On the torsion function with Robin or Dirichlet boundary conditions

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Abstract

For $p \in (1, +\infty)$ and $b \in (0, +\infty]$ the p-torsion function with Robin boundary conditions associated to an arbitrary open set $\Omega \subset \mathbb{R}^m$ satisfies formally the equation $-\Delta_p = 1$ in Ω and $|\nabla u|^{p-2} \frac{\partial u}{\partial n} + b|u|^{p-2}u = 0$ on $\partial\Omega$. We obtain bounds of the L^∞ norm of u *only* in terms of the bottom of the spectrum (of the Robin p-Laplacian), b and the dimension of the space in the following two extremal cases: the linear framework (corresponding to $p = 2$) and arbitrary $b > 0$, and the non-linear framework (corresponding to arbitrary $p > 1$) and Dirichlet boundary conditions ($b = +\infty$). In the general case, $p \neq 2, p \in (1, +\infty)$ and $b > 0$ our bounds involve also the Lebesgue measure of Ω.

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1 Introduction

Let Ω be an open set in Euclidean space $\mathbb{R}^m$ with non-empty boundary $\partial\Omega$, and let the torsion function $u : \Omega \to \mathbb{R}$ be the unique weak solution of

$$-\Delta u = 1, \quad u|_{\partial\Omega} = 0.$$

If Ω has finite measure the solution is obtained in the usual framework of the Lax-Milgram theorem, while if Ω has infinite measure then $1 \notin H^{-1}(\Omega)$ and u is defined as the supremum over all balls of the torsion functions associated to $\Omega \cap B$.

The torsional rigidity is the set function defined by

$$P(\Omega) = \int_\Omega u. \tag{1}$$

Since $u \geq 0$ we have that $P(\Omega)$ takes values in the non-negative extended real numbers, and that $P(\Omega) = \|u\|_{L^1(\Omega)}$, whenever u is integrable. Both the torsion function and the torsional rigidity arise in many areas of mathematics, for example in elasticity theory [2, 19, 23], in heat conduction [5], in the definition of gamma convergence [8], in the study of minimal submanifolds [21] etc. The connection with probability theory is as follows. Let $(B(s), s \geq 0; \mathbb{P}_x, x \in \mathbb{R}^m)$ be Brownian motion with generator Δ, and let

$$T_\Omega = \inf\{s \geq 0 \colon B(s) \in \mathbb{R}^m \setminus \Omega\}$$

be the first exit time of Brownian motion from Ω. Then [24]

$$u(x) = \mathbb{E}_x [T_\Omega], \quad x \in \Omega.$$

Let λ be the bottom of the spectrum of the Dirichlet Laplacian acting in $L^2(\Omega)$.

In [6, 7] it was shown that $u \in L^\infty(\Omega)$ if and only if $\lambda > 0$. If $\lambda > 0$ then

$$\lambda^{-1} \leq \|u\|_{L^\infty(\Omega)} \leq (4 + 3m \log 2)\, \lambda^{-1}. \tag{2}$$

Previous results of this nature were obtained in Theorem 1 of [3] for open, simply connected, planar sets Ω. The question of the sharp constant in the upper bound in the right hand side of (2) for these sets was addressed in [3, 4].

In this paper we consider the torsion function u_b for the Laplacian with Robin boundary conditions. The Robin Laplacian is generated by the quadratic form

$$\mathcal{Q}_b(u, v) = \int_\Omega \nabla u.\nabla v + b \int_{\partial\Omega} uv d\mathcal{H}^{m-1}, \; u, v \in W^1_{2,2}(\Omega, \partial\Omega),$$

where $\mathcal{H}^{m-1}$ denotes the $(m-1)$-dimensional Hausdorff measure on $\partial\Omega$, and b is a strictly positive constant. This quadratic form defined on $W^1_{2,2}(\Omega, \partial\Omega)$ is closed. The unique self-adjoint operator generated by $\mathcal{Q}_b$ is the Robin Laplacian which formally satisfies the boundary condition

$$\frac{\partial u}{\partial n} + bu = 0, \quad x \in \partial\Omega, \tag{3}$$

where n denotes the outward unit normal, and $\frac{\partial}{\partial n}$ is the normal derivative. The torsion function u_b is the unique weak solution of $-\Delta u = 1$ with boundary condition (3). For convenience we put $q_b(u) = \mathcal{Q}_b(u, u)$. It is well known that $W^1_{2,2}(\Omega) = W^{1,2}(\Omega)$ if Ω is bounded and $\partial\Omega$ is Lipschitz. See [22] for details. However, as all our results are for arbitrary open sets in $\mathbb{R}^m$ we will not rely on this identity. We denote the bottom of the spectrum of the Robin Laplacian acting in $L^2(\Omega)$ by

$$\lambda(\Omega, b) = \inf\{q_b(u) : \|u\|_{L^2(\Omega)} = 1, u \text{ smooth in } \overline{\Omega}\}. \tag{4}$$

The main results of this paper are the following.

Theorem 1. *Let Ω be an open set in $\mathbb{R}^m, m = 2, 3, \cdots$. The torsion function u_b is bounded if and only if $\lambda(\Omega, b) > 0$. In that case we have that*

$$\lambda(\Omega, b)^{-1} \le \|u_b\|_{L^\infty(\Omega)} \le 6m\lambda(\Omega, b)^{-1} \log\left(2^{11}3\sqrt{3}m(1 + b^{-1}\lambda(\Omega, b)^{1/2})\right). \tag{5}$$

For $b \to \infty$ we have that $\lambda(\Omega, b) \to \lambda$, and we recover (2) with an albeit worse constant. For $b \to 0$ we have that $\lambda(\Omega, b) \to 0$. However, in the case where Ω is a $C^{0,1}$ domain it was shown in [17] that

$$\lim_{b \to 0} b^{-1}\lambda(\Omega, b) = |\Omega|^{-1}\mathcal{H}^{m-1}(\partial\Omega), \tag{6}$$

where $|\Omega| = \int_\Omega 1$. (The upper bound in (6) follows by choosing the test function $u = |\Omega|^{-1/2}$ in (4).) So for these domains $b^{-1}\lambda(\Omega, b)^{1/2} \asymp b^{-1/2}$, and the upper bound in Theorem 1 has an extra factor $\log b$ compared with the Dirichlet regime $b \to \infty$. It is unclear whether this additional $\log b$ factor is in fact sharp.

