# A FABER-KRAHN INEQUALITY FOR THE CHEEGER CONSTANT OF $N$-GONS

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Abstract. We prove that the regular $N$-gon minimizes the Cheeger constant among polygons with a given area and $N$ sides.

## 1. Introduction

The Cheeger constant of a set $\Omega \subset \mathbb{R}^2$ having finite measure and a Lipschitz boundary is defined by

$$h(\Omega) := \inf \Big\{ \frac{\mathrm{Per}(A, \mathbb{R}^2)}{|A|} \ : \ A \text{ measurable}\,,\ A \subseteq \Omega \Big\}\,. \tag{1}$$

Here and below, $\mathrm{Per}(A, \mathbb{R}^2)$ denotes the perimeter of $A$ in the sense of De Giorgi and $|A|$ denotes the volume or Lebesgue measure of $A$.
The minimization problem (1), named after Cheeger who introduced it in [11], has attracted a lot of interest in recent years; without any attempt of completeness, a list of related works is [1, 2, 8, 9, 10, 14, 16, 17, 21, 24, 25, 28]. Here we limit ourselves to recall that, for $\Omega$ as above, there exists at least a solution to (1), which is called a Cheeger set of $\Omega$, and in general is not unique (unless $\Omega$ is convex, see [1]). Let us also mention that the Cheeger constant can be interpreted as the first Dirichlet eigenvalue of the 1-Laplacian (see [22, 23]), as the relaxed formulation of problem (1) reads

$$\inf \Big\{ \frac{|Du|(\mathbb{R}^2)}{\int_\Omega |u|} \ : \ u \in BV(\mathbb{R}^2) \setminus \{0\}\,,\ u = 0 \text{ on } \mathbb{R}^2 \setminus \Omega \Big\}\,.$$

It readily follows from definition (1) and the isoperimetric inequality that the ball minimizes the Cheeger constant under a volume constraint. Indeed, denoting by $\Omega^*$ a ball with the same volume as $\Omega$, by $C(\Omega)$ a Cheeger set of $\Omega$, and by $C^*(\Omega) \subseteq \Omega^*$ a ball with the same volume as $C(\Omega)$, it holds

$$h(\Omega) = \frac{\mathrm{Per}(C(\Omega), \mathbb{R}^2)}{|C(\Omega)|} \geq \frac{\mathrm{Per}(C^*(\Omega), \mathbb{R}^2)}{|C^*(\Omega)|} \geq h(\Omega^*)\,. \tag{2}$$

In this paper we prove the following discrete version of the isoperimetric inequality (2):

**Theorem 1.** *Among all simple polygons with a given area and at most $N$ sides, the regular $N$-gon minimizes the Cheeger constant.*

The main motivation which led us to study the minimization of $h(\Omega)$ over the class of polygons with prescribed area and number of sides came from a long-standing conjecture

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by Pólya and Szegö about discrete versions of classical isoperimetric inequalities having the ball as optimal domain.
Actually, like (2), well known isoperimetric-type inequalities state that the ball is optimal when minimizing a shape functional under a volume constraint. This is clearly the case of perimeter, by the classical isoperimetric inequality, but also of many relevant shape functionals in the Calculus of Variations, such as the first Dirichlet eigenvalue of the Laplacian, by the Faber-Krahn inequality (see [18, Section 3]), or the torsional rigidity and the logarithmic capacity.
Thus, a very natural question is whether these symmetry results admit a discrete version, namely whether the optimal shape still obeys symmetry in the polygonal case.
In case of perimeter, an affirmative answer comes from the classical isoperimetric inequality for polygons in $\mathbb{R}^2$, which states that the regular $N$-gon minimizes the perimeter among simple polygons with given area and $N$ sides, see *e.g.* [7].
More than fifty years ago, Pólya and Szegö conjectured that the same property holds for the principal frequence, namely that the regular $N$-gon is the unique domain minimizing the first Dirichlet eigenvaule of the Laplacian among polygons with a given area and $N$ sides. Analogous conjectures were formulated also for the torsional rigidity and for the logarithmic capacity. For $N = 3$ and $N = 4$ these conjectures were proved by Pólya and Szegö themselves [26, p. 158], via the classical tool of Steiner symmetrization. For $N \geq 5$, as explained in [18, Section 3.3], Steiner symmetrization cannot be applied because it may increase the number of sides, and, though easy-to-state, Pólya and Szegö conjecture can be included into the class of challenging problems. To the best of our knowledge, at present the unique solved case is the one of logarithmic capacity, which was settled by Solynin and Zalgaller in the notable paper [27], whereas the cases of the first eigenvalue and torsional rigidity are currently open. In this respect, let us mention incidentally the recent paper [15], where it was proved that that the regular $N$-gon maximizes the torsional rigidity among the subclass of convex polygons, with a given area and $N$ sides, for which a suitable notion of "asymmetry measure" exceeds a critical threshold.
Theorem 1 provides another case, besides logarithmic capacity, in which there is preservation of symmetry when passing from minimization in the "continuum setting" of arbitrary domains to minimization in the "discrete setting" of polygons.
Similarly as it occurs for Pólya and Szegö conjecture, the main difficulties in order to obtain Theorem 1 are the impossibility of determining explicitly the Cheeger constant of a general polygon, and the failure of Steiner symmetrization as soon as $N \geq 5$. On the other hand, with respect to the case of the first eigenvalue, we can take advantage of the fact that the shape functional $h(\Omega)$ can be formulated without invoking a pde, so that Theorem 1 can be established by means of a careful geometric analysis.
Finally, let us point out that Theorem 1 can be seen as an extreme case of Pólya and Szegö conjecture, formulated for the $p$-Laplacian. In fact, the Cheeger constant is related to the $p$-Laplace eigenvalue problem as $p \to 1^+$ through the equality

$$\lim_{p \to 1^+} \lambda_p(\Omega) = h(\Omega)$$

where $\lambda_p$ denotes the first Dirichlet eigenvalue of the $p$-Laplacian. (A similar convergence result has been recently obtained in [6] in terms of $p$-torsion functions). The limit on the other extreme is given by (*cf.* [20])

$$\lim_{p \to +\infty} \lambda_p^{1/p}(\Omega) = \lambda_\infty(\Omega) := \frac{1}{\max_{x \in \overline{\Omega}} \operatorname{dist}(x, \partial\Omega)} .$$

Clearly the infimum of $\lambda_\infty(\Omega)$ among polygons with a given area and $N$ sides is attained, as well, at the regular $N$-gon.
A bit of mathematical faith, taken from [13], is that *"One important principle of mathematics is that extreme cases reveal interesting structure."* In this perspective, we believe that Theorem 1 brings some evidence to Pólya and Szegö conjecture, and hopefully can be of some help in order to prove it.
The following short outline of the paper summarizes how the proof of Theorem 1 proceeds. We point out that a much simpler proof, essentially based on the isoperimetric inequality for convex polygons, would allow to settle the case of simple *convex* polygons (*cf.* Remark 32).

