilon \cdot \mathbf{n}|_{\partial\Omega_\varepsilon} = [\mathbf{D}[\mathbf{u}_\varepsilon] \cdot \mathbf{n}]_{tan}|_{\partial\Omega_\varepsilon} = 0, \tag{7}$$

where we have set

$$\mathbf{D}[\mathbf{u}_\varepsilon] = \frac{1}{2}(\nabla \mathbf{u}_\varepsilon + \nabla^t \mathbf{u}_\varepsilon).$$

Our aim is to investigate the asymptotic behavior of the solutions $\{\mathbf{u}_\varepsilon\}_{\varepsilon>0}$ for $\varepsilon \to 0$. It is easy to check that the functions $\mathbf{u}_\varepsilon$, suitably extended to the whole set D, converge to a limit $\mathbf{u}$ that satisfies equations (4), (5) in $(0, T) \times \Omega$ in the sense of distributions. The principal issue addressed in this study is identifying the boundary conditions satisfied by the limit function $\mathbf{u}$. We show that $\mathbf{u}$ satisfies the so-called *friction-driven* boundary conditions, where at each point of the boundary a certain (possibly empty) component of the velocity vanishes while its complementary part satisfies a kind of the Navier boundary conditions (3). More specifically, we find a family of vector spaces

$$\{V(x)\}_{x\in\partial\Omega},\ V(x) \perp \mathbf{n}(x)\ \ (V(x) \text{ is a subspace of the tangent plane})$$

such that $\mathbf{u}(x) \in V(x)$ for q.e. $x \in \partial\Omega$. Moreover, there exist a finite positive Borel measure μ, supported in $\partial\Omega$ and absolutely continuous with respect to the (newtonian) capacity, and a symmetric positively definite matrix $A = \{a_{i,j}\}_{i,j=1}^n$ of real valued Borel functions such that

$$[\mathbf{D}[\mathbf{u}](x) \cdot \mathbf{n}(x) + \mu A \mathbf{u}(x)] \cdot \mathbf{v}|_{\partial\Omega} = 0 \text{ for all } \mathbf{v} \in V(x).$$

Note that the complete slip boundary conditions (7) correspond to the choice $V \approx$ tangent space, $\mu \equiv 0$.

It turns out that the necessary piece of information in order to identify the asymptotic limit is contained in the leading terms of the associated *elliptic* part represented by the stationary *Stokes system*. As the latter possesses a variational structure, we take advantage of the methods based on $\Gamma-$convergence developed in the monograph by Dal Maso [10]. Specifically, our method relies on a representation theory for general C^1-convex and quadratic functionals in the spirit of Dal Maso et al. [11], [12]. Although the basic ingredients of the representation theory are formally the same as in [11], [12], there are two principal stumbling blocks to be handled in the present setting: (i) the variable underlying spatial domains, (ii) the incompressibility constraint (5).

Our approach is based on "artificial compressibility", where the unconstrained energy functional is perturbed by a singular term penalizing non-adiabatic motions of the fluid.

The $\Gamma-$limit is then identified in two steps by letting the singular parameter explode. Note that our technique and results are qualitatively different from those obtained by Ansini and Garroni [3], where the incompressibility constraint yields partial compactness of a (lower order) energy functional.

As already pointed out, the bulk of the paper is devoted to the analysis of the underlying *Stokes system* regarded as a family of variational problems on Ω_ε (see Section 2). We study the associated $\Gamma-$limit (see Section 3) and identify the asymptotic form of the boundary conditions in Section 4. Finally, the abstract theory is applied to the Navier-Stokes system in Section 5. Note that, unlike the Stokes problem, the Navier-Stokes system does not admit a variational structure, not even in the stationary case.

2 Stokes system

Given $\mathbf{f} \in L^2(D, R^N)$, $N = 2, 3$, we consider a (perturbed) Stokes problem in the form:

$$\left\{ \begin{array}{rcl} -\mathrm{div}\ \mathbf{D}[\mathbf{u}_\varepsilon] + \mathbf{u}_\varepsilon + \nabla p_\varepsilon & = & \mathbf{f} \text{ in } \Omega_\varepsilon \\ \mathrm{div}\ \mathbf{u}_\varepsilon & = & 0 \text{ in } \Omega_\varepsilon \\ \mathbf{u}_\varepsilon \cdot \mathbf{n} & = & 0 \text{ on } \partial\Omega_\varepsilon \\ (\mathbf{D}[\mathbf{u}_\varepsilon] \cdot \mathbf{n}_\varepsilon)_{tan} & = & 0 \text{ on } \partial\Omega_\varepsilon \end{array} \right\} \tag{8}$$

Problem (8) is understood in the variational sense, where the solutions are regarded as minimizers of the associated energy functional

$$\mathbf{v} \mapsto \frac{1}{2} \int_{\Omega_\varepsilon} \Big(|\mathbf{D}[\mathbf{v}]|^2 + |\mathbf{v}|^2 \Big) dx - \int_{\Omega_\varepsilon} \mathbf{f} \cdot \mathbf{v} dx \tag{9}$$

in the class

$$\mathbf{v} \in H^1(\Omega_\varepsilon; R^N),\ \mathrm{div}\ \mathbf{v} = 0 \text{ in } \Omega_\varepsilon,\ \mathbf{v} \cdot \mathbf{n}|_{\partial\Omega} = 0. \tag{10}$$

As a straightforward consequence of Korn's inequality and the classical Lax-Milgram theorem, problem (9) possesses a unique minimizer $\mathbf{u}_\varepsilon$ in the class (10). Moreover, since Korn's inequality holds uniformly (with the same constant) on all domains Ω_ε (see for instance [22]), there exists a constant M such that

$$\|\mathbf{u}_\varepsilon\|_{H^1(\Omega_\varepsilon, R^N)} \leq M.$$

Consequently, letting $\varepsilon \to 0$ we may infer that

$$\mathbf{u}_\varepsilon \to \mathbf{u} \text{ weakly in } W^{1,2}(D, R^N),$$

$$1_{\Omega_\varepsilon} \mathbf{u}_\varepsilon \to 1_\Omega\ \mathbf{u} \text{ (strongly) in } L^2(D; R^N),$$

and

$$1_{\Omega_\varepsilon} \nabla \mathbf{u}_\varepsilon \to 1_\Omega\ \nabla \mathbf{u} \text{ weakly in } L^2(D, R^{N \times N})$$

at least for a suitable subsequence, where we have tacitly assumed that the functions $\mathbf{u}_\varepsilon$ have been extended to the whole domain D. Indeed there is a family of uniformly bounded extension operators acting on $H^1(\Omega_\varepsilon)$ with values in $H^1(D)$. The remaining part of the above assertion follows from the compact embedding of $H^1(D)$ into $L^2(D)$ and the pointwise convergence of the characteristic functions of the domains.

Our aim is to identify the limit problem satisfied by the function $\mathbf{u}$ for any admissible right hand side $\mathbf{f}$ as the Γ-limit of the energy functional associated to the Stokes problem. For simplicity, we consider the functions $\mathbf{f} \in L^2(D, R^N)$, however, extension to larger spaces is straightforward (see Remark 4.4).

As already pointed out in the introductory part, our method leans on the representation theory of C^1-convex and quadratic functionals associated to vector valued functions. We refer the reader to [10] for a comprehensive introduction to the Γ-convergence theory, and to [11], [12] for the representation results concerning C^1-convex functionals.

We adopt the following strategy:

- We identify the $\overline{\Gamma}$-limit of a family of modified energy functionals, where a penalization is used to substitute for the incompressibility condition; it turns out that these functionals are C^1-convex.

- By means of a diagonalization procedure, we find the Γ-limit when the penalization grows to infinity. For the sake of clarity, this step is performed in a direct fashion while observing that the same result can be obtained as a direct consequence of metrizability of the Γ-convergence.

- We extend the Γ-limit result to the subspaces of divergenceless (solenoidal) functions by using uniform estimates obtained by application of the so-called Bogovskii operator (inverse divergence). Here, the energy functionals restricted to solenoidal fields are no longer C^1-convex.

- We find the limit and identitfy the boundary conditions for the sequence of solutions to the Stokes problem.

To begin, we introduce the penalized energy functionals. Let $\mathcal{A}(D)$ denote the family of all open subsets of D. For a fixed $\lambda > 0$, we define

$$E_\varepsilon^\lambda : H^1(D; R^N) \times \mathcal{A}(D) \to R \cup \{\infty\},$$

$$E_\varepsilon^\lambda(\mathbf{v}, A) = F_\varepsilon^\lambda(\mathbf{v}, A) + G_\varepsilon^\lambda(\mathbf{v}, A), \tag{11}$$

where

$$F_\varepsilon^\lambda(\mathbf{v}, A) = \int_{A\cap\Omega_\varepsilon} \Big(|\mathbf{D}[\mathbf{v}]|^2 + |\mathbf{v}|^2\Big)dx + \lambda \int_{A\cap\Omega_\varepsilon} |\mathrm{div}\ \mathbf{v}|^2 dx,$$

$$G_\varepsilon^\lambda(\mathbf{v}, A) = \left\{ \begin{array}{l} 0 \text{ if } \mathbf{v}\cdot\mathbf{n}|_{\partial\Omega_\varepsilon\cap A} = 0 \\ \infty \text{ otherwise.} \end{array} \right.$$

For reader's convenience, we recall the definition of $\overline{\Gamma}$-convergence introduced in [10, Definitions 16.2 and 15.5].