The proof of Theorem 1 depends very heavily on the availability of Gaussian upper bounds for the Robin heat kernel. These were obtained in great generality in [13]. We note that the estimates obtained in [12] and [14] for elliptic Robin boundary value problems do not seem explicit enough to keep track of the geometric data of Ω. The remainder of this paper is organised as follows. In Section 2 we prove Theorem 1. In Section 3 we obtain some bounds for Robin eigenfunctions in the case where Ω has finite measure.

In Section 4 we study the torsion function and torsional rigidity for the p-Laplacian with Dirichlet and Robin boundary conditions respectively. In particular in Theorem 9 we will obtain an L^∞ estimate for the torsion function of the p-Laplacian with Dirichlet boundary conditions for an arbitrary open set in terms of the corresponding spectral bottom. This extends the upper bound in (2) for $p = 2$ to all $p > 1$. Moreover it extends the results of [11, Theorem 13] for convex sets to arbitrary open sets with finite or infinite measure. In the very general case of the p-Laplacian with Robin boundary conditions, we obtain L^∞ bounds which hold only on open sets with finite measure, and the constant involve the measure of Ω as well. This last result is probably not optimal, since one may expect that the Lebesgue measure should not enter into the constant, but we are not able to overcome a series of technical points.

2 Proof of Theorem 1

The main ingredient in the proof of Theorem 1 is a Gaussian bound for the Robin heat kernel $p_b(x, y; t), x \in \Omega, y \in \Omega, t > 0$, for arbitrary open sets Ω in

$\mathbb{R}^m, m = 2, 3, \cdots$. In the special case where Δ is the standard Laplacian and $b > 0$ is constant on $\partial\Omega$, Theorem 6.1 in [13] reads as follows.

For all $0 < \epsilon \leq 1$ and for all $x \in \Omega, y \in \Omega, t > 0$

$$p_b(x, y; t) \leq C_{2m}(\alpha\epsilon)^{-m} C^m t^{-m} e^{-|x-y|^2/(4\omega(1+\epsilon)t)},$$

where $\alpha = \min\{1, b\}$, and C_{2m} and ω are constants depending on m only. C is the constant which appears in the Nash inequality

$$\|u\|_{L^2(\Omega)}^{2+\frac{2}{m}} \leq C q_1(u) \|u\|_{L^1(\Omega)}^{\frac{2}{m}},$$

It is straightforward to trace the m-dependence of ω. We find that upon consulting Lemma 6.3 and its proof in [13],

$$\omega = 1 + m^{1/2} + 4m.$$

Similarly we find that using Corollary 5.3 and Lemma 5.7 and their proofs that

$$C_{2m} = (192m)^m. \tag{7}$$

We can also verify that if $\mathcal{N}_b$ is a constant in the Nash inequality

$$\|u\|_{L^2(\Omega)}^{2+\frac{2}{m}} \leq \mathcal{N}_b q_b(u) \|u\|_{L^1(\Omega)}^{\frac{2}{m}}, \tag{8}$$

and if we choose $\epsilon = 1$ then we infer, by the previous lines, that

$$p_b(x, y; t) \leq C_{2m} \mathcal{N}_b^m t^{-m} e^{-|x-y|^2/(8\omega t)}. \tag{9}$$

In Lemmas 2, 3, 4 and 5 below we prove the Nash inequality (8) with a constant $\mathcal{N}_b$ depending upon $b, \lambda(\Omega, b)$ and m only. As a first step we shall consider only open and bounded sets Ω with a smooth boundary. By a standard density argument we obtain the full SBV-case in a second step.

Lemma 2. *There exists a constant $C(m)$ depending upon m only such that for all $v \in BV(\mathbb{R}^m)$*

$$\|v\|_{L^{m/(m-1)}(\mathbb{R}^m)} \leq C(m)|Dv|(\mathbb{R}^m),$$

where $|Dv|(\mathbb{R}^m)$ is the total variation of v on $\mathbb{R}^m$, and $C(m)$ is the isoperimetric constant given by

$$C(m) = m^{-1}\pi^{-1/2}(\Gamma((2+m)/2))^{1/m}. \tag{10}$$

For a proof we refer to Theorem 3.4.7 in [1], or for an elementary proof with (non-sharp) constant 1 instead of $C(m)$ to Theorem 1 in Section 4.5.1 in [16].

Lemma 3. *Let Ω be an open bounded set in $\mathbb{R}^m$ and with smooth boundary, and let $u \in H^1(\Omega)$. Then*

$$\|u\|^2_{L^{2m/(m-1)}(\Omega)} \leq C(m) \left(2\int_\Omega |u||\nabla u| + \int_{\partial\Omega} u^2 d\mathcal{H}^{m-1} \right).$$

Proof. Let $u \in H^1(\Omega)$ and observe that $u^2 \in BV(\mathbb{R}^m)$. In fact we have that $u^2 \in SBV(\mathbb{R}^m)$, where u is extended by 0 on $\mathbb{R}^m \setminus \Omega$. Indeed $u^2 \in L^1(\mathbb{R}^m)$ and for any open set $A \subset \mathbb{R}^m$,

$$Du^2(A) = 2\int_A u\nabla u + \int_{\partial\Omega\cap A} u^2 \overrightarrow{n}\, d\mathcal{H}^{m-1}.$$

So

$$|Du^2|(\mathbb{R}^m) \le 2\int_\Omega |u||\nabla u| + \int_{\partial\Omega} u^2 d\mathcal{H}^{m-1}.$$

By Lemma 2

$$\|u^2\|_{L^{m/(m-1)}(\Omega)} \le C(m)\left(2\int_\Omega |u||\nabla u| + \int_{\partial\Omega} u^2 d\mathcal{H}^{m-1}\right),$$

which implies the lemma. □

Lemma 4. *For all $b > 0$ and all $u \in H^1(\Omega)$,*

$$\|u\|^2_{L^{2m/(m-1)}(\Omega)} \le C(m)\left(\frac{1}{b} + \frac{b}{\lambda(\Omega, b)}\right) q_b(u), \tag{11}$$

and for all $b \ge \lambda(\Omega, b)^{1/2}$ and $u \in H^1(\Omega)$,

$$\|u\|^2_{L^{2m/(m-1)}(\Omega)} \le 2C(m)\lambda(\Omega, b)^{-1/2} q_b(u). \tag{12}$$

Proof. In order to prove (12) we use Cauchy-Schwarz and obtain that

$$2\int_\Omega |u||\nabla u| \le b^{-1}\int_\Omega |\nabla u|^2 + b\int_\Omega u^2. \tag{13}$$

So by Lemma 3

$$\begin{aligned}\|u\|^2_{L^{2m/(m-1)}(\Omega)} &\le C(m)\left(b^{-1}\int_\Omega |\nabla u|^2 + b\int_\Omega u^2 + \int_{\partial\Omega} u^2 d\mathcal{H}^{m-1}\right)\\ &\le C(m)b^{-1}\left(q_b(u) + b^2\int_\Omega u^2\right)\\ &\le C(m)b^{-1}\left(q_b(u) + b^2\lambda(\Omega, b)^{-1} q_b(u)\right),\end{aligned} \tag{14}$$

which yields (11). In order to prove (12) we replace b by $\lambda(\Omega, b)^{1/2}$ in (13) and (14) respectively. This gives that

$$\|u\|^2_{L^{2m/(m-1)}(\Omega)} \le 2C(m)\lambda(\Omega, b)^{-1/2} q_{\lambda(\Omega,b)^{1/2}}(u) \le 2C(m)\lambda(\Omega, b)^{-1/2} q_b(u),$$

by monotonicity of $b \mapsto q_b$. □

Finally we obtain the following Nash inequality.