- In Section 2, in order to obtain an existence result, we enlarge the class of admissible polygons, by taking, in a suitable sense, the closure of simple polygons with at most $N$ sides; in particular, polygons lying in this larger class may present self-intersections. For such generalized polygons, we introduce a new, *ad hoc* conceived by a natural relaxation procedure, notion of "Neumann-Cheeger constant", which reduces to the classical Cheeger constant in the case of simple polygons. In this framework, we obtain the existence of a generalized polygon which minimizes the Neumann-Cheeger constant under a constraint on the volume and on the number of sides. Moreover, we are able to provide a representation formula for the Neumann-Cheeger constant of such an optimal generalized polygon, which is used as a crucial tool in the sequel.
- In Section 3, we derive some stationarity conditions satisfied by a generalized polygon which minimizes the Neumann-Cheeger constant under a constraint on the volume and on the number of sides. To that aim, we perform first order shape derivatives with respect to suitable perturbations, namely rotations and parallel movements of one side of an optimal generalized polygon. By this way, we are able to deduce some relevant information about the length of the sides which do not contain self-intersections and on the measures of the angles formed by them.
- Relying on the results obtained in Section 3, in Section 4 we are able to exclude the possibility that the boundary of an optimal generalized polygon contains self-intersections and the possibility that it contains reflex angles. We are thus reduced to the case of simple convex polygons, among which the regular gon turns out to be the unique solution.
- The conclusion of the proof, along with the stronger form of Theorem 1 that it actually entails, is given in Section 5, where we also postpone some related remarks and open questions.

## 2. Existence of an optimal generalized polygon and representation of its Cheeger constant

Firstly, let us precise what is meant by "simple polygons with at most $N$ sides" in the statement of Theorem 1.

**Definition 2.** A *simple polygon* is the open bounded planar region $\Omega$ delimited by a finite number of not self-intersecting line segments (called *sides*) which are pairwise joined (at their endpoints called *vertices*) to form a closed path. We denote by $\mathcal{P}_N$ the class of simple polygons with at most $N$ sides.

Then our object of study is the following shape optimization problem:

$$\min\Big\{h(\Omega)\ :\ \Omega\in\mathcal{P}_N\,,\quad |\Omega|=c\Big\}\,, \tag{3}$$

where $c$ is a positive constant.
In order to gain the existence of an optimal domain, we are led to enlarge the class of admissible polygons in the above shape optimization problem.
Let us begin by recalling that the *Hausdorff complementary distance* between two open sets $\Omega_1,\Omega_2\subset\mathbb{R}^2$ is defined by

$$d_{H^c}(\Omega_1,\Omega_2):=\sup_{x\in\mathbb{R}^2}\big|\mathrm{dist}(x,\Omega_1^c)-\mathrm{dist}(x,\Omega_2^c)\big|\,,$$

where $\Omega_i^c$ denotes the complement of $\Omega_i$ and $\mathrm{dist}(\cdot,\Omega_i^c)$ is the Euclidean distance from the closed set $\Omega_i^c$.
Given a sequence of open sets $\{\Omega_h\}$ and an open set $\Omega$, by writing

$$\Omega_h\xrightarrow{H^c}\Omega\qquad\text{and}\qquad\Omega_h\xrightarrow{H^c_{\rm loc}}\Omega\,,$$

we mean respectively that $\lim_h d_{H^c}(\Omega_h,\Omega)=0$ and $\lim_h d_{H^c}(\Omega_h\cap B,\Omega\cap B)=0$ for every ball $B$.
For the properties of the Hausdorff complementary topology, we refer the reader to [4, 19].

**Definition 3.** A *generalized polygon with at most $N$-sides* is the limit in the $H^c_{\rm loc}$ topology of a sequence $\{\Omega_h\}\subset\mathcal{P}_N$ such that $\limsup_h|\Omega_h|<+\infty$. The class of generalized polygons with at most $N$ sides is denoted by $\overline{\mathcal{P}_N}$.

*Remark* 4. Let $\Omega\in\overline{\mathcal{P}_N}$. Then:

- (i) $\Omega$ is an open set;
- (ii) $\Omega$ is simply connected, since $\Omega^c$ is connected [19, Remark 2.2.18];
- (iii) $\Omega$ may be disconnected; each connected component of $\Omega$ is delimited by a finite number of line segments (still called the sides of $\Omega$), which are pairwise joined at their endpoints (still called vertices of $\Omega$) to form a closed path, possibly containing self-intersections;
- (iv) $\Omega$ has finite Lebesgue measure [19, Proposition 2.2.21];
- (v) $\Omega$ is bounded (otherwise, since $\Omega$ has at most $N$ sides, necessarily it would have two parallel sides, contradicting item (iv)).