Definition 2.1 *A sequence of functionals $\{E_\varepsilon^\lambda\}_{\varepsilon>0}$ $\overline{\Gamma}$-converges to E^λ for $\varepsilon \to 0$ if E^λ is the inner regular envelope of both $\Gamma - \liminf_{\varepsilon\to 0} E_\varepsilon^\lambda(\cdot, A)$ and $\Gamma - \limsup_{\varepsilon\to 0} E_\varepsilon^\lambda(\cdot, A)$ on $H^1(D, R^N) \times \mathcal{A}(D)$, where the $\Gamma-$limits are understood in the topology of $L^2(D, R^N)$.*

Specifically, for any open set $A \in \mathcal{A}(D)$

$$E^\lambda(\cdot, A) = \sup_B \{\Gamma - \liminf_{\varepsilon \to 0} E^\lambda_\varepsilon(\cdot, B)\} = \sup_B \{\Gamma - \limsup_{\varepsilon \to 0} E^\lambda_\varepsilon(\cdot, B)\},$$

where the supremum is taken over the family of sets $B \in \mathcal{A}(D)$ such that $\overline{B} \subseteq A$. We notice that $E^\lambda_\varepsilon, E^\lambda$ are nondecreasing with respect to the set inclusion for a fixed $\mathbf{u}$ and lower semi-continuous in $L^2(D, R^N)$ for any fixed set A.

By virtue of [12, Proposition 1.4], the $\overline{\Gamma}-$convergence of the sequence E^λ_ε to E^λ is equivalent to the existence of a rich set $\mathcal{R}_\lambda \subseteq \mathcal{A}(D)$ such that

$$E^\lambda_\varepsilon(\cdot, A) \xrightarrow{\Gamma} E^\lambda(\cdot, A) \text{ in } L^2(D) \text{ as } \varepsilon \to 0 \text{ for any } A \in \mathcal{R}_\lambda. \tag{12}$$

We report the following result that may be regarded as a direct consequence of [10, Theorems 16.9 and 8.5].

Theorem 2.2 *The sequence $\{E^\lambda_\varepsilon\}_{\varepsilon>0}$ possesses a subsequence that $\overline{\Gamma}$-converges in $L^2(D, R^N)$.*

Denoting by E^λ the $\overline{\Gamma}-$limit of the family $\{E^\lambda_\varepsilon\}_{\varepsilon>0}$, we may write, at least formally,

$$E^\lambda(\mathbf{v}, A) = F^\lambda(\mathbf{v}, A) + G^\lambda(\mathbf{v}, A) \text{ for any } A \in \mathcal{A}(D), \tag{13}$$

where

$$F^\lambda(\mathbf{v}, A) = \int_{A\cap\Omega} \Big(|D[\mathbf{v}]|^2 + |\mathbf{v}|^2\Big) dx + \lambda \int_{A\cap\Omega} |\text{div } \mathbf{v}|^2 dx.$$

If the value $F^\lambda(\mathbf{v}, A)$ is finite for some $\mathbf{v} \in L^2(D, R^N)$ and $A \in \mathcal{A}(D)$, then $G^\lambda(\mathbf{v}, A)$ can be identified with the difference $E^\lambda(\mathbf{v}, A) - F^\lambda(\mathbf{v}, A)$. Otherwise we put $G^\lambda(\mathbf{v}, A) = \infty$. Our goal is to find a representation formula for $G^\lambda(\mathbf{v}, A)$. In the context of problem (8), the functional G^λ captures the behavior of weak limits of the sequence $\{\mathbf{u}_\varepsilon\}_{\varepsilon>0}$ at the boundary and plays a crucial role in the reconstruction of the limit boundary conditions satisfied by $\mathbf{u}$.

3 $\overline{\Gamma}$-convergence of C^1-convex functionals

Our next goal is to express the limit functional $G^\lambda : H^1(D, R^N) \times \mathcal{A}(D) \to [0, \infty]$ in terms of the representation formula obtained in [11, Corollary 7.4] (see also [12, 13]). To this end, we have to verify that G^λ enjoys the following properties:

(P1) (*lower semi-continuity*) The mapping $\mathbf{v} \mapsto G^\lambda(\mathbf{v}, A)$ is lower semi-continuous in $H^1(D, R^N)$ for all $A \in \mathcal{A}(D)$.

(P2) (*measure property*) The mapping

$$\mathcal{A}(D) \ni A \mapsto G^\lambda(\mathbf{v}, A)$$

is a trace of a Borel measure for all $\mathbf{v} \in H^1(D, R^N)$.

(P3) (*locality property*) For all $\mathbf{u}, \mathbf{v} \in H^1(D, R^N)$, $A \in \mathcal{A}(D)$, we have

$$G^\lambda(\mathbf{u}, A) = G^\lambda(\mathbf{v}, A).$$

whenever $\mathbf{u}|_A = \mathbf{v}|_A$.

(P4) (C^1-*convexity*) For any given $A \in \mathcal{A}(D)$, the function

$$H^1(D, R^N) \ni \mathbf{v} \mapsto G^\lambda(\mathbf{v}, A)$$

is convex in $H^1(D, R^N)$. Moreover, we have

$$G^\lambda(\varphi \mathbf{u} + (1-\varphi)\mathbf{v}, A) \leq G^\lambda(\mathbf{u}, A) + G^\lambda(\mathbf{v}, A)$$

for all $\varphi \in C^1(D) \cap W^{1,\infty}(D)$, $0 \leq \varphi \leq 1$ in D and for all $\mathbf{u}, \mathbf{v} \in H^1(D, R^N)$.

(P5) (*quadraticity*) For any fixed set $A \in \mathcal{A}(D)$, the functional $H^1(D, R^N) \ni \mathbf{v} \mapsto G^\lambda(\mathbf{v}, A) \in [0, \infty]$ is quadratic, i.e., there exists a linear subspace $Y_A \subseteq H^1(D, R^N)$ and a symmetric bilinear form $B_A : H^1(D, R^N) \times H^1(D, R^N) \to R$ such that

$$G^\lambda(\mathbf{v}, A) = \begin{cases} B_A(\mathbf{v}, \mathbf{v}) & \text{for all } \mathbf{v} \in Y_A, \\ \infty & \text{for all } \mathbf{v} \in H^1(D, R^N) \setminus Y_A. \end{cases}$$

According to [11, Corollary 7.4], if the functional $G^\lambda : H^1(D, R^N) \times \mathcal{A}(D) \to [0, \infty]$ satisfies **(P1 - P5)**, then there exist (i) a finite Borel measure μ on D absolutely continuous with respect to the (newtonian) capacity; (ii) a symmetric matrix $A = \{a_{i,j}\}_{i,j=1}^N$ of Borel functions $a_{i,j} : D \to R$ such that

$$\sum_{i,j=1}^N a_{i,j}(x)\xi_i\xi_j \geq 0 \text{ for (quasi) any } x \in D, \quad \text{for all } \xi \in R^N;$$

(iii) a family of linear vector spaces $\{V(x)\}_{x \in D} \subseteq R^N$ with the following properties:

- if $G^\lambda(\mathbf{v}, A) < \infty$, then $\mathbf{v}(x) \in V(x)$ for q.a. $x \in A$;
- if $\mathbf{v} \in H^1(D, R^N)$, $A \in \mathcal{A}(D)$, and $\mathbf{v}(x) \in V(x)$ for q.a. $x \in A$, then $G^\lambda(\mathbf{v}, A) < \infty$, and, moreover,

$$G^\lambda(\mathbf{v}, A) = \int_A \sum_{i,j=1}^N a_{ij}(x)\mathbf{v}_i(x)\mathbf{v}_j(x)d\mu. \tag{14}$$

Remark 3.1 Theorem 2.2, together with the properties **(P1 - P5)**, were used in [12] and [13] to represent the Γ-limits for the elasticity problems on varying regions with the homogeneous Dirichlet boundary conditions. In the present context, however, several new difficulties appear:

- we consider Navier's slip boundary conditions involving oscillatory normal fields;

- we consider Stokes problem including the incompressibility constraint which *does not obey* the C^1-convexity property;

- the main energy term in the varying functionals is variable, being summed over a variable domain;

- our target problem - the Navier-Stokes system - is not of variational type.

Remark 3.2 In the present problem, we show that the measure μ is concentrated on $\partial\Omega$ so that the functional G^λ determines the limit boundary conditions through the Euler equation associated to the energy functional. Specifically, the constraint $\mathbf{v}(x) \in V(x)$ yields the *driven* part of the boundary conditions while the boundary integral (14) represents a *friction* term.

Remark 3.3 In any case, we get $\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0$ for the limit problem, in other words, the space $V(x)$ lies in the tangent space for any $x \in \partial\Omega$. Note that the friction term may vanish even if the spaces $V(x)$ are non-trivial as in the case of riblet-like boundaries studied in [8].

Remark 3.4 We point out that a family of spaces $\{V(x)\}_{x\in\partial\Omega}$ of nonzero dimension may still give rise to the homogeneous *no-slip* boundary condition. This is a consequence of the quasi (and fine) continuity of H^1-functions, namely, the *no-slip* boundary conditions appear whenever the mapping $\partial\Omega \ni x \mapsto V(x)$ is *severely discontinuous* in a suitable sense.