Lemma 5. *For all $u \in H^1(\Omega)$ we have that*

$$\|u\|^{2+\frac{2}{m}}_{L^2(\Omega)} \le \mathcal{N}_b q_b(u) \|u\|^{\frac{2}{m}}_{L^1(\Omega)},$$

where

$$\mathcal{N}_b = \begin{cases} C(m)(b^{-1} + \lambda(\Omega, b)^{-1} b) & , \quad b > 0, \\ 2C(m)\lambda(\Omega, b)^{-1/2} & , \quad b \ge \lambda(\Omega, b)^{1/2}. \end{cases} \tag{15}$$

Proof. The lemma follows by Lemma 4 and the following interpolation inequality

$$\|u\|_{L^2(\Omega)} \le \|u\|^{m/(m+1)}_{L^{2m/(m-1)}(\Omega)} \|u\|^{1/(m+1)}_{L^1(\Omega)}.$$

□

For non-smooth Ω one can follow the same arguments if instead of the Sobolev traces we consider $u^2 \in SBV(\mathbb{R}^m)$, and pointwise traces, or the Mazya trace (which has a priori higher L^2 norm). See page 940 lines -10 to -1 and page 941 lines 1 to 6 in [10].

In the lemma below we will use the heat equation techniques from [6] that were used to obtain bounds for the torsion function with Dirichlet boundary conditions. We abbreviate $K = C_{2m}\mathcal{N}_b^m$.

Lemma 6. *Suppose that $\lambda(\Omega, b) > 0$ and that (9) holds. Let $T > 0$ be arbitrary. Then the torsion function with Robin boundary conditions satisfies*

$$\|u\|_{L^\infty(\Omega)} \le T + 2^{-1}(256\pi\omega)^{m/2} K\lambda(\Omega, b)^{-1} T^{-m/2} e^{-T\lambda(\Omega,b)/4}.$$

Proof. First note that Lemma 1 in [6] holds for heat kernels with Robin boundary conditions. By choosing $\beta = 1/2$ in that lemma we obtain that

$$p_b(x, x; t) \le e^{-t\lambda(\Omega,b)/2}\, p_b\big(x, x; t/2).$$

Next note that by the heat semigroup property and the Cauchy-Schwarz inequality

$$\begin{aligned} p_b(x, y; t) &= \int_\Omega p_b(x, z; t/2)\, p_b(z, y; t/2) dz \\ &\le \left(\int_\Omega p_b(x, z; t/2)^2 dz\right)^{1/2} \left(\int_\Omega p_b(z, y; t/2)^2\, dz\right)^{1/2} \\ &= (p_b(x, x; t)\, p_b(y, y; t))^{1/2}. \end{aligned}$$

So putting the above two estimates together with (9) gives that

$$\begin{aligned} p_b(x, y; t) &\le (p_b(x, y; t))^{1/2} (p_b(x, x; t) p_b(y, y; t))^{1/4} \\ &\le K 2^{m/2} e^{-t\lambda(\Omega,b)/4} t^{-m} e^{-|x-y|^2/(16\omega t)}. \end{aligned}$$

We obtain that, by extending the region of integration to all of $\mathbb{R}^m$,

$$\int_\Omega dy p_b(x, y; t) \le K(32\pi\omega)^{m/2} t^{-m/2} e^{-t\lambda(\Omega,b)/4},$$

and

$$\begin{aligned} \int_{[T,\infty)} dt \int_\Omega dy p_b(x, y; t) &\le 4(32\pi\omega)^{m/2} K\lambda(\Omega, b)^{-1} T^{-m/2} e^{-T\lambda(\Omega,b)/4} \\ &\le 2^{-1}(256\pi\omega)^{m/2} K\lambda(\Omega, b)^{-1} T^{-m/2} e^{-T\lambda(\Omega,b)/4}. \end{aligned} \tag{16}$$

Furthermore

$$v(x; t) = \int_\Omega dy p_b(x, y; t)$$

is the solution of $\Delta v = \frac{\partial v}{\partial t}$ with initial condition $v(x;0) = 1$ and Robin boundary conditions. By the maximum principle we have that $0 \le v(x;t) \le 1$. Hence

$$\int_{[0,T]} dt \int_{\Omega} dy p_b(x,y;t) \le T,$$

and the lemma follows by (16) since the torsion function can be represented by

$$u(x) = \int_{[0,\infty)} dt \int_{\Omega} dy p_b(x,y;t).$$

□

Proof of Theorem 1. We choose T to be the unique positive root of

$$(256\pi\omega)^{m/2} K\lambda(\Omega,b)^{-1}T^{-m/2}e^{-T\lambda(\Omega,b)/4} = T. \tag{17}$$

We rewrite this, using the numerical value of K, as follows.

$$(T\lambda(\Omega,b))^{(2+m)/2}e^{T\lambda(\Omega,b)/4} = (2^{20}3^2m^2\mathcal{N}_b^2\lambda(\Omega,b)\pi\omega)^{m/2}. \tag{18}$$

It is easily seen that $\mathcal{N}_b^2\lambda(\Omega,b) \ge 1$ for all b and all $\lambda(\Omega,b)$, and that the right hand side of (18) is at least $e^{1/4}$. We conclude that $T\lambda(\Omega,b) \ge 1$. Hence $e^{T\lambda(\Omega,b)/4} \le (2^{20}3^2m^2\mathcal{N}_b^2\lambda(\Omega,b)\pi\omega)^{m/2}$, and

$$T \le 2m\lambda(\Omega,b)^{-1}\log(2^{20}3^2m^2\mathcal{N}_b^2\lambda(\Omega,b)\pi\omega). \tag{19}$$

By Lemma 6, (17) and (19) we find that

$$\|u\|_{L^\infty(\Omega)} \le 3m\lambda(\Omega,b)^{-1}\log(2^{20}3^2m^2\mathcal{N}_b^2\lambda(\Omega,b)\pi\omega). \tag{20}$$

To estimate the numerical constant under the log in the right hand side of (20) we first note that by (15),

$$\mathcal{N}_b^2\lambda(\Omega,b) \le 4C^2(m)\left(1 + b^{-1}\lambda(\Omega,b)^{1/2}\right)^2. \tag{21}$$

By (10), (20), (21) and the bounds $\omega \le 6m$ and $(\Gamma((2+m)/2))^{2/m} \le m/2$ we find that

$$\|u\|_{L^\infty(\Omega)} \le 3m\lambda(\Omega,b)^{-1}\log\left(2^{22}3^3m^2(1+b^{-1}\lambda(\Omega,b)^{1/2})^2\right).$$

This completes the proof of the right hand side in (5).