We now introduce a new notion of Neumann-Cheeger constant. As well as the classical notion (1), it can be given for every subset $\Omega$ of $\mathbb{R}^2$ having finite measure and a Lipschitz boundary; actually, we shall use it only for generalized polygons. We stress that we need to introduce this notion of Neumann-Cheeger constant just for technical reasons, that is to say in order to handle the possible self-intersections of generalized polygons (which in turn cannot be avoided to have an existence result). On the other hand, it will be clear from the definition that our notion of Neumann-Cheeger constant reduces to the classical Cheeger constant for simple polygons.
Let us prepone the following new notion of Neumann-perimeter relative to $\Omega$:

**Definition 5.** Let $\Omega\subset\mathbb{R}^2$ be a set having finite Lebesgue measure and a Lipschitz boundary, and let $A$ be a measurable subset of $\Omega$. Then we define the *Neumann-perimeter of $A$ relative to* $\Omega$ by

$$\overline{\mathrm{Per}}(A,\Omega):=\sup\Big\{\int_A \mathrm{div}V\,dx\ :\ V\in W^{1,2}(\Omega;\mathbb{R}^2)\cap C(\Omega;\mathbb{R}^2)\,,\ \|V\|_{L^\infty}\leq 1\Big\}\,.$$

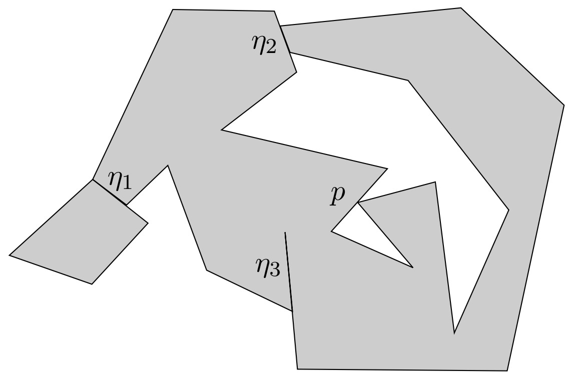


FIGURE 1. An example of generalized polygon, with 2 connected components, which self-intersects at the point $p$ and at the segments $\eta_1$, $\eta_2$, $\eta_3$.

*Remark* 6. We point out that, with respect to the classical definitions of perimeter appearing in (1) and of perimeter relative to $\Omega$, which read respectively

$$\mathrm{Per}(A,\mathbb{R}^2) := \sup\Big\{ \int_A \mathrm{div} V\, dx \ : \ V \in C_0^\infty(\mathbb{R}^2;\mathbb{R}^2)\,,\ \|V\|_{L^\infty} \leq 1 \Big\}$$
$$\mathrm{Per}(A,\Omega) := \sup\Big\{ \int_A \mathrm{div} V\, dx \ : \ V \in C_0^\infty(\Omega;\mathbb{R}^2)\,,\ \|V\|_{L^\infty} \leq 1 \Big\}\,,$$

the crucial difference appearing in Definition 5 is the different choice of the class of test fields (which is the reason why we have chosen the terminology "Neumann-perimeter"). For simple polygons $\Omega \in \mathcal{P}_N$, $\overline{\mathrm{Per}}(A,\Omega)$ agrees with $\mathrm{Per}(A,\mathbb{R}^2)$; the same holds true also when $\Omega \in \overline{\mathcal{P}_N}$ provided $\overline{A} \subset \Omega$. In spite, when $\Omega \in \overline{\mathcal{P}_N}$ and $\overline{A} \not\subset \Omega$, in general it holds $\mathrm{Per}(A,\Omega) \leq \overline{\mathrm{Per}}(A,\Omega)$, with possibly strict inequality. More precisely, if there is a line segment $S$ contained into two sides of $\Omega$, the $\mathcal{H}^1$-measure of the points of $A$ having density 1 and lying on $S$ is counted twice in the computation of $\overline{\mathrm{Per}}(A,\Omega)$ (wheres it does not appear at all in the computation of $\mathrm{Per}(A,\Omega)$).

**Definition 7.** Let $\Omega \subset \mathbb{R}^2$ be a set with finite Lebesgue measure and a Lipschitz boundary. We define the *Neumann-Cheeger constant* of $\Omega$ by

$$\overline{h}(\Omega) := \inf\Big\{ \frac{\overline{\mathrm{Per}}(A,\Omega)}{|A|} \ : \ A \text{ measurable}\,,\ A \subseteq \Omega \Big\}\,, \tag{4}$$

**Proposition 8.** *On the class of generalized polygons, the Neumann-Cheeger constant enjoys the following properties:*

(i) *It is monotone decreasing with respect to inclusion.*
(ii) *It is homogeneous of degree $-1$ by homotheties.*
(iii) *For every $\Omega \in \overline{\mathcal{P}_N}$, there exists at least a Neumann-Cheeger set of $\Omega$, namely a measurable subset of $\Omega$ at which the infimum in (4) is attained.*
(iv) *If $C(\Omega)$ is a Neumann-Cheeger set of $\Omega \in \overline{\mathcal{P}_N}$, the set $\partial C(\Omega) \cap \Omega$ is made by arcs of circle of curvature $\overline{h}(\Omega)$; moreover, $\partial C(\Omega)$ necessarily meets $\partial\Omega$, this occurs either tangentially or at a vertex, and $\partial C(\Omega) \cap \partial\Omega$ may contain self-intersections.*

*Proof.* (i) Let $\Omega_1 \subseteq \Omega_2$ be two generalized polygons. Then

$$\begin{aligned}
\overline{h}(\Omega_2) &= \inf\Big\{\frac{\overline{\mathrm{Per}}(A,\Omega_2)}{|A|}\ :\ A \text{ measurable},\ A\subseteq \Omega_2\Big\}\\
&\le \inf\Big\{\frac{\overline{\mathrm{Per}}(A,\Omega_2)}{|A|}\ :\ A \text{ measurable},\ A\subseteq \Omega_1\Big\}\\
&\le \inf\Big\{\frac{\overline{\mathrm{Per}}(A,\Omega_1)}{|A|}\ :\ A \text{ measurable},\ A\subseteq \Omega_1\Big\}\\
&= \overline{h}(\Omega_1)\,,
\end{aligned}$$