The first result of this paper reads as follows.

Theorem 3.5 *The functional G^λ identified in* (13) *satisfies* **(P1 - P5)**.

Proof We follow the principal ideas of [12, Proposition 3.1], where the homogeneous Dirichlet boundary conditions are considered on variable domains.

First of all, we notice that G^λ is non-negative. Indeed if $A \in \mathcal{R}_\lambda$ (cf. (12)), $\mathbf{v}_\varepsilon \to \mathbf{v}$ in $L^2(D, R^N)$, where $\mathbf{v}_\varepsilon, \mathbf{v} \in H^1(D, R^N)$, then

$$F^\lambda(\mathbf{v}, A) \le \liminf_{\varepsilon\to 0} F^\lambda_\varepsilon(\mathbf{v}_\varepsilon, A).$$

Accordingly, the desired conclusion follows for any $A \in \mathcal{R}_\lambda$, and, by means of inner regularity, for any $A \in \mathcal{A}(D)$.

Proof of (P1)

Consider $A \in \mathcal{A}(D)$. The functional E^λ, being a $\overline{\Gamma}$-limit, is lower semi-continuous in $L^2(D, R^N)$ (see for instance [12, Remark 1.3]). Moreover, the F^λ-component of E^λ is continuous in $H^1(D, R^N)$; whence $\mathbf{v} \mapsto G^\lambda(\mathbf{v}, A)$ is lower semi-continuous in $H^1(D, R^N)$.

Proof of (P2)

According to [10, Theorem 18.5], $E^\lambda(\mathbf{v}, \cdot)$ is a measure provided it satisfies the so-called *uniform fundamental estimate* (see below). In addition, since the F^λ-part of E^λ is a measure, and, moreover, G^λ is positive, we conclude that G^λ is a measure.

The *uniform fundamental estimate* mentioned above (see [10, Definition 18.2]) reads as follows:

Given $\delta > 0$ and a trio of open sets $A', A'', B \in \mathcal{A}(D)$, $\overline{A}' \subseteq A''$, there exists $M > 0$ with the following property: for any $\mathbf{u}, \mathbf{v} \in L^2(D, R^N)$ there exists $\varphi \in C_0^\infty(A'', [0,1])$, $\varphi = 1$ in a neighborhood of A', such that

$$E_\varepsilon^\lambda\Big(\varphi\mathbf{u} + (1-\varphi)\mathbf{v}, A' \cup B\Big) \leq$$

$$(1+\delta)\Big(E_\varepsilon^\lambda(\mathbf{u}, A'') + E_\varepsilon^\lambda(\mathbf{v}, B)\Big) + \delta\Big(\|\mathbf{u}\|_{L^2(S,R^N)} + \|\mathbf{v}\|_{L^2(S,R^N)} + 1\Big) + M\|\mathbf{u} - \mathbf{v}\|_{L^2(S,R^N)},$$

where $S = (A'' \setminus A') \cap B$. Here $M = M(\delta, A', A'', B)$ while φ may depend on E_ε^λ, $\mathbf{u}$, $\mathbf{v}$.

In order to verify the uniform fundamental estimate, and, to identify the constant M, together with the function φ, we may assume that both $E_\varepsilon^\lambda(\mathbf{u}, A'')$ and $E_\varepsilon^\lambda(\mathbf{v}, B)$ are finite. Observe that the value of $E_\varepsilon^\lambda(\varphi\mathbf{u} + (1-\varphi)\mathbf{v}, A' \cup B)$ is also finite. Indeed suppose that $x \in (A' \cup B) \cap \partial\Omega_\varepsilon$. Since $\mathbf{u} \cdot \mathbf{n}|_{A' \cap \partial\Omega_\varepsilon} = 0$ and $\varphi|_{A'} = 1$, we have

$$(\varphi(x)\mathbf{u}(x) + (1-\varphi(x))\mathbf{v}(x)) \cdot \mathbf{n}(x) = 0 \text{ whenever } x \in A' \cap \partial\Omega_\varepsilon.$$

On the other hand, if $x \in (B \setminus A') \cap \partial\Omega_\varepsilon$, we evaluate $\varphi(x)\mathbf{u}(x) \cdot \mathbf{n}(x)$ using the fact that $\mathbf{v} \cdot \mathbf{n}|_{(B\setminus A')\cap\partial\Omega_\varepsilon} = 0$ and noticing that if $x \in (A'' \setminus A') \cap \partial\Omega_\varepsilon$, then $\mathbf{u}(x) \cdot \mathbf{n}(x) = 0$, while if $x \in (B \setminus A'') \cap \partial\Omega_\varepsilon$, then $\varphi(x) = 0$.

Thus it remains to show the fundamental estimate for the integral part. As the proof is trivial for the L^2-norm component, we focus on the gradient part that may be handled exactly as in [10, Chapter 18]:

$$\int_{(A'\cup B)\cap\Omega_\varepsilon} |\mathbf{D}[\varphi\mathbf{u} + (1-\varphi)\mathbf{v}]|^2 dx + \lambda \int_{(A'\cup B)\cap\Omega_\varepsilon} |\mathrm{div}\ (\varphi\mathbf{u} + (1-\varphi)\mathbf{v})|^2 dx \leq$$

$$\frac{1}{1-\delta}\Big(\int_{A''\cap\Omega_\varepsilon} |\mathbf{D}[\mathbf{u}]|^2 dx + \lambda \int_{A''\cap\Omega_\varepsilon} |\mathrm{div}\ \mathbf{u}|^2 dx + \int_{B\cap\Omega_\varepsilon} |\mathbf{D}[\mathbf{v}]|^2 dx + \lambda \int_{B\cap\Omega_\varepsilon} |\mathrm{div}\ \mathbf{v}|^2 dx\Big) +$$
$$\frac{(1+\lambda)\max|\nabla\varphi|^2}{\delta} \int_{(A''\setminus A')\cap B\cap\Omega_\varepsilon} |\mathbf{u} - \mathbf{v}|^2 dx.$$

Proof of (P3)

Assume that $A \in \mathcal{A}(D)$ and $\mathbf{u}|_A = \mathbf{v}|_A$. In order to prove that

$$E^\lambda(\mathbf{u}, A) \leq E^\lambda(\mathbf{v}, A)$$

it is enough to show

$$\tilde{E}^\lambda(\mathbf{u}, A_1) \leq \tilde{E}^\lambda(\mathbf{v}, A_2) \text{ for all } A_1 \subset \overline{A}_1 \subset A_2 \subset \overline{A}_2 \subset A, \tag{15}$$

where $\tilde{E}^\lambda(\cdot, C) = \Gamma - \liminf_{\varepsilon\to 0} E_\varepsilon^\lambda(\cdot, C)$.

Consider $\mathbf{v}_\varepsilon \to \mathbf{v}$ in $L^2(D, R^N)$ such that

$$\tilde{E}^\lambda(\mathbf{v}, A_2) = \liminf_{\varepsilon\to 0} E_\varepsilon^\lambda(\mathbf{v}_\varepsilon, A_2).$$

Let $\varphi \in C_0^1(A_2,[0,1])$, $\varphi = 1$ on A_1. Put

$$\mathbf{u}_\varepsilon = \varphi \mathbf{v}_\varepsilon + (1-\varphi)\mathbf{u}$$

so that $\mathbf{u}_\varepsilon$ converges to $\varphi\mathbf{v} + (1-\varphi)\mathbf{u}$ in $L^2(D,R^N)$ which is equal to $\mathbf{v} = \mathbf{u}$ on A_1.

Consequently,

$$\tilde{E}^\lambda(\mathbf{u}, A_1) \le \liminf_{\varepsilon\to 0} E_\varepsilon^\lambda(\mathbf{u}_\varepsilon, A_1) =$$

$$\liminf_{\varepsilon\to 0} E_\varepsilon^\lambda(\mathbf{v}_\varepsilon, A_1) \le \liminf_{\varepsilon\to 0} E_\varepsilon^\lambda(\mathbf{v}_\varepsilon, A_2) = \tilde{E}^\lambda(\mathbf{v}, A_2).$$

Using the inner regularity of E^λ (see [12]) we get $E^\lambda(\mathbf{u}, A) \le E^\lambda(\mathbf{v}, A)$. The opposite inequality is obtained by changing $\mathbf{u}$ and $\mathbf{v}$.