To prove the lower bound in (5) we let $B(p,R) = \{x : |x-p| < R\}$, $\Omega_R = \Omega \cap B(p,R)$, and we denote by $p_{b,R}(x,y;t)$ the heat kernel with Robin boundary conditions on $(\partial\Omega) \cap B(p,R)$ and Dirichlet boundary conditions on $(\partial\Omega_R) \setminus ((\partial\Omega) \cap B(p,R))$. The region Ω_R has finite volume and the spectrum of the Laplacian with the corresponding mixed boundary conditions is discrete. Denote the first eigenvalue by $\tilde{\lambda}(\Omega_R,b)$ with corresponding eigenfunction $\tilde{\phi}_{b,R}$. In Proposition 8 below we will see that the first Robin eigenfunction on an open set with finite Lebesgue measure is bounded. Following the proof of Proposition 8 we will show that $\tilde{\phi}_{b,R}$ is also bounded. We then have that

$$\begin{aligned}
u_{b,R}(x) &= \int_{[0,\infty)} dt \int_{\Omega} dy p_{b,R}(x,y;t) \\
&\ge \int_{[0,\infty)} dt \int_{\Omega} dy p_{b,R}(x,y;t) \frac{\tilde{\phi}_{b,R}(y)}{\|\tilde{\phi}_{b,R}\|_{L^\infty(\Omega_R)}} \\
&= \int_{[0,\infty)} dt e^{-t\tilde{\lambda}(\Omega_R,b)} \frac{\tilde{\phi}_{b,R}(x)}{\|\tilde{\phi}_{b,R}\|_{L^\infty(\Omega_R)}} \\
&= \tilde{\lambda}(\Omega_R,b)^{-1} \frac{\tilde{\phi}_{b,R}(x)}{\|\tilde{\phi}_{b,R}\|_{L^\infty(\Omega_R)}}.
\end{aligned} \tag{22}$$

Taking first the supremum over all $x \in \Omega_R$ in the left hand side of (22), and subsequently the supremum over all $x \in \Omega_R$ in the right hand side of (22) gives that

$$\|u_{b,R}\|_{L^\infty(\Omega_R)} \ge \tilde{\lambda}(\Omega_R,b)^{-1}. \tag{23}$$

Taking first the limit $R \to \infty$ followed by the same limit in the right hand side of (23) yields the lower bound in (5). This completes the proof of Theorem 1.

3 Robin eigenfunctions

In this section we obtain some estimates for eigenfunctions of the Robin Laplacian.

Proposition 7. *Let Ω be an open set in $\mathbb{R}^m$ with finite measure $|\Omega|$, and suppose that $\lambda(\Omega,b) > 0$. Then the spectrum of the Robin Laplacian acting in $L^2(\Omega)$ is discrete and for all $t > 0$*

$$\sum_{j=1}^{\infty} e^{-t\lambda_j(\Omega,b)} \le C_{2m}\mathcal{N}_b^m |\Omega| t^{-m}, \tag{24}$$

where C_{2m} and $\mathcal{N}_b$ are the constants in (7) *and* (15) *respectively and $\{\lambda_j(\Omega,b) : j \in \mathbb{N}\}$ are the eigenvalues of the Robin Laplacian. Note that $\lambda(\Omega,b) = \lambda_1(\Omega,b)$ in this case.*

Proof. Integrating the diagonal element of the heat kernel over Ω shows that the Robin heat semigroup is trace class. Hence the Robin spectrum is discrete and this in turn implies (24). □

It is well known that upper bounds on the heat kernel imply bounds for eigenfunctions. See Example 2.1.8 in [15]. The following below is another such instance.

Proposition 8. *Let Ω be an open set in $\mathbb{R}^m$ with finite measure $|\Omega|$, and suppose that $\lambda(\Omega,b) > 0$. Let $\{\phi_j : j \in \mathbb{N}\}$ denote an orthonormal set of eigenfunctions corresponding to the eigenvalues in Proposition 7. Then $\phi_j \in L^1(\Omega) \cap L^\infty(\Omega)$ and for all $j \in \mathbb{N}$*

$$\|\phi_j\|_{L^\infty(\Omega)} \le (C_{2m}\mathcal{N}_b^m e^m m^{-m})^{1/2} \lambda_j(\Omega,b)^{m/2}. \tag{25}$$

Proof. By Cauchy-Schwarz and orthonormality we have that $\|\phi_j\|_{L^1(\Omega)} \le |\Omega|^{1/2}$. Since $|\Omega| < \infty$ the heat semigroup is trace class, and so

$$e^{-t\lambda_j(\Omega,b)}\phi_j^2(x) \le \sum_{j=1}^{\infty} e^{-t\lambda_j(\Omega,b)}\phi_j^2(x) \le C_{2m}\mathcal{N}_b^m t^{-m}.$$

Hence

$$|\phi_j(x)|^2 \le C_{2m}\mathcal{N}_b^m e^{t\lambda_j(\Omega,b)}t^{-m}. \tag{26}$$

Taking the supremum over all $x \in \Omega$ in the left hand side of (26) followed by taking the infimum over all $t > 0$ in the right hand side gives the bound in (25). $\square$

To see that $\tilde{\phi}_{b,R}$, defined above (22), is bounded we note that by (5), $p_{b,R}(x,y;t) \le p_b(x,y;t) \le C_{2m}\mathcal{N}_b^m t^{-m}$. So $e^{-t\lambda(\Omega_R,b)}\phi_{b,R}^2(x) \le p_{b,R}(x,x;t) \le C_{2m}\mathcal{N}_b^m t^{-m}$. This shows that $\phi_{b,R}$ is bounded.