where the first inequality comes directly from the assumption $\Omega_1 \subseteq \Omega_2$, and the second one from the fact that, due to the inclusion of fields in $W^{1,2}(\Omega_2;\mathbb{R}^2)\cap C(\Omega_2;\mathbb{R}^2)$ into $W^{1,2}(\Omega_1;\mathbb{R}^2)\cap C(\Omega_1;\mathbb{R}^2)$, we have $\overline{\mathrm{Per}}(A,\Omega_2)\le \overline{\mathrm{Per}}(A,\Omega_1)$ for any measurable set $A\subseteq\Omega_1$.
(ii) The fact that $\overline{h}$ is homogeneous of degree $-1$ follows from the fact that, for every measurable set $A\subset\Omega\in\overline{\mathcal{P}_N}$, it holds $\overline{\mathrm{Per}}(\lambda A,\lambda\Omega)=\lambda\overline{\mathrm{Per}}(A,\Omega)$, and $|\lambda A|=\lambda^2|A|$.
(iii) Let $\{A_n\}$ be a minimizing sequence for problem (4). If we are able to prove that it admits a minimizing sequence which converges in $L^1(\Omega)$, we are done. Indeed, it readily follows from its definition as the supremum of a family functionals which are continuous in $L^1(\Omega)$, that $\overline{\mathrm{Per}}(\cdot,\Omega)$ is lower semicontinuous in $L^1(\Omega)$. Hence, the set $A$ which is the $L^1$-limit of $\{A_n\}$ will be a solution to (4):

$$\overline{h}(\Omega)\le\frac{\overline{\mathrm{Per}}(A,\Omega)}{|A|}\le\liminf_n\frac{\overline{\mathrm{Per}}(A_n,\Omega)}{|A_n|}=\overline{h}(\Omega)\,.$$

Let us show that $\{A_n\}$ admits a subsequence which converges in $L^1(\Omega)$. For $k\in\mathbb{N}\setminus\{0\}$, set $\Omega^k:=\{x\in\Omega\ :\ \mathrm{dist}(x,\partial\Omega)\ge\frac{1}{k}\}$. Since $\{A_n\}$ is a minimizing sequence for problem (4), we have $\sup_n\overline{\mathrm{Per}}(A_n,\Omega)<+\infty$, and hence we also have $\sup_n\overline{\mathrm{Per}}(A_n\cap\Omega^k,\Omega)<+\infty$ for every fixed $k$. Now we observe that

$$\begin{aligned}
\mathrm{Per}(A_n\cap\Omega^k,\Omega^k) &= \sup\Big\{\int_{A_n\cap\Omega^k}\mathrm{div}V\,dx\ :\ V\in C_0^\infty(\Omega^k;\mathbb{R}^2)\,,\ \|V\|_{L^\infty}\le 1\Big\}\\
&\le \sup\Big\{\textstyle\int_{A_n\cap\Omega^k}\mathrm{div}V\,dx\ :\ V\in W^{1,2}(\Omega;\mathbb{R}^2)\cap C(\Omega;\mathbb{R}^2)\,,\ \|V\|_{L^\infty}\le 1\Big\}\\
&= \overline{\mathrm{Per}}(A_n\cap\Omega^k,\Omega)\,.
\end{aligned}$$

By the compact embedding of $BV(\Omega^k)$ into $L^1(\Omega^k)$, we deduce that, for every fixed $k$, the sequence $\{A_n\cap\Omega^k\}$ admits a subsequence which converges in $L^1(\Omega^k)$. Since $\lim_k|\Omega\setminus\Omega_k|=0$, we conclude that $\{A_n\}$ admits a subsequence which converges in $L^1(\Omega)$.
(iv) Since $\overline{\mathrm{Per}}(A,\Omega)$ agrees with the usual perimeter of $A$ in $\mathbb{R}^2$ for all sets $A$ such that $\overline{A}\subset\Omega$, all the properties stated here for a Neumann-Cheeger set of a generalized polygon (except for the possible presence of self-intersections in $\partial C(\Omega)\cap\partial\Omega$) are readily inherited from well-known properties of a Cheeger set of a simple polygon. For more details, we refer the reader to [25, Section 4] and references therein. Finally, since $\partial C(\Omega)$ meets necessarily $\partial\Omega$, the possible presence of self-intersections in $\partial C(\Omega)\cap\partial\Omega$ is an immediate consequence of the possible presence of self-intersections in $\partial\Omega$. □

We are now ready to prove an existence result for the following generalized version of problem (3):

$$\min\Big\{\overline{h}(\Omega)\ :\ \Omega\in\overline{\mathcal{P}_N}\quad|\Omega|=c\Big\}\,, \tag{5}$$

**Proposition 9.** *The shape optimization problem* (5) *admits at least a solution.*

*Remark* 10. An equivalent formulation of problem (5), which is convenient in order to drop the volume constaint and deal deal with a scaling invariant shape functional, is:

$$\min\left\{|\Omega|\overline{h}^2(\Omega)\ :\ \Omega\in\overline{\mathcal{P}_N}\right\}. \tag{6}$$

Namely, if $\Omega$ solves problem (5), it solves also problem (6); viceversa, if $\Omega$ solves problem (6), a suitable homothety of $\Omega$ solves (5) (*cf.* [18, Proposition 1.2.9]).

*Proof of Proposition 9.* Let $\{\Omega_n\}$ be a minimizing sequence for problem (5). For every $\Omega_n$ we select a connected Cheeger set $C_n$ of $\Omega_n$. Using the fact that $\{\Omega_n\}$ is a minimizing sequence, the relationship between the Neumann-perimeter and the classical one, and the isoperimetric inequality, we infer that there exist positive constants $k_1, k_2$ such that

$$|C_n|\ge k_1\overline{\mathrm{Per}}(C_n,\Omega_n)\ge k_1\mathrm{Per}(C_n,\mathbb{R}^2)\ge k_2|C_n|^{1/2}. \tag{7}$$

On the other hand, using the fact that $\Omega_n$ are admissible domains in (5), and the same inequality above, we infer that

$$c=|\Omega_n|\ge|C_n|\ge k_1\mathrm{Per}(C_n,\mathbb{R}^2). \tag{8}$$

From (7) and (8), we see respectively that $\liminf_n|C_n|>0$ and that $\limsup_n\mathrm{Per}(C_n,\mathbb{R}^2)<+\infty$. Since $C_n$ are connected, we deduct that they remain uniformly bounded, namely we can translate the sets $\Omega_n$ such that all $C_n$ lie in a fixed, sufficiently large ball $B$.
By (8) and the compact embedding of $BV(B)$ into $L^1(B)$, there exists a measurable set $C$ such that

$$C_n\xrightarrow{L^1}C. \tag{9}$$

By the compactness and lower semicontinuity properties of the Hausdorff complementary topology [19, Corollary 2.2.24 and Proposition 2.2.21], up to passing to a (not relabeled) subsequence, there exists $\Omega\in\overline{\mathcal{P}_N}$, with $|\Omega|\le c$, such that

$$\Omega_n\xrightarrow{H^c_{\mathrm{loc}}}\Omega. \tag{10}$$

Clearly, $C\subseteq\Omega$. Then it is enough to prove that

$$\overline{\mathrm{Per}}(C,\Omega)\le\liminf_n\overline{\mathrm{Per}}(C_n,\Omega_n). \tag{11}$$