Proof of (P4)

It is enough to prove the C^1-convexity property on smooth sets $A \in \mathcal{R}_\lambda$ and to use the inner regularity of E^λ. We start by proving the convexity of the mapping

$$\mathbf{v} \mapsto G^\lambda(\mathbf{v}, A).$$

Consider $\mathbf{u}, \mathbf{v} \in H^1(D,R^N)$ satisfying $G^\lambda(\mathbf{u},A), G^\lambda(\mathbf{v},A) < \infty$. There exist $\mathbf{u}_\varepsilon, \mathbf{v}_\varepsilon \in H^1(D,R^N)$

$$\mathbf{u}_\varepsilon \to \mathbf{u},\ \ \mathbf{v}_\varepsilon \to \mathbf{v} \text{ (strongly) in } L^2(D,R^N)$$

such that

$$E_\varepsilon^\lambda(\mathbf{u}_\varepsilon, A) \to E^\lambda(\mathbf{u}, A),\ \ E_\varepsilon^\lambda(\mathbf{v}_\varepsilon, A) \to E^\lambda(\mathbf{v}, A).$$

Consequently, $G_\varepsilon^\lambda(\mathbf{u}_\varepsilon, A) = G_\varepsilon^\lambda(\mathbf{v}_\varepsilon, A) = 0$, and

$$1_{\Omega_\varepsilon}\nabla\mathbf{u}_\varepsilon \to 1_\Omega\nabla\mathbf{u}, \quad 1_{\Omega_\varepsilon}\nabla\mathbf{v}_\varepsilon \to 1_\Omega\nabla\mathbf{v} \text{ weakly in } L^2(D,R^{N\times N}).$$

Following [12, Theorem 4.1] we can write

$$G^\lambda(\varphi\mathbf{u} + (1-\varphi)\mathbf{v}, A) \le \lim_{\varepsilon\to 0}[F_\varepsilon^\lambda(\varphi\mathbf{u}_\varepsilon + (1-\varphi)\mathbf{v}_\varepsilon, A) - F^\lambda(\varphi\mathbf{u} + (1-\varphi)\mathbf{v}, A)] =$$

$$\lim_{\varepsilon\to 0}\int_{A\cap\Omega_\varepsilon}\Big|\varphi\mathbf{D}[\mathbf{u}_\varepsilon - \mathbf{u}] + (1-\varphi)\mathbf{D}[\mathbf{v}_\varepsilon - \mathbf{v}]\Big|^2 dx +$$

$$\lambda\lim_{\varepsilon\to 0}\int_{A\cap\Omega_\varepsilon}\Big|\varphi\text{div }(\mathbf{u}_\varepsilon - \mathbf{u}) + (1-\varphi)\text{div }(\mathbf{v}_\varepsilon - \mathbf{v})\Big|^2 dx := L.$$

In order to prove convexity, assume that $0 \le \varphi \le 1$ is a constant. Consequently,

$$L \le \varphi\limsup_{\varepsilon\to 0}\Big(\int_{A\cap\Omega_\varepsilon}|\mathbf{D}[\mathbf{u}_\varepsilon - \mathbf{u}]|^2 dx + \lambda\int_{A\cap\Omega_\varepsilon}|\text{div }(\mathbf{u}_\varepsilon - \mathbf{u})|^2 dx\Big) +$$

$$(1-\varphi)\limsup_{\varepsilon\to 0}\Big(\int_{A\cap\Omega_\varepsilon}|\mathbf{D}[\mathbf{v}_\varepsilon - \mathbf{v}]|^2 dx + \lambda\int_{A\cap\Omega_\varepsilon}|\text{div }(\mathbf{v}_\varepsilon - \mathbf{v})|^2 dx\Big) =$$

$$\varphi\limsup_{\varepsilon\to 0}\Big(\int_{A\cap\Omega_\varepsilon}|\mathbf{D}[\mathbf{u}_\varepsilon]|^2 dx + \lambda\int_{A\cap\Omega_\varepsilon}|\text{div }\mathbf{u}_\varepsilon|^2 dx\Big) - \varphi\Big(\int_{A\cap\Omega}|D[\mathbf{u}]|^2 dx + \lambda\int_{A\cap\Omega}|\text{div }\mathbf{u}|^2 dx\Big) +$$

$$(1-\varphi)\limsup_{\varepsilon\to 0}\Big(\int_{A\cap\Omega_\varepsilon}|\mathbf{D}[\mathbf{v}_\varepsilon]|^2dx+\lambda\int_{A\cap\Omega_\varepsilon}|\text{div }\mathbf{v}_\varepsilon|^2dx\Big)-$$

$$(1-\varphi)\Big(\int_{A\cap\Omega}|\mathbf{D}[\mathbf{v}]|^2dx+\lambda\int_{A\cap\Omega}|\text{div }\mathbf{v}|^2dx\Big)=\varphi G^\lambda(\mathbf{u},A)+(1-\varphi)G^\lambda(\mathbf{v},A).$$

Finally, to show C^1-convexity, take $\varphi\in C^1(D,[0,1])$. Keeping the previous notation we obtain

$$L\le\limsup_{\varepsilon\to 0}\Big(\int_{A\cap\Omega_\varepsilon}|\mathbf{D}[\mathbf{u}_\varepsilon-\mathbf{u}]|^2dx+\lambda\int_{A\cap\Omega_\varepsilon}|\text{div }(\mathbf{u}_\varepsilon-\mathbf{u})|^2dx\Big)+$$

$$\limsup_{\varepsilon\to 0}\Big(\int_{A\cap\Omega_\varepsilon}|\mathbf{D}[\mathbf{v}_\varepsilon-\mathbf{v}]|^2dx+\lambda\int_{A\cap\Omega_\varepsilon}|\text{div }(\mathbf{v}_\varepsilon-\mathbf{v})|^2dx\Big)=$$

$$G^\lambda(\mathbf{u},A)+G^\lambda(\mathbf{v},A).$$

Proof of (P5)

From the basic properties of $\overline{\Gamma}$-convergence we may infer that G^λ is nonnegative. Moreover, it follows from [10, Theorem 11.10] that G^λ is quadratic. Indeed E^λ is quadratic as a Γ-limit of quadratic functionals, and so is G^λ.

□

Remark 3.6 For $\mathbf{u}\in C_c^\infty(\Omega,R^N)$ we get $F_\varepsilon^\lambda(\mathbf{u},A)\to F^\lambda(\mathbf{u},A)$ hence $G^\lambda(\mathbf{u},A)=0$. Moreover, the same conclusion holds if $\mathbf{u}\in C_c^\infty(D\setminus\overline{\Omega},R^N)$. In such a case,

$$\int_A\sum_{i,j=1}^N a_{i,j}u_iu_j\ d\mu=0;$$

whence we may assume that the measure μ vanishes on the set $D\setminus\partial\Omega$ without changing the functional G^λ. In particular, the support of μ is concentrated in $\partial\Omega$. Finally, we remark that it is only the behavior $a_{i,j}$ on $\partial\Omega$ that can influence the value of G^λ.

4 Analysis of Stokes system with friction-driven boundary conditions

Let μ be a finite positive Borel measure concentrated on $\partial\Omega$ and absolutely continuous with respect to capacity. In addition, we are given a family of linear spaces $\mathcal{V}:=\{V(x)\}_{x\in\partial\Omega}$, where $V(x)$ is a subspace of the tangent hyperplane at $x\in\partial\Omega$, in particular, the dimension of $V(x)$ does not exceed $N-1$. Furtermore, let $a_{i,j}:\partial\Omega\to R$, $1\le i,j\le N$, be Borel functions such that $a_{i,j}=a_{j,i}$, and $\sum_{i,j=1}^N a_{ij}\xi_i\xi_j\ge 0$ for all $\xi\in R^N$. We set $A=\{a_{i,j}\}_{i,j=1}^N$. Finally, we take $\mathbf{f}\in L^2(D,R^N)$.

We say that $\mathbf{u}$ is a solution to Stokes system with friction-driven boundary conditions determined by means of the trio $\{\mu,A,\mathcal{V}\}$ if $\mathbf{u}$ solves the minimization problem

$$\mathcal{J}(\mathbf{u})=\min_{\mathbf{v}\in\mathcal{C}}\mathcal{J}(\mathbf{v}),\tag{16}$$

where

$$\mathcal{J}(\mathbf{v}) := \frac{1}{2}\int_\Omega \Big(|\mathbf{D}[\mathbf{v}]|^2 + |\mathbf{v}|^2\Big)dx + \frac{1}{2}\int_{\partial\Omega} A\mathbf{v}\cdot\mathbf{v}d\mu - \int_\Omega \mathbf{f}\cdot\mathbf{v}dx, \tag{17}$$

and

$$\mathcal{C} := \Big\{\mathbf{v}\in H^1(\Omega, R^N)\ \Big|\ \text{div}\ \mathbf{v} = 0 \text{ in } \Omega,\ \mathbf{v}(x)\in V(x) \text{ for q. a. } x\in\partial\Omega\Big\}.$$

As $\mathcal{C}$ is a closed subspace of $H^1(\Omega, R^N)$, the classical Lax-Milgram theorem together with Korn's inequality yield the following result.

Theorem 4.1 *Problem* (16) *has a unique solution.*

Remark 4.2 Writing the Euler equation associated to the minimization problem (16) we *formally* get

$$\begin{cases} -\text{div}\ \mathbf{D}[\mathbf{u}] + \mathbf{u} + \nabla p &= \mathbf{f} \text{ in } \Omega \\ \text{div}\ \mathbf{u} &= 0 \text{ in } \Omega \\ \mathbf{u}(x) &\in V(x) \text{ for q.e. } x\in\partial\Omega \\ \Big[\mathbf{D}[\mathbf{u}]\cdot\mathbf{n} + \mu A\mathbf{u}\Big]\cdot\mathbf{v} &= 0 \text{ for any } \mathbf{v}\in V(x),\ x\in\partial\Omega. \end{cases} \tag{18}$$

The *driven* part of the boundary conditions is given by the family of spaces $\{V(x)\}_{x\in\partial\Omega}$ while the *friction* part is determined by the matrix $A = \{a_{i,j}\}_{i,j=1}^N$ and the measure μ.