We note that the L^∞ estimate in (25) together with (22) implies the following comparison estimate between torsion function and first eigenfunction.

$$u_b(x) \ge (C_{2m}\mathcal{N}_b^m e^m m^{-m})^{-1/2}\lambda(\Omega,b)^{-1-\frac{m}{2}}\phi_1(x). \tag{27}$$

For $b \ge \lambda(\Omega,b)^{1/2}$ we use the second inequality in (15) to obtain that

$$u_b(x) \ge C\lambda(\Omega,b)^{-1-\frac{m}{4}}\phi_1(x),$$

for some constant C depending on m only. This jibes with Theorem 5.1 in [7]. In general one cannot expect however, that u_b and ϕ_1 are comparable. See Theorem 6 and the discussion in Section 3 in [7]. Similarly Theorem 1 and Proposition 8 show that for $b \ge \lambda(\Omega,b)^{1/2}$,

$$\|u_b^{m/4}\phi_1\|_{L^\infty(\Omega)} \le C',$$

where C' depends on m only. This jibes with Theorem 5.2 in [7], and completes the analogy with the Dirichlet case in this regime.

4 Torsion Function and torsional rigidity for the p-Laplacian

4.1 Dirichlet boundary conditions

In this section we consider the p-Laplacian for $1 < p < +\infty$ with Dirichlet boundary conditions, corresponding formally to $b = +\infty$. Let Ω be an open and bounded set of $\mathbb{R}^m$ and w_Ω (or simply w) the torsion function of the p-Laplacian with Dirichlet boundary conditions. It is the unique solution of

$$\min_{u\in W_0^{1,p}(\Omega)} \frac{1}{p}\int_\Omega |\nabla u|^p - \int_\Omega u.$$

Let

$$\lambda := \min\{\frac{\int_\Omega |\nabla u|^p}{\int_\Omega |u|^p} : u \in W_0^{1,p}(\Omega)\setminus\{0\}\}$$

be the first eigenvalue of the p-Laplacian with Dirichlet boundary conditions on Ω.

These notions extend to every open set, not necessarily bounded, by replacing the torsion function and the first eigenvalue with suitable definitions on unbounded sets. The first eigenvalue has to be replaced by the spectral bottom, and the torsion function by the following Borel function (possibly infinite valued) obtained by inner approximations

$$w_\Omega(x) = \sup_{R>0} w_{\Omega \cap B(0,R)}(x).$$

We shall prove the following.

Theorem 9. *There exists a constant $C_{m,p}$ such that for any open set $\Omega \subseteq \mathbb{R}^m$*

$$\|w\|_{L^\infty(\Omega)} \le C_{m,p} \lambda^{-\frac{1}{p-1}}. \tag{28}$$

Theorem 9 extends the inequality obtained in [6] and [7] to the p-Laplacian, for which we give an elliptic proof. We refer the reader to [11, Theorem 13], where this inequality is proved for convex sets.

Proof of Theorem 9. It is enough to consider a smooth, bounded open set Ω and approach a general open set with an increasing sequence of smooth inner sets. We extend the torsion function w to all of $\mathbb{R}^m$ by 0, and denote the new function again by w. Then w satisfies

$$-\Delta_p w \le 1,$$

in the sense that

$$\forall \varphi \in C_c^\infty(\mathbb{R}^m), \varphi \ge 0, \int_{\mathbb{R}^m} |\nabla w|^{p-2} \nabla w \nabla \varphi \le \int_{\mathbb{R}^m} \varphi. \tag{29}$$

We prove the following Cacciopoli type inequality.

Lemma 10. *For every $c_1 > 2^{p-1}$ there exists c_2 depending on c_1, m and p such that for every $\theta \in W^{1,\infty}(\mathbb{R}^m)$ we have*

$$\int_{\mathbb{R}^m} |\nabla(w\theta)|^p \le c_1 \int_{\mathbb{R}^m} w|\theta|^p + c_2 \int_{\mathbb{R}^m} |\nabla \theta|^p w^p.$$

Proof. Without loss of the generality, we may assume that $\theta \ge 0$. By taking $\varphi = w\theta^p$ as a test function in (29) it suffices to prove that

$$\int_{\mathbb{R}^m} |\nabla(w\theta)|^p \le c_1 \int_{\mathbb{R}^m} |\nabla w|^{p-2} \nabla w \nabla(w|\theta|^p) + c_2 \int_{\mathbb{R}^m} |\nabla \theta|^p w^p.$$

Since

$$\int_{\mathbb{R}^m} |\nabla(w\theta)|^p \le 2^{p-1} \int_{\mathbb{R}^m} \left(|\nabla w|^p \theta^p + |\nabla \theta|^p w^p\right),$$

and

$$\int_{\mathbb{R}^m} |\nabla w|^{p-2} \nabla w \nabla(w|\theta|^p) = \int_{\mathbb{R}^m} |\nabla w|^p \theta^p + p \int_{\mathbb{R}^m} |\nabla w|^{p-2} \nabla w \nabla \theta \theta^{p-1} w$$

it suffices to prove that

$$-pc_1\int_{\mathbb{R}^m}|\nabla w|^{p-2}\nabla w\nabla\theta\,\theta^{p-1}w$$
$$\leq (c_1-2^{p-1})\int_{\mathbb{R}^m}|\nabla w|^p\theta^p+(c_2-2^{p-1})\int_{\mathbb{R}^m}|\nabla\theta|^p w^p,$$

or even

$$\int_{\mathbb{R}^m}|\nabla w|^{p-1}\theta^{p-1}|\nabla\theta|w\leq\frac{c_1-2^{p-1}}{pc_1}\int_{\mathbb{R}^m}|\nabla w|^p\theta^p+\frac{c_2-2^{p-1}}{pc_1}\int_{\mathbb{R}^m}|\nabla\theta|^p w^p.$$

This last inequality is a consequence of Young's inequality, for c_2 given by

$$c_2=2^{p-1}+c_1\Big(\frac{(p-1)c_1}{c_1-2^{p-1}}\Big)^{p-1}.$$

□

In order to get a pointwise bound of w in terms of the average of w on balls, we recall the following result from [20] (see also [25]).