Indeed, if (11) holds, we get

$$\overline{h}(\Omega)=\frac{\overline{\mathrm{Per}}(C,\Omega)}{|C|}\le\liminf_n\frac{\overline{\mathrm{Per}}(C_n,\Omega_n)}{|C_n|}\le\liminf_n\overline{h}(\Omega_n),$$

which readily implies that an homothety of $\Omega$ (precisely $\sqrt{c|\Omega|^{-1}}\Omega$) solves problem (5) and achieves the proof.
It remains to prove (11). To that aim we observe that

$$\Omega_n\xrightarrow{L^1_{\mathrm{loc}}}\Omega. \tag{12}$$

Indeed, the $L^1_{\mathrm{loc}}$ convergence in (12) follows from the $H^c_{\mathrm{loc}}$-convergence in (10) by applying Theorem 4.2 in [5]. (In fact, for every ball $B$, we can apply such result to the sequence $\Omega_n\cap B$ because both the number of connected components of $(\Omega_n\cap B)^c$ and the perimeter of $\Omega_n\cap B$ remain bounded from above: the former because the sets $\Omega_n$ are simply connected,

and the latter because, since $\Omega_n$ has at most $N$ sides, it can be estimated from above by $\mathrm{Per}(B, \mathbb{R}^2) + N\mathrm{diam}\,(B)$.)
We are now in a position to prove the lower semicontinuity property (11). If we are able to approximate any field $V \in W^{1,2}(\Omega;\mathbb{R}^2) \cap C(\Omega;\mathbb{R}^2)$ with $\|V\|_{L^\infty} \leq 1$, in the strong $W^{1,2}$ topology, by a sequence of fields $V_n$ in $W^{1,2}(\Omega_n;\mathbb{R}^2) \cap C(\Omega_n;\mathbb{R}^2)$ with $\|V\|_{L^\infty} \leq 1$, we get the required lower semicontinuity property; indeed, using also (9), we shall have

$$\overline{\mathrm{Per}}(C,\Omega) \leq \int_C \mathrm{div} V\, dx = \lim_n \int_{C_n} \mathrm{div} V_n\, dx \leq \liminf_n \overline{\mathrm{Per}}(C_n,\Omega_n)\,.$$

A sequence $\{V_n\}$ which gives the approximation above (which is related to the first condition in the Mosco-convergence of the Sobolev spaces $W^{1,2}(\Omega_n;\mathbb{R}^2)$ to $W^{1,2}(\Omega;\mathbb{R}^2)$) can be constructed by using the same arguments as in Section 3 of [5], to which we refer for more details. In short, the procedure works as follows. We first reduce ourselves to the case when the approximating sequence $\{\Omega_n\}$ is contained into a fixed ball. We stress that this is possible thanks to the fact that $\Omega$ is bounded (*cf.* Remark 4 (v)) and that the sequence $\{\Omega_n\} \subseteq \overline{P_N}$ satisfies (10)-(12): these conditions ensure that one can modify the generalized polygons $\Omega_n$ into a new sequence of generalized polygons $\widetilde{\Omega}_n$ which satisfy the same convergence properties as $\Omega_n$ and in addition are all contained into a fixed ball (such modification can be done by arguing as in the proof of Lemma 3.6 in [5]). Once we are reduced to the case when the approximating sequence $\{\Omega_n\}$ is contained into a fixed ball, we are in a position to apply Lemma 3.7 in [5] in order to get a sequence of fields $V_n \in W^{1,2}(\Omega_n;\mathbb{R}^2)$ which converge strongly in $W^{1,2}$ to $V$. Finally, the fact that these fields $V_n$ can be constructed in order to satisfy also the constraint $\|V_n\|_{L^\infty} \leq 1$ can be checked by inspection of the proof of Lemma 3.4 in [5]. □

The next result collects some easy-to-obtain qualitative properties of an optimal generalized polygon:

**Proposition 11.** *Let $\Omega \in \overline{\mathcal{P}_N}$ be a solution to problem* (5). *Then:*

(i) *$\Omega$ has exactly $N$ sides;*
(ii) *$\Omega$ is connected;*
(iii) *the generalized polygon whose boundary is obtained by eliminating from $\partial\Omega$ all line segments possibly contained into two consecutive sides of $\Omega$ is still a solution to problem* (5).

*Remark* 12. In view of Proposition 11 (iii), in the sequel when dealing with a solution $\Omega$ to problem (5), we shall directly assume, with no loss of generality, that there is no line segment contained into two consecutive sides of $\Omega$. For brevity, we shall call such a solution a *reduced optimal polygon*. For instance, if $\Omega$ would be the connected component of the generalized polygon represented in Figure 1 containing the point $p$ on its boundary, the corresponding reduced polygon would be obtained by eliminating the line segment $\eta_3$.

For the proof of Proposition 11 (i), we use in particular the following elementary fact, that we prefer to state separately since it which will be repeatedly used in the paper.