Here is the main result of the paper.

Theorem 4.3 *Let Ω_ε converge to Ω in the sense specified in Section 2, and let $\mathbf{f}\in L^2(D, R^N)$ be given. Let $\{\mathbf{u}_\varepsilon\}_{\varepsilon>0}$ be the family of (weak) solutions to problem* (8) *in Ω_ε.*

Then, at least for a suitable subsequence,

$$1_{\Omega_\varepsilon}\mathbf{u}_\varepsilon \to 1_\Omega\mathbf{u} \textit{ (strongly) in } L^2(D, R^N),$$

$$1_{\Omega_\varepsilon}\nabla\mathbf{u}_\varepsilon \to 1_\Omega\nabla\mathbf{u} \textit{ weakly in } L^2(D, R^{N\times N}),$$

where $\mathbf{u}$ is a solution of the minimization problem (16), or, equivalently, a weak solution of (18) in Ω, for a suitable trio $\{\mu, A, \mathcal{V}\}$ independent of the driving force $\mathbf{f}$.

A remarkable fact is that the boundary conditions for the limit problem are *independent* of the right-hand side $\mathbf{f}$; they are related only to the boundary variations of solutions, and, of course, to the specific choice of the differential operator supplemented with the incompressibility constraint. Moreover, the same result still holds if the driving force $\mathbf{f} = \mathbf{f}_\varepsilon$ is allowed to vary with ε provided the sequence $\{\mathbf{f}_\varepsilon\}_{\varepsilon>0}$ belongs to the dual space $[H^1(\Omega_\varepsilon, R^N)]'$ and is precompact in a suitable sense (see Remark 4.4).

Proof (i) For a fixed $\lambda > 0$ we consider the functionals E_ε^λ introduced in (11). In accordance with Theorem 2.2, we may assume that E_ε^λ $\overline{\Gamma}$-converge in $L^2(D, R^N)$ to some functional E^λ as $\varepsilon\to 0$, where the limit can be written in the form

$$E^\lambda(\mathbf{v}, A) = F^\lambda(\mathbf{v}, A) + G^\lambda(\mathbf{v}, A)$$

with

$$F^{\lambda}(\mathbf{v}, A) = \int_{A\cap\Omega} \Big(|D\mathbf{v}|^2 + |\mathbf{v}|^2 \Big) dx + \lambda \int_{A\cap\Omega} |\mathrm{div}\ \mathbf{v}|^2 . dx$$

Moreover, G^{λ} satisfies conditions **(P1 - P5)**.

(ii) Next, we consider sequences $\{\varepsilon_m\}_{m=1}^{\infty}$, $\{\lambda_m\}_{m=1}^{\infty}$, $\varepsilon_m \searrow 0$, $\lambda_m \nearrow \infty$, and use the diagonalization method to identify the Γ-limit of the sequence of functionals $\{E_{\varepsilon_m}^{\lambda_m}(\cdot, D)\}_{m>0}$. Extracting subsequences and relabeling indices, we may assume that

$$E_{\varepsilon_n}^{\lambda_m} \xrightarrow{\overline{\Gamma}} E^{\lambda_m} := F^{\lambda_m} + G^{\lambda_m} \text{ as } n \to \infty$$

for any fixed m.

(iii) Now, observe that

$$G^{\lambda}(\mathbf{v}, A) \leq G^{\lambda'}(\mathbf{v}, A) \text{ whenever } \lambda < \lambda' \tag{19}$$

for all $\mathbf{v} \in H^1(\Omega, R^N)$, $A \in \mathcal{A}(D)$. Indeed consider a sequence $\mathbf{v}_n \to \mathbf{v}$ strongly in $L^2(D, R^N)$, together with a (smooth) set $A \in \mathcal{R}_{\lambda'}$, such that

$$E_{\varepsilon_n}^{\lambda'}(\mathbf{v}_n, A) \to E^{\lambda'}(\mathbf{v}, A) < \infty.$$

Seeing that

$$\liminf_{n\to\infty} E_{\varepsilon_n}^{\lambda}(\mathbf{v}_n, A) \geq E^{\lambda}(\mathbf{v}, A)$$

we obtain

$$(\lambda' - \lambda) \limsup_{n\to\infty} \int_{\Omega_{\varepsilon_n}\cap A} |\mathrm{div}\ \mathbf{v}_n|^2 dx \leq (\lambda' - \lambda) \int_{\Omega\cap A} |\mathrm{div}\ \mathbf{v}|^2 dx + G^{\lambda'}(\mathbf{v}, A) - G^{\lambda}(\mathbf{v}, A). \tag{20}$$

Since

$$\mathbf{v}_n \cdot \mathbf{n}|_{\partial\Omega_{\varepsilon}\cap A} = \mathbf{v} \cdot \mathbf{n}|_{\partial\Omega\cap A} = 0$$

we have

$$\int_{\Omega_{\varepsilon_n}\cap A} |\mathrm{div}\ \mathbf{v}_n|^2 dx = \int_A |\mathrm{div}\ \tilde{\mathbf{v}}_n|^2 dx, \ \int_{\Omega\cap A} |\mathrm{div}\ \mathbf{v}|^2 dx = \int_A |\mathrm{div}\ \tilde{\mathbf{v}}|^2 dx,$$

where

$$\tilde{\mathbf{v}}_n = \left\{ \begin{array}{l} \mathbf{v}_n \text{ in } \Omega_{\varepsilon_n}, \\ 0 \text{ in } D \setminus \Omega_{\varepsilon_n}, \end{array} \right. , \ \tilde{\mathbf{v}} = \left\{ \begin{array}{l} \mathbf{v} \text{ in } \Omega, \\ 0 \text{ in } D \setminus \Omega. \end{array} \right.$$

Consequently, relation (20) implies (19) at least for any $A \in \mathcal{R}_{\lambda'}$, and, by virtue of the inner regularity property, for any $A \in \mathcal{A}(D)$.

As a byproduct of the previous considerations, we have $V^{\lambda'}(x) \subseteq V^{\lambda}(x)$ for q.a. $x \in \partial\Omega$ as the space $V^{\lambda'}(x)$ can be replaced by $V^{\lambda'}(x) \cap V^{\lambda}(x)$ without changing the functional $G^{\lambda'}$.

(iv) At this stage, we introduce

$$G(\mathbf{v}, A) := \lim_{m\to\infty} G^{\lambda_m}(\mathbf{v}, A) = \sup_m G^{\lambda_m}(\mathbf{v}, A), \tag{21}$$

together with

$$V(x) = \cap_m V_{\lambda_m}(x) \text{ for q.a. } x \in \partial\Omega.$$

As G is a supremum of functionals satisfying **(P1 - P5)**, G obeys the same conditions.

Finally, we introduce a functional H,

$$H(\mathbf{v}) = \begin{cases} \int_\Omega \Big(|\mathbf{D}[\mathbf{v}]|^2 + |\mathbf{v}|^2\Big) dx + G(\mathbf{v}, D) & \text{if } \mathbf{v} \in H^1(D, R^N), \ \text{div } \mathbf{v} = 0 \text{ in } \Omega, \\ \infty & \text{otherwise,} \end{cases} \tag{22}$$

together with

$$H_n(\mathbf{v}) = \begin{cases} \int_{\Omega_{\varepsilon_n}} \Big(|\mathbf{D}\mathbf{v}|^2 + |\mathbf{v}|^2\Big) dx & \text{if } \mathbf{v} \in H^1(D, R^N), \ \text{div } \mathbf{v} = 0 \text{ in } \Omega_{\varepsilon_n}, \ \mathbf{v} \cdot \mathbf{n}|_{\partial\Omega_{\varepsilon_n}} = 0, \\ \infty & \text{otherwise.} \end{cases} \tag{23}$$

We claim that H_n Γ-converges to H in $L^2(D, R^N)$. To this end, we have to examine two cases:

(v) Γ**-liminf**

Let $\mathbf{v}_n \to \mathbf{v}$ in $L^2(D, R^N)$. We restrict ourselves to the case

$$\liminf_{n\to\infty} H_n(\mathbf{v}_n) < \infty,$$

in particular,

$$\text{div } \mathbf{v}_n = 0 \text{ in } \Omega_{\varepsilon_n}, \quad \mathbf{v}_n \cdot \mathbf{n}|_{\partial\Omega_{\varepsilon_n}} = 0.$$

Since

$$H_n(\mathbf{v}_n) = E^{\lambda_m}_{\varepsilon_n}(\mathbf{v}_n, D) \text{ for all } m,$$

we deduce that

$$\liminf_{n\to\infty} H_n(\mathbf{v}_n) = \liminf_{n\to\infty} E^{\lambda_m}_{\varepsilon_n}(\mathbf{v}_n, D)$$
$$\geq \int_\Omega \Big(|D\mathbf{v}|^2 + |\mathbf{v}|^2\Big) dx + \lambda_m \int_\Omega |\text{div } \mathbf{v}|^2 dx + G^{\lambda_m}(\mathbf{v}, D).$$

Moreover, $\mathbf{v}(x) \in V_{\lambda_m}(x)$ for q.e. $x \in \partial\Omega$.