Lemma 11. *Let $p\in(1,m]$ and $u\in W^{1,p}(\mathbb{R}^m)$, $u\geq 0$ and $-\Delta_p u\leq 1$. Let $\gamma\in(p-1,\frac{N(p-1)}{N-(p-1)})$. Then there exist two constants C,C' independent of u such that*

$$u(0)\leq C\Big(\int_{B(0,1)}u^\gamma\Big)^{\frac{1}{\gamma}}+C'.$$

Proof. This is a consequence of [20, Theorem 3.3]. □

We now continue our proof of Theorem 9. Clearly, by re-scaling we get that

$$u(0)\leq C\Big(\frac{1}{r^m}\int_{B(0,r)}u^\gamma\Big)^{\frac{1}{\gamma}}+C'r^{\frac{p}{p-1}}.$$

We shall choose $\gamma\in(p-1,p)$, close to $p-1$. For such a γ we have by Hölder's inequality

$$\Big(\frac{1}{r^m}\int_{B(0,r)}u^\gamma\Big)^{\frac{1}{\gamma}}\leq\Big(\frac{1}{r^m}\int_{B(0,r)}u^p\Big)^{\frac{1}{p}}\omega_m^{1-\frac{\gamma}{p}}.$$

So changing the constant C we have for $p\in(1,m]$

$$u(0)\leq C\Big(\frac{1}{r^m}\int_{B(0,r)}u^p\Big)^{\frac{1}{p}}+C'r^{\frac{p}{p-1}}.\tag{30}$$

If $p>m$, this inequality holds as well. This is a consequence of the continuous embedding of $W^{1,p}(B_1(0))$ in $L^\infty(B_1(0))$ and of Lemma 10. Indeed, from Lemma 10, there exist constants (which may change from line to line) such that

$$\int_{B(0,1/2)}|\nabla u|^p\leq c_1\int_{B(0,1)}u+c_2\int_{B(0,1)}u^p$$
$$\leq C\int_{B(0,1)}u^p+C'.$$

On the other hand

$$\|u\|_{L^\infty(B(0,1/2))} \le C\|u\|_{L^p(B(0,1/2))} + C\|\nabla u\|_{L^p(B(0,1/2))},$$

so that

$$\|u\|_{L^\infty(B(0,1/2))} \le C\|u\|_{L^p(B(0,1))} + C'.$$

By re-scaling, we obtain inequality (30).

Let now $\theta \in C_c^\infty(B_2(0)$, $0 \le \theta \le 1$, $\theta \equiv 1$ on $B_1(0)$. Let $\theta_R(x) = \theta(\frac{x}{R})$. Then $w\theta \in W_0^{1,p}(\Omega)$, so

$$\lambda(\Omega) \le \frac{\displaystyle\int_\Omega |\nabla(w\theta)|^p}{\displaystyle\int_\Omega w^p\theta^p}$$

and from Lemma 10

$$\lambda(\Omega) \le \frac{c_1\displaystyle\int_\Omega w\theta^p + c_2\int_\Omega |\nabla\theta|^p w^p}{\displaystyle\int_\Omega w^p\theta^p}.$$

Using inequality (30) we have for R small enough

$$\frac{R^m}{C^p}\Big(w(0) - C'R^{\frac{p}{p-1}}\Big)^p \le \int_{B(0,R)} w^p.$$

At the same time

$$\int_{B(0,2R)} \theta_R^p \le \omega_m 2^m R^m \ \text{ and } \int_{B(0,2R)} |\nabla\theta_R|^p \le C''R^{m-p}.$$

As a consequence, renaming constants, we get

$$\lambda(\Omega) \le \frac{Cw(0)R^m + C'w(0)^pR^{m-p}}{R^m\Big(w(0) - C''R^{\frac{p}{p-1}}\Big)^p}.$$

We choose R such that

$$w(0) = 2C''R^{\frac{p}{p-1}},$$

and so

$$\lambda(\Omega) \le \frac{\tilde{C}}{R^p} = \frac{2^{p-1}C''^{p-1}\tilde{C}}{w(0)^{p-1}},$$

where

$$\tilde{C} = \frac{2CC'' + (2C'')^pC'}{C''^p}.$$

This concludes the proof of inequality (28). $\square$

4.2 Robin boundary conditions

The p-torsion function with Dirichlet boundary conditions on the ball $B(0, R)$ is given by

$$u(x) = (p-1)p^{-1}m^{-(p-1)^{-1}}(R^{p/(p-1)} - |x|^{p/(p-1)}),\ |x| < R.$$

Hence the solution is bounded, positive and regular. It follows by the comparison- and regularity theorems that if $\Omega \subset B(0, R)$ then the p-torsion function for Ω with Dirichlet boundary conditions satisfies

$$\|u\|_{L^\infty(\Omega)} \le (p-1)p^{-1}m^{-(p-1)^{-1}}R^{p/(p-1)}.$$

See [11] and the references therein. In this section we obtain some results for the p-torsion function with Robin boundary conditions and the corresponding torsional rigidity.

Let $p > 1$ and $\Omega \subset \mathbb{R}^m$ be an open set of finite measure. We introduce the torsion function for the p-Laplacian with Robin boundary conditions relaying on the $W^1_{p,p}(\Omega, \partial\Omega)$-spaces (see [22]). All results of this section can be rephrased in the framework introduced in [10], where the Robin problem in non-smooth sets is defined by using the SBV- spaces (see Remark 1 at the end of the section).

The torsion function u_b is the unique weak solution of

$$-\Delta_p u = 1 \text{ in } \Omega, \quad |\nabla u|^{p-2}\frac{\partial u}{\partial n} + b|u|^{p-2}u = 0 \text{ on } \partial\Omega,$$

which is the minimizer in $W^1_{p,p}(\Omega, \partial\Omega)$ of the energy

$$v \mapsto \int_\Omega |\nabla v|^p + b\int_{\partial\Omega} |v|^p d\mathcal{H}^{m-1} - p\int_\Omega v. \tag{31}$$

Let us notice that u is non-negative and continuous in Ω. We introduce the family of open sets $U_t = \{u > t, t \ge 0\}$ and denote by $\lambda(\Omega, b)$ the first Robin eigenvalue for the open set Ω associated to the Robin constant b, which is defined as

$$\lambda(\Omega, b) = \inf\Big\{\frac{\int_\Omega |\nabla v|^p + b\int_{\partial\Omega} |v|^p d\mathcal{H}^{m-1}}{\int_\Omega |v|^p} : v \in W^1_{p,p}(\Omega, \partial\Omega), v \ne 0\Big\}. \tag{32}$$

Throughout this section we suppress the p-dependence of the first Robin eigenvalue, torsion function etc. The isoperimetric inequality for the first eigenvalue of the Robin p-Laplacian in a non-smooth setting (see [10]) states that

$$\lambda(U_t, b) \ge \lambda(U_t^*, b), \tag{33}$$

where we adopt the usual notation: for any measurable set A with finite Lebesgue measure A^* stands for the ball of measure $|A|$ centered at 0.