**Lemma 13.** *Let $C(\Omega)$ be a Neumann-Cheeger set of $\Omega$, and let $\widetilde{\Omega}$ be such that $C(\Omega) \subset \widetilde{\Omega} \subset \Omega$. Then $\overline{h}(\Omega) = \overline{h}(\widetilde{\Omega})$.*

*Proof.* Since $\widetilde{\Omega} \subset \Omega$, and $\overline{h}$ is monotone decreasing with respect to inclusions, there holds $\overline{h}(\Omega) \leq \overline{h}(\widetilde{\Omega})$. On the other hand, since $C(\Omega) \subset \widetilde{\Omega}$, $C(\Omega)$ is an admissible set for the Neumann-Cheeger problem in $\widetilde{\Omega}$, so that $\overline{h}(\widetilde{\Omega}) \leq \frac{|\partial C(\Omega)|}{|C(\Omega)|} = \overline{h}(\Omega)$. □

*Proof of Proposition 11.* (i) Assume by contradiction that $\Omega$ is a solution to problem (5) with strictly less than $N$ sides, and let $C(\Omega)$ denote one of its Cheeger sets. Then by "cutting" an angle of $\Omega$ which measures less than $\pi$, one can construct a polygon $\widetilde{\Omega}$ with one side more than $\Omega$ (thus, with at most $N$ sides), such that $C(\Omega) \subset \widetilde{\Omega} \subset \Omega$. Clearly $|\widetilde{\Omega}| < |\Omega|$ whereas, by Lemma 13, $\overline{h}(\widetilde{\Omega}) = \overline{h}(\Omega)$. We conclude that $|\Omega|\overline{h}^2(\Omega) > |\widetilde{\Omega}|\overline{h}^2(\widetilde{\Omega})$, so that $\Omega$ cannot be a solution to problem (6) and hence neither to problem (5), contradiction.
(ii) Assume by contradiction that $\Omega$ is disconnected. Denote by $\Omega_1$ a connected component of $\Omega$, and set $\Omega_2 := \Omega \setminus \Omega_1$. Let $C(\Omega)$ be a Neumann-Cheeger set of $\Omega$, and set $C_1 := C(\Omega) \cap \Omega_1$ and $C_2 := C(\Omega) \cap \Omega_2$. Since

$$\overline{\mathrm{Per}}(C(\Omega), \Omega) = \overline{\mathrm{Per}}(C_1, \Omega_1) + \overline{\mathrm{Per}}(C_2, \Omega_2) \qquad \text{and} \qquad |C(\Omega)| = |C_1| + |C_2| \,,$$

we have

$$\begin{aligned} \overline{h}(\Omega) &= \frac{\overline{\mathrm{Per}}(C_1, \Omega_1) + \overline{\mathrm{Per}}(C_2, \Omega_2)}{|C_1| + |C_2|} \\ &\geq \min\left\{ \frac{\overline{\mathrm{Per}}(C_1, \Omega_1)}{|C_1|}, \frac{\overline{\mathrm{Per}}(C_2, \Omega_2)}{|C_2|} \right\} \geq \min\left\{ \overline{h}(\Omega_1), \overline{h}(\Omega_2) \right\}. \end{aligned}$$

On the other hand, we have $|\Omega| > \max\{|\Omega_1|, |\Omega_2|\}$. Hence for at least one among the indices $i = 1$ and $i = 2$ it holds $|\Omega_i|^2\overline{h}(\Omega_i) < |\Omega|^2\overline{h}(\Omega)$. This shows that $\Omega$ cannot be a solution to problem (6) and hence neither to problem (5), contradiction.
(iii) Consider the generalized polygon $\Omega'$ whose boundary is obtained by eliminating from $\partial\Omega$ all line segments contained into two consecutive sides of $\Omega$. Then, we still have $\Omega' \in \overline{\mathcal{P}_N}$. Clearly, $\Omega$ and $\Omega'$ have the same volume, whereas by Proposition 8 (i) we have $\overline{h}(\Omega') \leq \overline{h}(\Omega)$. We infer that $|\Omega|^2\overline{h}(\Omega) = |\Omega'|^2\overline{h}(\Omega')$. Hence $\Omega'$ is still a solution to problem (5).

□

Now, in order to provide a representation formula for the Cheeger constant of an optimal generalized polygon, we need to introduce some additional definitions. By a *convex angle* we mean an angle $\theta \in (0, \pi)$, whereas by a *reflex angle* we mean an angle $\theta \in (\pi, 2\pi)$.

**Definition 14.** Given a generalized polygon $\Omega$, we set:

– $\Theta(\Omega)$ := the class of inner angles of $\Omega$, namely the angles $\theta$ formed at the interior of $\Omega$ by two consecutive sides of $\partial\Omega$.

– $\Theta_C(\Omega)$, $\Theta_R(\Omega)$ := the subclasses of convex/reflex angles in $\Theta(\Omega)$.

– $\mathcal{S}(\Omega)$ := the family of all sides of $\Omega$.

– $\mathcal{F}(\Omega)$:= the family of the *free sides* of $\Omega$, intended as the sides $S \in \mathcal{S}(\Omega)$ such that $S$ does not contain self-intersections, namely such that the only other sides which meet $S$ are its two consecutive sides, and this occurs only at the endpoints of $S$.

– $\mathcal{F}_{CC}(\Omega)$, $\mathcal{F}_{CR}(\Omega)$, $\mathcal{F}_{RR}(\Omega)$ := the subclass of sides $S \in \mathcal{F}(\Omega)$ such that the two angles of $\Theta(\Omega)$ formed by $S$ and its two consecutives sides are respectively convex-convex, convex-reflex, and reflex-reflex.

**Definition 15.** Given a generalized polygon $\Omega$, we set

$$\tau(\Omega) := \sum_{\alpha \in \Theta_C(\Omega)} \left[ \tan\left(\frac{\pi - \alpha}{2}\right) - \left(\frac{\pi - \alpha}{2}\right) \right].$$

*Remark* 16. From the inequality $\tan x > x$ for all $x \in (0, \frac{\pi}{2})$, it follows that $\tau(\Omega) > 0$ for any generalized polygon $\Omega$. Notice also that, if $\Omega$ is a simple convex polygon, there holds

$$\tau(\Omega) = \sum_{\alpha \in \Theta(\Omega)} \Big[ \tan \Big( \frac{\pi - \alpha}{2} \Big) \Big] - \pi \,.$$

**Proposition 17.** *Let $\Omega \in \overline{\mathcal{P}_N}$ be a reduced optimal polygon. Then there exists a unique Cheeger set $C(\Omega)$, which is determined by the equality*

$$\partial C(\Omega) \cap \Omega = \bigcup \Big\{ \Gamma_\alpha \ : \ \alpha \in \Theta_C(\Omega) \Big\} \,, \tag{13}$$