Thus, necessarily, div $\mathbf{v} = 0$ in Ω and $\mathbf{v}(x) \in V(x)$ for q.a. $x \in \partial\Omega$, and, letting $m \to \infty$ we conclude that

$$\liminf_{n\to\infty} H_n(\mathbf{v}_n) \geq H(\mathbf{v}).$$

(vi) Γ**-limsup:**

Consider $\mathbf{v} \in L^2(D, R^N)$ such that

$$H(\mathbf{v}) < \infty,$$

in particular, div $\mathbf{v} = 0$ in Ω, and $\mathbf{v}(x) \in V(x)$ for q.a. $x \in \partial\Omega$. Since $E^{\lambda_m}_{\varepsilon_n}$ $\overline{\Gamma}$-converges to E^{λ_m}, the diagonalization procedure produces a sequence $\{\mathbf{w}_n\}_{n>0}$ such that

$$\mathbf{w}_n \to \mathbf{v} \text{ in } L^2(D, R^N) \text{ as } n \to \infty$$

and

$$\limsup_{n\to\infty} E^{\lambda_n}_{\varepsilon_n}(\mathbf{w}_n, D) \leq E^{\lambda_m}(\mathbf{v}, D) = \int_\Omega \Big(|\mathbf{D}[\mathbf{v}]|^2 + |\mathbf{v}|^2 \Big) dx + G(\mathbf{v}, D).$$

In addition, we have

$$\int_{\Omega_{\varepsilon_n}} |\mathrm{div}\ \mathbf{w}_n|^2 dx \leq \frac{1}{\lambda_n}.$$

At this stage, we introduce a family of auxiliary functions

$$\mathbf{h}_n \in H^1(\Omega_{\varepsilon_n}, R^N),\ \mathbf{h}_n|_{\partial\Omega_{\varepsilon_n}} = 0,\ \mathrm{div}\ \mathbf{h}_n = \mathrm{div}\ \mathbf{w}_n,$$

where the functions $\mathbf{h}_n \approx \mathrm{div}^{-1}(\mathrm{div}\ \mathbf{w}_n)$ are determined by means of the integral formula proposed by Bogovskii [5]. A remarkable feature of this construction is the estimate

$$\|\mathbf{h}_n\|_{H^1(\Omega_{\varepsilon_n}, R^N)} \leq c \|\mathrm{div}\ \mathbf{w}_n\|_{L^2(\Omega_{\varepsilon_n}, R^N)} \leq \frac{c}{\sqrt{\lambda_n}}, \tag{24}$$

where the constant c is *independent* of n provided the family of sets $\{\Omega_{\varepsilon_n}\}_{n>0}$ satisfies the uniform cone condition (see Galdi [14, Chapter 3]).

Setting

$$\mathbf{v}_n = \mathbf{h}_n - \mathbf{w}_n$$

we arrive at the desired conclusion

$$\limsup_{n\to\infty} H_n(\mathbf{v}_n) = \limsup_{n\to\infty} \int_{\Omega_{\varepsilon_n}} \Big(|\mathbf{D}[\mathbf{v}_n]|^2 + |\mathbf{v}_n|^2 \Big) dx =$$

$$\limsup_{n\to\infty} \int_{\Omega_{\varepsilon_n}} \Big(|\mathbf{D}\mathbf{w}_n|^2 + |\mathbf{w}_n|^2 \Big) dx \leq \limsup_{n\to\infty} E^{\lambda_n}_{\varepsilon_n}(\mathbf{w}_n, D)$$

$$= \int_\Omega \Big(|\mathbf{D}[\mathbf{v}]|^2 + |\mathbf{v}|^2 \Big) dx + G(\mathbf{v}, D).$$

(vii) We are in a position to complete the proof of the theorem. Let $\mathbf{u}_n$ be the solution of the Stokes problem in Ω_{ε_n}. As we have seen, the energy functionals

$$\mathbf{v} \mapsto \frac{1}{2} H_n(\mathbf{v}) - \int_{\Omega_{\varepsilon_n}} \mathbf{f} \cdot \mathbf{v}\ dx$$

Γ-converge in $L^2(D, R^N)$ to

$$\mathbf{v} \mapsto \frac{1}{2} H(\mathbf{v}) - \int_\Omega \mathbf{f} \cdot \mathbf{v}\ dx.$$

Let $\tilde{\mathbf{u}}_n$ be extensions of $\mathbf{u}_n$ onto $D \setminus \Omega_n$ such that $\|\tilde{\mathbf{u}}_n\|_{H^1} \leq C \|\mathbf{u}_n\|_{H^1}$, with a constant C independent of n. Then $\{\tilde{\mathbf{u}}_n\}_{n>0}$ is bounded in $H^1(D, R^N)$ and we may assume (at least for a subsequence) that $\tilde{\mathbf{u}}_n$ converges weakly in $H^1(D, R^N)$ to some function $\mathbf{v}$. However, as a consequence of Γ-convergence, the limit $\mathbf{v}$ is an extension of $\mathbf{u}$ which completes the proof. □

Remark 4.4 Under the hypotheses of Theorem 4.3, assume that the right-hand side $\mathbf{f} = \mathbf{f}_\varepsilon$, where the family $\{\mathbf{f}_\varepsilon\}_{\varepsilon>0}$ belongs to the dual space $[H^1(\Omega_\varepsilon, R^N)]'$ and as such can be represented as

$$< \mathbf{f}_\varepsilon, \varphi >_{[H^1(\Omega_\varepsilon,R^N)]'\times H^1(\Omega_\varepsilon,R^N)} = \int_{\Omega_\varepsilon} \mathbf{f}^0_\varepsilon \cdot \varphi \; dx + \sum_{i=1}^n \int_{\Omega_\varepsilon} \mathbf{f}^i_\varepsilon \cdot \partial_{x_i}\varphi \; dx$$

for certain $\mathbf{f}^0_\varepsilon$, $\mathbf{f}^i_\varepsilon$, $i = 1, \dots, N$ belonging to $L^2(D, R^N)$.

In addition, assume that

$$\mathbf{f}^0_\varepsilon \to \mathbf{f}^0 \text{ weakly in } L^2(D, R^N),\ \mathbf{f}^i_\varepsilon \to \mathbf{f}^i \text{ strongly in } L^2(D, R^N),\ i = 1, \dots, N.$$

Then the conclusion of Theorem 4.3 is still valid, meaning the limit $\mathbf{u}$ solves (16) with $\mathbf{f}$ determined in terms of $\mathbf{f}^0$, $\mathbf{f}^i$. Indeed the energy functionals still Γ-converge since

$$< \mathbf{f}_\varepsilon, \varphi_\varepsilon|_{\Omega_\varepsilon} >_{[H^1(\Omega_\varepsilon,R^N)]'\times H^1(\Omega_\varepsilon,R^N)} \to < \mathbf{f}, \varphi|_\Omega >_{[H^1(\Omega,R^N)]'\times H^1(\Omega,R^N)}$$

whenever $\varphi_\varepsilon \rightharpoonup \varphi$ weakly in $H^1(D, R^N)$.

5 Navier-Stokes system

Our ultimate goal is to apply Theorem 4.3 to the evolutionary *Navier-Stokes system*. Similarly to Section 2, we consider the problem

$$\left\{ \begin{array}{rcl} \partial_t \mathbf{u}_\varepsilon + \mathrm{div}\,(\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) - \nu \mathrm{div}\, \mathbf{D}[\mathbf{u}_\varepsilon] + \nabla p_\varepsilon &=& \mathbf{g} \text{ in } (0,T)\times\Omega_\varepsilon,\ \nu > 0 \\ \mathrm{div}\, \mathbf{u}_\varepsilon &=& 0 \text{ in } (0,T)\times\Omega_\varepsilon \\ \mathbf{u}_\varepsilon \cdot \mathbf{n}|_{\partial\Omega_\varepsilon} &=& 0 \\ (\mathbf{D}[\mathbf{u}_\varepsilon]\cdot \mathbf{n}_\varepsilon)_{tan}|_{\partial\Omega_\varepsilon} &=& 0 \\ \mathbf{u}_\varepsilon(0,\cdot) &=& \mathbf{u}_0 \end{array} \right\} \tag{25}$$

We say that $\mathbf{u}_\varepsilon$ is a weak (variational) solution to problem (25) if

$$\mathbf{u}_\varepsilon \in L^\infty(0,T; L^2(\Omega_\varepsilon, R^N) \cap L^2(0,T; H^1(\Omega_\varepsilon, R^N)),\ \mathbf{u}_\varepsilon \cdot \mathbf{n}|_{\partial\Omega_\varepsilon} = 0,\ \mathrm{div}\, \mathbf{u}_\varepsilon = 0,$$

and the integral identity

$$\int_0^T \int_{\Omega_\varepsilon} (\mathbf{u}_\varepsilon \cdot \partial_t \varphi + \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon : \nabla\varphi - \nu \mathbf{D}[\mathbf{u}_\varepsilon] : \mathbf{D}[\varphi]) \; dx \; dt \tag{26}$$

$$= - \int_{\Omega_\varepsilon} \mathbf{u}_0 \cdot \varphi(0,\cdot) \; dx - \int_0^T \int_{\Omega_\varepsilon} \mathbf{g} \cdot \varphi \; dx \; dt$$

holds for any test function φ such that

$$\varphi \in C^1_c([0,T) \times \overline{\Omega}_\varepsilon; R^N),\ \varphi \cdot \mathbf{n}|_{\partial\Omega_\varepsilon} = 0,\ \mathrm{div}\, \varphi = 0.$$

In addition, we focus on the class of *turbulent* weak solutions in the sense of Leray that satisfy the *energy inequality*

$$\int_{\Omega_\varepsilon} \frac{1}{2}|\mathbf{u}_\varepsilon|^2(\tau,\cdot)\ dx + \nu \int_0^\tau \int_{\Omega_\varepsilon} |\mathbf{D}[\mathbf{u}_\varepsilon]|^2\ dx\ dt \leq \int_{\Omega_\varepsilon} \frac{1}{2}|\mathbf{u}_0|^2\ dx + \int_0^\tau \int_{\Omega_\varepsilon} \mathbf{g}\cdot\mathbf{u}_\varepsilon\ dx\ dt \tag{27}$$

for a.a. $\tau \in (0,T)$. The reader may consult the standard monographs by Ladyzhenskaya [19], Sohr [25], or Temam [26] for the basic properties and the existence theory for the evolutionary Navier-Stokes system in the framework of weak solutions.