Let us define the constants

$$c_1 = \frac{m}{m(p-1)+1},$$

$$c_2 = p^{\frac{m}{m(p-1)+1}}(m(p-1)+1),$$

$$c_3 = \frac{1}{(p-1)(m(p-1)+1)},$$

$$h(t) = \frac{|\Omega|^{\frac{1}{m}}}{|U_t|^{\frac{1}{m}}}.$$

We put $U_t^\sharp = h(t)U_t$ so that $|U_t^\sharp| = |\Omega|$. We prove the following result.

Theorem 12. *For every open set of finite measure, the torsion function u_b belongs to $L^\infty(\Omega)$ and*

$$\int_0^{\|u_b\|_{L^\infty(\Omega)}} \left(h(t)^{p-1}\lambda(\Omega^*, \frac{b}{h(t)^{p-1}}) \right)^{c_1} dt \leq \frac{c_2}{\lambda(\Omega, b)^{c_3}}. \tag{34}$$

In the proof of Theorem 12 we will need the following.

Lemma 13. *Let B be any open ball in $\mathbb{R}^m$ with finite Lebesgue measure. Then*

$$\lim_{\alpha \downarrow 0} \frac{\lambda(B, b\alpha)}{\alpha} = mb.$$

Proof. Let u be the first normalized eigenfunction on $B(0,1)$. The mapping $[0,1] \ni r \mapsto b(r) = \frac{|\nabla u(r)|^{p-1}}{u(r)^{p-1}}$ is increasing and continuous (see for instance [9, Proposition 4.2]). Moreover, we have

$$\lambda(B(0,r), b(r)) = \lambda(B(0,1), b).$$

By re-scaling we get that

$$\frac{1}{r^p}\lambda(B(0,1), b(r)r^{p-1}) = \lambda(B(0,1), b),$$

so that

$$\frac{1}{b(r)r^{p-1}}\lambda(B(0,1), b(r)r^{p-1}) = \frac{r}{b(r)}\lambda(B(0,1), b).$$

It remains to prove that

$$\lim_{r \downarrow 0} \frac{r}{b(r)}\lambda(B(0,1), b) = \frac{\mathcal{H}^{m-1}(\partial B(0,1))}{|B(0,1)|}.$$

Multiplying the identity $-\Delta_p u = \lambda(B(0,1), b)|u|^{p-2}u$ with u and integrating on $B(0,r)$ we get

$$\int_{B(0,r)} |\nabla u|^p + b(r)\int_{\partial B(0,r)} |u|^p d\mathcal{H}^{m-1} = \lambda(B(0,1), b)\int_{B(0,r)} |u|^p.$$

Dividing by r^m and passing to the limit $r \downarrow 0$, we get

$$\lim_{r \downarrow 0} \frac{b(r)}{r}\mathcal{H}^{m-1}(\partial B(0,1))|u(0)|^p = \lambda(B(0,1), b)|u(0)|^p|B(0,1)|,$$

where we have used that $\nabla u(0) = 0$ and $u(0) \neq 0$ by the regularity of u inside the ball. □

Lemma 13 above extends (5) to the p-Laplacian with Robin boundary conditions in the special case of a ball.
Proof of Theorem 12. As usual in the search of L^∞ estimates, we choose $u\wedge t := \min\{u,t\}$ as a test function in (31). We have that

$$\begin{aligned}&\int_\Omega |\nabla u|^p + b\int_{\partial\Omega}|u|^p d\mathcal{H}^{m-1} - p\int_\Omega u\\ &\le \int_\Omega |\nabla(u\wedge t)|^p + b\int_{\partial\Omega}|u\wedge t|^p d\mathcal{H}^{m-1} - p\int_\Omega (u\wedge t).\end{aligned}$$

But

$$\int_\Omega |\nabla(u\wedge t)|^p = \int_{\{0<u<t\}}|\nabla u|^p = \int_\Omega |\nabla u|^p - \int_{U_t}|\nabla u|^p,$$

$$\int_\Omega (u\wedge t) = \int_{\Omega\setminus U_t} u + \int_{U_t} t = \int_\Omega u - \int_\Omega (u-t)^+,$$

and

$$\begin{aligned}\int_{\partial\Omega}|u\wedge t|^p d\mathcal{H}^{m-1} &= \int_{\partial\Omega\cap\{0<u<t\}}|u|^p d\mathcal{H}^{m-1} + \int_{\partial\Omega\cap\{u\ge t\}} t^p d\mathcal{H}^{m-1}\\ &= \int_{\partial\Omega}|u|^p d\mathcal{H}^{m-1} - \int_{\partial\Omega\cap\{u\ge t\}}(|u|^p - t^p)d\mathcal{H}^{m-1}\\ &= \int_{\partial\Omega}|u|^p d\mathcal{H}^{m-1} - \int_{\partial U_t}(|u|^p - t^p)d\mathcal{H}^{m-1}.\end{aligned}$$

This last equality is a consequence of the fact that for every $x\in\partial U_t\cap\Omega$, we have (from the continuity of u) that $u(x)=t$. Finally,

$$\int_{U_t}|\nabla(u-t)^+|^p + b\int_{\partial U_t}(|(u-t)^+ + t|^p - t^p)d\mathcal{H}^{m-1} \le p\int_{U_t}(u-t)^+,$$

so that by the definition of the first Robin eigenvalue

$$\begin{aligned}\lambda(U_t,b)\int_{U_t}|(u-t)^+|^p &\le \int_{U_t}|\nabla(u-t)^+|^p + b\int_{\partial U_t}|(u-t)^+|^p d\mathcal{H}^{m-1}\\ &\le p\int_{U_t}(u-t)^+.\end{aligned}$$

Let $f(t)=\int_{U_t}(u-t)^+, t\ge 0$, then Hölder's inequality gives that

$$\lambda(U_t,b)f(t)^p\frac{1}{|U_t|^{\frac{p}{p'}}} \le pf(t).$$

Equivalently, introducing the re-scaling of U_t, we get that

$$f(t)^{p-1}h(t)^p\lambda(U_t^\sharp,\frac{b}{h(t)^{p-1}}) \le p|U_t|^{p-1},$$

or

$$f(t)^{p-1}h(t)^{p-1}\lambda(U_t^\sharp,\frac{b}{h(t)^{p-1}})|\Omega|^{\frac{1}{m}} \le p|U_t|^{p-1+\frac{1}{m}}.$$