*where $\Gamma_\alpha$ is an arc of circumference of radius $(\overline{h}(\Omega))^{-1}$ which is tangent to the two sides of $\partial\Omega$ forming the angle $\alpha$.*
*Moreover, the Neumann-Cheeger constant of $\Omega$ is given by*

$$\overline{h}(\Omega) = \frac{|\partial\Omega| + \Delta(\Omega)}{2|\Omega|} \qquad \textit{with} \ \ \Delta(\Omega) := \sqrt{|\partial\Omega|^2 - 4|\Omega|\tau(\Omega)} > 0 \,, \tag{14}$$

*where $|\partial\Omega|$ is intended as $\overline{\mathrm{Per}}(\Omega, \Omega)$.*

*Remark* 18. (i) It may happen that, for two (or more) consecutive angles $\alpha_i \in \Theta_C(\Omega)$, the arcs $\Gamma_{\alpha_i}$ appearing in (13) lie on the same circumference.
(ii) By equality (13), reflex corners of $\partial\Omega$ are contained into $\partial C(\Omega)$.
(iii) Formula (14) already appeared in [21], where it was established to hold for simple convex polygons $\Omega$ whose Cheeger set meets all sides of $\partial\Omega$.

*Proof of Proposition 17.* The fact that $\partial C(\Omega) \cap \Omega$ is made by arcs or circumference of curvature $h(\Omega)$ is well-known, as well as the fact that $\partial C(\Omega) \cap \Omega$ must meet tangentially $\partial\Omega$, if this occurs at points where $\partial\Omega$ is $C^1$, see for instance [25, Section 4] and references therein.
Assume now that $\Omega \in \overline{\mathcal{P}_N}$ is a solution to problem (5), and let $C(\Omega)$ be a Neumann-Cheeger set of $\Omega$. Then it is readily seen that $C(\Omega)$ must touch every side of $\Omega$, that is

$$\partial C(\Omega) \cap S \neq \emptyset \qquad \forall\, S \in \mathcal{S}(\Omega) \,. \tag{15}$$

Namely, assume by contradiction that there exists a side $S$ which is not touched by $C(\Omega)$. In this case it is possible to construct a domain $\widetilde{\Omega}$, still belonging to $\overline{\mathcal{P}_N}$, such that that $C(\Omega) \subset \widetilde{\Omega} \subset \Omega$. Then $|\widetilde{\Omega}| < |\Omega|$ and $\overline{h}(\widetilde{\Omega}) = \overline{h}(\Omega)$ by Lemma 13, so that $|\Omega|\overline{h}^2(\Omega) > |\widetilde{\Omega}|\overline{h}^2(\widetilde{\Omega})$. Hence $\Omega$ cannot be a solution to problem (6), nor to problem (5).
As a consequence of (15) and of the connectedness of $C(\Omega)$, we obtain that all the arcs of circumference contained into $\partial C(\Omega) \cap \Omega$ must be of the form $\Gamma_\alpha$ for some $\alpha \in \Theta_C(\Omega)$, that is, each arc must meet two sides of $\partial\Omega$ forming a convex angle.
We have thus shown the inclusion $\subseteq$ in (13). To get the opposite inclusion, we have to prove that boundary of a Neumann-Cheeger set of $\Omega$ cannot contain any convex angle, namely that, for every $\alpha \in \Theta_C(\Omega)$, there exists an arc of the form $\Gamma_\alpha$ such that $\Gamma_\alpha \subseteq \partial C(\Omega) \cap \Omega$.
Let $\alpha \in \Theta_C(\Omega)$ be fixed, and let $\Omega_{\alpha,r}$ be the domain obtained by "smoothing" the corner $\alpha$ by means of an arc of cirumference of radius $r$, tangent to the two sides of $\partial\Omega$ forming the angle $\alpha$. It is readily seen by geometric arguments that, for $r$ sufficiently small,

$$\overline{\mathrm{Per}}(\Omega_{\alpha,r}, \Omega) = |\partial\Omega| - 2r \cot\big(\frac{\alpha}{2}\big) + (\pi - \alpha) r$$

and

$$|\Omega_{\alpha,r}| = |\Omega| - r^2 \cot\big(\frac{\alpha}{2}\big) + \big(\frac{\pi-\alpha}{2}\big) r^2 .$$

Then,

$$\frac{\overline{\mathrm{Per}}(\Omega_{\alpha,r},\Omega)}{|\Omega_{\alpha,r}|} = \frac{|\partial\Omega| - 2r\big[\tan\big(\frac{\pi-\alpha}{2}\big) - \big(\frac{\pi-\alpha}{2}\big)\big]}{|\Omega| - r^2\big[\tan\big(\frac{\pi-\alpha}{2}\big) - \big(\frac{\pi-\alpha}{2}\big)\big]} .$$

Since the term in squared parenthesis is positive, we see that the inequality $\frac{\overline{\mathrm{Per}}(\Omega_{\alpha,r},\Omega)}{|\Omega_{\alpha,r}|} < \frac{|\partial\Omega|}{|\Omega|}$ is satisfied for $r$ sufficiently small (precisely, for $r < \frac{2|\Omega|}{|\partial\Omega|}$).

Let us now prove (14). In view of the equality (13), repeating the above argument at every $\alpha \in \Theta_C(\Omega)$, and setting

$$f(r) := \frac{|\partial\Omega| - 2r \sum\limits_{\alpha\in\Theta_C(\Omega)} \big[\tan\big(\frac{\pi-\alpha}{2}\big) - \big(\frac{\pi-\alpha}{2}\big)\big]}{|\Omega| - r^2 \sum\limits_{\alpha\in\Theta_C(\Omega)} \big[\tan\big(\frac{\pi-\alpha}{2}\big) - \big(\frac{\pi-\alpha}{2}\big)\big]} = \frac{|\partial\Omega| - 2r\tau(\Omega)}{|\Omega| - r^2\tau(\Omega)} ,$$

we have that $r_\Omega$ minimizes $f(r)$ over the interval $\big[0, \frac{2|\Omega|}{|\partial\Omega|}\big]$. Imposing $f'(r_\Omega) = 0$ we obtain that $r_\Omega$ solves the second order equation

$$\tau(\Omega) r^2 - |\partial\Omega| r + |\Omega| = 0 .$$

We infer that $\Delta(\Omega) := |\partial\Omega|^2 - 4|\Omega|\tau(\Omega) \geq 0$, and that $r_\Omega$ is equal to one of the two roots

$$r_\pm := \frac{|\partial\Omega| \pm \sqrt{|\partial\Omega|^2 - 4|\Omega|\tau(\Omega)}}{2\tau(\Omega)} = \frac{2|\Omega|}{|\partial\Omega| \mp \sqrt{|\partial\Omega|^2 - 4|\Omega|\tau(\Omega)}} .$$