Assuming $\mathbf{u}_0 \in L^2(D,R^N)$, and, say, $\mathbf{g} \in L^\infty(0,T;L^2(D,R^3))$, we may infer from (27) that

$$\mathbf{u}_\varepsilon \to \mathbf{u} \text{ weakly in } L^2(0,T;H^1(D,R^N)),$$

and

$$\mathbf{u}_\varepsilon \to \mathbf{u} \text{ weakly-(*) in } L^\infty(0,T;L^2(D,R^N))$$

provided $\mathbf{u}_\varepsilon$ were extended on the set D.

In order to exploit Theorem 4.3, we consider the regularized functions

$$[\mathbf{u}_\varepsilon]^\delta(t,\cdot) = \int_R \xi_\delta(t-s)\mathbf{u}_\varepsilon(s,\cdot)\ ds,$$

where $\{\xi_\delta\}_{\delta>0}$ is a standard family of regularizing kernels, supp$[\xi_\delta] \subset (-\delta,\delta)$, and where $\mathbf{u}_\varepsilon(t,\cdot)$ has been set $\mathbf{u}_0$ for $t \leq 0$.

Accordingly, the quantities $[\mathbf{u}_\varepsilon]^\delta(t,\cdot)$ can be regarded as the (unique) solution of the Stokes minimization problem (16), supplemented with the driving force

$$< \mathbf{f}_\varepsilon(t), \varphi >_{[H^1(\Omega_\varepsilon,R^N)]'\times H^1(\Omega_\varepsilon,R^N)}$$

$$= \int_{\Omega_\varepsilon} \Big([\mathbf{g}]^\delta(t,\cdot)\cdot\varphi + [\mathbf{u}_\varepsilon\otimes\mathbf{u}_\varepsilon]^\delta(t,\cdot) : \nabla\varphi + [\mathbf{u}_\varepsilon]^\delta(t,\cdot)\cdot\varphi - \partial_t[\mathbf{u}_\varepsilon]^\delta(t,\cdot)\cdot\varphi \Big)\ dx$$

for any fixed time $t \in [\delta, T-\delta]$. Note that, as a direct consequence of the standard compact embedding relation $H^1(D,R^N) \to L^p(D,R^N)$, $p < 2N/(N-2)$, we may infer that $\{[\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon]^\delta(t,\cdot)\}_{\varepsilon>0}$ is precompact in L^2 for any fixed δ, $t \in [\delta, T-\delta]$, $N = 2,3$.

Thus, by virtue of Theorem 4.3 and Remark 4.4, we can pass to the limit first for $\varepsilon \to 0$ and then $\delta \to 0$, in order to conclude that there exists a trio $\{\mu, A, \mathcal{V}\}$ such that the limit velocity field $\mathbf{u}$ belongs to the class

$$\mathbf{u} \in L^2(0,T;H^1(D,R^N)) \cap L^\infty(0,T;L^2(D,R^N)), \tag{28}$$

$$\text{div}\ \mathbf{u} = 0 \text{ in } (0,T)\times\Omega,\ \mathbf{u}(t,\cdot)|_{\partial\Omega} \in \mathcal{V} \text{ for a.a. } t \in (0,T),$$

and satisfies the integral identity

$$\int_0^T \int_\Omega \left(\mathbf{u}\cdot\partial_t\varphi + \overline{[\mathbf{u}\otimes\mathbf{u}]} : \nabla\varphi - \nu\mathbf{D}[\mathbf{u}] : \mathbf{D}[\varphi]\right)\ dx\ dt \tag{29}$$

$$= -\int_\Omega \mathbf{u}_0\cdot\varphi(0,\cdot)\ dx - \int_0^T \int_\Omega \mathbf{g}\cdot\varphi\ dx\ dt + \int_0^T \int_\Omega A\mathbf{u}\cdot\varphi\ d\mu$$

for any test function

$$\varphi \in C_c^1([0,T) \times \overline{\Omega}; R^N), \text{ div } \varphi = 0, \ \varphi|_{\partial\Omega} \in \mathcal{V}, \tag{30}$$

where $\overline{\mathbf{u} \otimes \mathbf{u}}$ denotes a weak L^1-limit of $\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon$.

Our final goal is to show that

$$\int_0^T \int_\Omega [\overline{\mathbf{u} \otimes \mathbf{u}}] : \nabla\varphi \ dx \ dt = \int_0^T \int_\Omega [\mathbf{u} \otimes \mathbf{u}] : \nabla\varphi \ dx \ dt \tag{31}$$

for any test function φ satisfying (30).

To begin, we show (31) for a smaller set of test functions φ, namely,

$$\varphi \in C_c^1((0,T) \times \Omega; R^N), \text{ div } \varphi = 0. \tag{32}$$

Given φ belonging to the class (32), we fix (smooth) domains $V \subset \overline{V} \subset B \subset \overline{B} \subset \Omega$, where V contains the support of φ. We introduce the *Helmholtz decomposition*

$$\mathbf{v} = \mathbf{H}[\mathbf{v}] + \mathbf{H}^\perp[\mathbf{v}]$$

defined for $\mathbf{v}$ in $L^2(B; R^N)$, specifically,

$$\mathbf{H}^\perp[\mathbf{v}] = \nabla\Psi, \text{ where } \int_B \Psi \ dx = 0, \ \int_B \nabla\Psi \cdot \nabla w \ dx = \int_B \mathbf{v} \cdot \nabla w \ dx \text{ for all } w \in C^1(\overline{B}).$$

As the family $\{\Omega_\varepsilon\}_{\varepsilon>0}$ converges in the sense of Hausdorff complementary topology, we have $B \subset \Omega_\varepsilon$ for all ε small enough. In particular, we may write

$$\mathbf{u}_\varepsilon|_B = \mathbf{H}[\mathbf{u}_\varepsilon] + \nabla\Psi_\varepsilon, \ \nabla\Psi_\varepsilon = \mathbf{H}^\perp[\mathbf{u}_\varepsilon].$$

As for the solenoidal components $\{\mathbf{H}[\mathbf{u}_\varepsilon]\}_{\varepsilon>0}$, we can use the standard Aubin-Lions argument to show that

$$\mathbf{H}[\mathbf{u}_\varepsilon] \to \mathbf{H}[\mathbf{u}] \text{ (strongly) in } L^2(0,T; L^2(B, R^N)).$$

On the other hand, we can write

$$\int_0^T \int_B [\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon] : \nabla\varphi \ dx \ dt = \tag{33}$$

$$\int_0^T \int_B \Big(\mathbf{H}[\mathbf{u}_\varepsilon] \otimes \mathbf{H}[\mathbf{u}_\varepsilon]\Big) : \nabla\varphi \ dx \ dt + \int_0^T \int_B \Big(\mathbf{H}[\mathbf{u}_\varepsilon] \otimes \nabla\Psi_\varepsilon\Big) : \nabla\varphi \ dx \ dt$$
$$+ \int_0^T \int_B \Big(\nabla\Psi_\varepsilon \otimes \mathbf{H}[\mathbf{u}_\varepsilon]\Big) : \nabla\varphi \ dx \ dt + \int_0^T \int_B (\nabla\Psi_\varepsilon \otimes \nabla\Psi_\varepsilon) : \nabla\varphi \ dx \ dt,$$

where

$$\int_0^T \int_B (\nabla\Psi_\varepsilon \otimes \nabla\Psi_\varepsilon) : \nabla\varphi \ dx \ dt = 0$$

Indeed by parts integration yields

$$\int_0^T \int_B (\nabla \Psi_\varepsilon \otimes \nabla \Psi_\varepsilon) : \nabla \varphi \; dx \; dt$$

$$= - \int_0^T \int_B \Delta \Psi_\varepsilon \nabla \Psi_\varepsilon \cdot \varphi \; dx \; dt - \frac{1}{2} \int_0^T \int_B \nabla |\nabla \Psi_\varepsilon|^2 \cdot \varphi = 0$$

as Ψ_ε is harmonic in B and φ is solenoidal with compact support in B.

Consequently, letting $\varepsilon \to 0$ in (33) and using strong convergence of $\mathbf{H}[\mathbf{u}_\varepsilon]$ in L^2 we obtain relation (31) for any test function φ satisfying (32).