Since $f'(t) = -|U_t|$, we get that

$$\frac{|\Omega|^{\frac{1}{m(p-1)+1}}}{p^{\frac{m}{m(p-1)+1}}}\left[h(t)^{p-1}\lambda(U_t^\sharp, \frac{b}{h(t)^{p-1}})\right]^{\frac{m}{m(p-1)+1}} \leq -\frac{f'(t)}{f(t)^{\frac{m(p-1)}{m(p-1)+1}}}.$$

Integrating this differential inequality between 0 and some value $T < \|u\|_{L^\infty(\Omega)}$, we get that

$$\frac{|\Omega|^{\frac{1}{m(p-1)+1}}}{p^{\frac{m}{m(p-1)+1}}}\int_0^T \left(h(t)^{p-1}\lambda(U_t^\sharp, \frac{b}{h(t)^{p-1}})\right)^{\frac{m}{m(p-1)+1}} dt$$
$$\leq (m(p-1)+1)(f(0)^{\frac{1}{m(p-1)+1}} - f(T)^{\frac{1}{m(p-1)+1}}).$$

By the isoperimetric inequality (33) and the positivity of f we obtain that

$$\frac{|\Omega|^{\frac{1}{m(p-1)+1}}}{p^{\frac{m}{m(p-1)+1}}}\int_0^T \left(h(t)^{p-1}\lambda(\Omega^*, \frac{b}{h(t)^{p-1}})\right)^{\frac{m}{m(p-1)+1}} dt$$
$$\leq (m(p-1)+1)f(0)^{\frac{1}{m(p-1)+1}}. \tag{35}$$

Since

$$\begin{aligned}\|u_b\|_{L^1(\Omega)} &= \int_\Omega |\nabla u_b|^p + b\int_{\partial\Omega} |u_b|^p d\mathcal{H}^{m-1} \\ &\geq \lambda(\Omega, b)\|u_b\|^p_{L^p(\Omega)} \\ &\geq \lambda(\Omega, b)\frac{\|u_b\|^p_{L^1(\Omega)}}{|\Omega|^{p-1}},\end{aligned} \tag{36}$$

we have that

$$\|u_b\|_{L^1(\Omega)} \leq \frac{|\Omega|}{\lambda(\Omega, b)^{\frac{1}{p-1}}}. \tag{37}$$

By (35), (36), (37) and the fact that $f(0) = \|u\|_{L^1(\Omega)}$ we conclude that

$$\frac{|\Omega|^{\frac{1}{m(p-1)+1}}}{p^{\frac{m}{m(p-1)+1}}}\int_0^T \left(h(t)^{p-1}\lambda(\Omega^*, \frac{b}{h(t)^{p-1}})\right)^{\frac{m}{m(p-1)+1}} dt \leq \frac{c_2}{\lambda(\Omega, b)^{c_3}}. \tag{38}$$

Since $h(t) \to +\infty$ as $t \uparrow \|u\|_{L^\infty(\Omega)}$, we have by Lemma 13 that

$$\lim_{t\uparrow\|u\|_\infty} h(t)^{p-1}\lambda(\Omega^*, \frac{b}{h(t)^{p-1}}) = mb.$$

Hence $\|u_b\|_{L^\infty(\Omega)} < \infty$. Then choosing $T = \|u_b\|_{L^\infty(\Omega)}$ in (38) completes the proof of Theorem 12. □

In analogy with (1) we define the torsional rigidity for the p-Robin torsion function by

$$P(\Omega, b) = \int_\Omega u_b,$$

where u_b is a minimiser of (31). It is easily seen that $u_b \geq 0$. Hence $P(\Omega, b) = \|u\|_{L^1(\Omega)}$.

Theorem 14. *If Ω is an open set in $\mathbb{R}^m, m = 2, 3, \cdots$ with $|\Omega| < \infty$, and if $p > 1$ and $b > 0$ then*

$$b^{-1/(p-1)}|\Omega|^{p/(p-1)}\mathcal{H}^{m-1}(\partial\Omega)^{-1/(p-1)} \leq P(\Omega, b) \leq \lambda(\Omega, b)^{-1/(p-1)}|\Omega|.$$

Proof. The infimum in (31) is attained by the p-torsion function $u_b \in W^1_{p,p}(\Omega)$. We have the following variational characterization.

$$P(\Omega, b)^{p-1} = \sup \left(\int_\Omega |\nabla v|^p + b \int_{\partial\Omega} |v|^p d\mathcal{H}^{m-1} \right)^{-1} \left| \int_\Omega v \right|^p,$$

where the supremum is over all $v \in W^1_{p,p}(\Omega) \setminus \{0\}$. To prove the lower bound we choose the test function $v = 1$. To prove the upper bound we have by Hölder's inequality that $\left(\int_\Omega v\right)^p \leq |\Omega|^{p-1} \int_\Omega |v|^p$. The variational characterization of $\lambda(\Omega, b)$ in (32) gives that $\int_\Omega |\nabla v|^p + b \int_{\partial\Omega} |v|^p d\mathcal{H}^{m-1} \geq \lambda(\Omega, b) \int_\Omega |v|^p$, which completes the proof. □

Remark 1. If Ω is an open set with non-smooth boundary then the space $W^1_{p,p}(\Omega, \partial\Omega)$ does not lead to the natural relaxation of the Robin problem. Precisely if Ω is a Lipschitz set from which one removes a Lipschitz crack, in the space $W^1_{p,p}(\Omega, \partial\Omega)$ all functions have the same trace on both sides of the crack, while one can write properly the Robin problem in the Sobolev space $W^{1,p}(\Omega)$ which is much larger than $W^1_{p,p}(\Omega, \partial\Omega)$. A suitable relaxation of the Robin problem, based on the special functions with bounded variations, was introduced in [10] to deal with these situations (see [1] for details). The torsional rigidity could be defined on the open bounded sets in $\mathbb{R}^m$ by $P(\Omega) = \int_\Omega u dx$, where u is any minimizer of

$$v \mapsto \int_\Omega |\nabla v|^p + b \int_{J_v} (|v^+|^p + |v^-|^p) d\mathcal{H}^{m-1} - p \int_\Omega v,$$

among all non-negative functions $v \in L^p(\mathbb{R}^m)$ such that $v^p \in SBV(\mathbb{R}^m)$, $v = 0$ a.e. on $\mathbb{R}^m \setminus \Omega$, $\mathcal{H}^{m-1}(J_v \setminus \partial\Omega) = 0$. Above J_v is the set of jump points of v and v^+ and v^- stand for the upper and lower approximate limits of v in a point of the jump set. A similar estimate as the one in (34) can be obtained in the SBV framework, by replacing the $W^1_{p,p}$-Robin eigenvalues with the SBV-Robin eigenvalues.

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