Finally, in order to extend (31) to the class of test functions specified in (30), we write

$$\int_0^T \int_\Omega [\overline{\mathbf{u} \otimes \mathbf{u}}] : \nabla \varphi \; dx \; dt = \lim_{\varepsilon \to 0} \int_0^T \int_{\Omega_\varepsilon} [\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon] : \nabla \varphi \; dx \; dt \tag{34}$$

$$= - \lim_{\varepsilon \to 0} \int_0^T \int_{\Omega_\varepsilon} [\nabla \mathbf{u}_\varepsilon \mathbf{u}_\varepsilon] \cdot \varphi \; dx \; dt,$$

where we have used the fact that $\mathbf{u}_\varepsilon$ are solenoidal fields with vanishing normal trace on $\partial \Omega_\varepsilon$.

On the other hand,

$$\lim_{\varepsilon \to 0} \int_0^T \int_{\Omega_\varepsilon} [\nabla \mathbf{u}_\varepsilon \mathbf{u}_\varepsilon] \cdot \varphi \; dx \; dt = \int_0^T \int_\Omega [\overline{\nabla \mathbf{u} \mathbf{u}}] \cdot \varphi \; dx \; dt, \tag{35}$$

where $\overline{\nabla \mathbf{u} \mathbf{u}}$ denotes a weak limit of $\{\nabla \mathbf{u}_\varepsilon \mathbf{u}_\varepsilon\}_{\varepsilon > 0}$ in L^1. In addition, as we have shown above,

$$\int_0^T \int_\Omega [\overline{\nabla \mathbf{u} \mathbf{u}}] \cdot \varphi \; dx \; dt = \int_0^T \int_\Omega [\nabla \mathbf{u} \mathbf{u}] \cdot \varphi \; dx \; dt = - \int_0^T \int_\Omega [\mathbf{u} \otimes \mathbf{u}] : \nabla \varphi \; dx \; dt \tag{36}$$

provided that the test function φ belongs to the class (32). Consequently, as the space of functions specified in (32) is *dense* in

$$\left\{ \varphi \in L^p((0,T) \times \Omega; R^N) \; \middle| \; \mathrm{div} \; \varphi = 0, \; \varphi \cdot \mathbf{n}|_{\partial \Omega} = 0 \right\}$$

for any finite p, we conclude that relation (36) holds for any φ as in (30). Thus we may infer, combining (34 - 36), that (31) is valid for any φ as in (30).

The results obtained in this section are summarized in the following theorem.

Theorem 5.1 *Let* $\{\Omega_\varepsilon\}_{\varepsilon > 0}$ *be a family of domains in* $D \subset R^N$, $N = 2, 3$, *satisfying the uniform cone condition. Let*

$$\mathbf{u}_\varepsilon \in L^\infty(0,T; L^2(\Omega_\varepsilon, R^N)) \cap L^2(0,T; H^1(\Omega_\varepsilon, R^N))$$

be a sequence of weak solutions to the Navier-Stokes system (25) satisfying the energy inequality (27).

Then there exists a trio $\{\mu, A, \mathcal{V}\}$ *such that, at least for a suitable subsequence,*

$$1_{\Omega_\varepsilon}\mathbf{u}_\varepsilon \to 1_\Omega \mathbf{u} \textit{ weakly-(*) in } L^\infty(0,T;L^2(D,R^N)),$$

$$1_{\Omega_\varepsilon}\nabla\mathbf{u}_\varepsilon \to 1_\Omega \nabla\mathbf{u} \textit{ weakly in } L^2(0,T;L^2(D,R^{N\times N})),$$

where $\mathbf{u}$ *is a weak solution of problem*

$$\begin{cases} \partial_t\mathbf{u} + \mathrm{div}(\mathbf{u}\otimes\mathbf{u}) - \nu\mathrm{div}\mathbf{D}[\mathbf{u}] + \nabla p & = & \mathbf{g} \textit{ in } (0,T)\times\Omega \\ \mathrm{div}\ \mathbf{u} & = & 0 \textit{ in } (0,T)\times\Omega \\ \mathbf{u}(x) & \in & V(x) \textit{ for q.a. } \ x\in\partial\Omega \\ \Big[\mathbf{D}[\mathbf{u}]\cdot\mathbf{n} + \mu A\mathbf{u}\Big](x)\cdot\mathbf{v} & = & 0 \textit{ for q.a. } x\in\partial\Omega,\ \mathbf{v}\in V(x) \\ \mathbf{u}(0,\cdot) & = & \mathbf{u}_0 \textit{ in } \Omega \end{cases}$$

in the sense specified in (28 - 30). Moreover, $V(x)\perp\mathbf{n}(x)$ *for any* $x\in\partial\Omega$*.*

References

[1] A. A. Amirat, D. Bresch, J. Lemoine, and J. Simon. Effect of rugosity on a flow governed by stationary Navier-Stokes equations. *Quart. Appl. Math.*, **59**:768–785, 2001.

[2] A. A. Amirat, E. Climent, E. Fernández-Cara, and J. Simon. The Stokes equations with Fourier boundary conditions on a wall with asperities. *Math. Models. Methods Appl.*, **24**:255–276, 2001.

[3] Nadia Ansini and Adriana Garroni. Γ-convergence of functionals on divergence-free fields. *ESAIM Control Optim. Calc. Var.*, 13(4):809–828 (electronic), 2007.

[4] A. Basson and D. Gérard-Varet. Wall laws for fluid flows at a boundary with random roughness. 2006. Preprint.

[5] M. E. Bogovskii. Solution of some vector analysis problems connected with operators div and grad (in Russian). *Trudy Sem. S.L. Sobolev*, **80**(1):5–40, 1980.

[6] G. Bouchitté, P. Seppecher. Cahn and Hilliard fluid on an oscillating boundary. *Motion by mean curvature and related topics* (Trento, 1992), 23–42, de Gruyter, Berlin, 1994.

[7] D. Bucur, E. Feireisl, Š. Nečasová, and J. Wolf. On the asymptotic limit of the Navier-Stokes system on domains with rough boundaries. *J. Differential Equations*, **244**:2890–2908, 2008.

[8] D. Bucur, E. Feireisl, Š. Nečasová. On the asymptotic limit of flows past a ribbed boundary *J. Math. Fluid Mech.* , 2008. To appear.

[9] J. Casado-Díaz, E. Fernández-Cara, and J. Simon. Why viscous fluids adhere to rugose walls: A mathematical explanation. *J. Differential Equations*, **189**:526–537, 2003.

[10] G. Dal Maso *An introduction to Γ-convergence.* Progress in Nonlinear Differential Equations and their Applications, 8. Birkhäuser Boston, Inc., Boston, MA, 1993.

[11] G. Dal Maso, A. Defranceschi, E Vitali. Integral representation for a class of C^1-convex functionals. *J. Math. Pures Appl.* (9) 73 (1994), no. 1, 1–46.

[12] A. Defranceschi, E Vitali. Limits of minimum problems with convex obstacles for vector valued functions. *Appl. Anal.* 52 (1994), no. 1-4, 1–33.

[13] M. El Jarroudi, A. Brillard. Relaxed Dirichlet problem and shape optimization within the linear elasticity framework. *NoDEA Nonlinear Differential Equations Appl.* 11 (2004), no. 4, 511–528.

[14] G. P. Galdi. *An introduction to the mathematical theory of the Navier - Stokes equations, I.* Springer-Verlag, New York, 1994.

[15] A. Henrot and M. Pierre *Variation et optimisation de formes. Une analyse géométrique.* Mathématiques & Applications (Berlin) 48. Berlin: Springer, 2005.

[16] T.I. Hesla. Collision of smooth bodies in a viscous fluid: A mathematical investigation. 2005. PhD Thesis - Minnesota.

[17] M. Hillairet. Lack of collision between solid bodies in a 2D incompressible viscous flow. 2006. Preprint - ENS Lyon.

[18] W. Jaeger and A. Mikelić. On the roughness-induced effective boundary conditions for an incompressible viscous flow. *J. Differential Equations*, **170**:96–122, 2001.

[19] O. A. Ladyzhenskaya. *The mathematical theory of viscous incompressible flow.* Gordon and Breach, New York, 1969.

[20] J. Málek and K. R. Rajagopal. Mathematical issues concerning the Navier-Stokes equations and some of its generalizations. In *Evolutionary equations. Vol. II*, Handb. Differ. Equ., pages 371–459. Elsevier/North-Holland, Amsterdam, 2005.

[21] H.K. Moffat. Viscous and resistive eddies near a sharp corner. *J. Fluid Mech.*, **18**:1, 1964.

[22] J.A. Nitsche. On Korn's second inequality. *RAIRO Anal. Numer.*, **15**:237–248, 1981.

[23] N. V. Priezjev, Darhuber A.A., and S.M. Troian. Slip behavior in liquid films on surfaces of patterned wettability:Comparison between continuum and molecular dynamics simulations. *Phys. Rev. E*, **71**:041608, 2005.

[24] N. V. Priezjev and S.M. Troian. Influence of periodic wall roughness on the slip behaviour at liquid/solid interfaces: molecular versus continuum predictions. *J. Fluid Mech.*, **554**:25–46, 2006.

[25] H. Sohr. *The Navier-Stokes equations: An elementary functional analytic approach.* Birkhäuser Verlag, Basel, 2001.

[26] R. Temam. *Navier-Stokes equations.* North-Holland, Amsterdam, 1977.