Since only $r_-$ falls into the interval $\big[0, \frac{2|\Omega|}{|\partial\Omega|}\big]$, we conclude that $r_\Omega = r_-$, and consequently that

$$\overline{h}(\Omega) = \frac{1}{r_\Omega} = \frac{1}{r_-} = \frac{|\partial\Omega| + \sqrt{|\partial\Omega|^2 - 4|\Omega|\tau(\Omega)}}{2|\Omega|} .$$

Finally, it remains to show that $\Delta(\Omega)$ is *strictly* positive. Assume by contradiction that $\Delta(\Omega) = 0$. In this case, by (14) we have $\overline{h}(\Omega) = \frac{|\partial\Omega|}{2|\Omega|}$. Thus,

$$\tau(\Omega) = \frac{|\partial\Omega|^2}{4|\Omega|} = \overline{h}(\Omega)\frac{|\partial\Omega|}{2} . \tag{16}$$

For $\alpha \in \Theta_C(\Omega)$, denote by $\ell_\alpha$ the length of the segment in $\partial\Omega$ joining the vertex of $\partial\Omega$ corresponding to the angle $\alpha$ with one of the points at which $\Gamma_\alpha$ is tangent to $\partial\Omega$. Then it holds $r_\Omega \cot\big(\frac{\alpha}{2}\big) = \ell_\alpha$ (with $r_\Omega = (\overline{h}(\Omega))^{-1}$). Summing over $\alpha \in \Theta_C(\Omega)$, we get

$$\sum_{\alpha\in\Theta_C(\Omega)} \tan\Big(\frac{\pi-\alpha}{2}\Big) = \overline{h}(\Omega) \sum_{\alpha\in\Theta_C(\Omega)} \ell_\alpha \leq \overline{h}(\Omega)\frac{|\partial\Omega|}{2} , \tag{17}$$

where the last equality holds since, for every $\alpha \in \Theta_C(\Omega)$, two segments of length $\ell_\alpha$ are contained into $\partial\Omega$. By combining (16) and (17), we get

$$\tau(\Omega) \geq \sum_{\alpha\in\Theta_C(\Omega)} \tan\Big(\frac{\pi-\alpha}{2}\Big) ,$$

which is readily seen to be in contradiction with Definition 15 of $\tau(\Omega)$.

□

## 3. Stationarity conditions and their consequences

In this section we rely on shape derivative arguments in order to get geometrical information on a solution to problem (5). The stationarity conditions we obtain are contained in Lemmas 23, 24, and 25 below. Their consequences on the length of the free sides of an optimal polygon, and on the measures of the angles formed by them, are given in Propositions 26, 27, and 28.
Let us begin by observing that, if $\Omega$ is a solution to problem (5), there exists a Lagrange multiplier $\mu$ such that $\Omega$ is stationary for the shape functional $\overline{h}(\Omega)+\mu|\Omega|$. For the sake of simplicity, up to replacing $\Omega$ by a dilate, we can take $\mu = 1$, and work with the stationarity condition written under the form

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\Big(\overline{h}(\Omega_\varepsilon) + |\Omega_\varepsilon|\Big) = 0\,, \tag{18}$$

where $\Omega_\varepsilon$ is a one-parameter family of deformations of $\Omega$.
We are going to work in particular with the following two kinds of deformations.

**Definition 19.** [*Rotations around the mid-point*]
For a fixed $S \in \mathcal{F}(\Omega)$, with consecutive sides $S_1$ and $S_2$, we denote by $\Phi_\varepsilon(\Omega)$, $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, the polygons obtained keeping fixed the other sides and replacing the three sides $(S, S_1, S_2)$ by the new sides $(S^\varepsilon, S_1^\varepsilon, S_2^\varepsilon)$ obtained in the following way (see Figure 2):

- $S^\varepsilon$ lies on the straight-line obtained by rotating of an oriented angle $\varepsilon$, around the mid-point of $S$, the straight-line containing $S$ (by oriented angle $\varepsilon$, we mean $+\varepsilon$ or $-\varepsilon$ according to whether, respectively, $S$ is rotated clock-wise or counter-clock-wise);
- $S_1^\varepsilon$ and $S_2^\varepsilon$ lie on the same straight-line containing respectively $S_1$ and $S_2$;
- the lengths of $S^\varepsilon$, $S_1^\varepsilon$ and $S_2^\varepsilon$, are chosen so that the three sides are consecutive (namely $(S^\varepsilon, S_1^\varepsilon)$ and $(S^\varepsilon, S^2_\varepsilon)$ have one point in common).

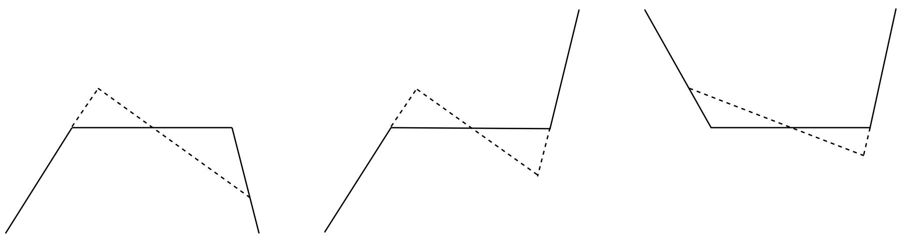

Figure 2. Rotation around the mid-point of a side in $\mathcal{F}_{AA}$, $\mathcal{F}_{AO}$, and $\mathcal{F}_{OO}$.

**Definition 20.** [*Parallel displacement*]
For a fixed $S \in \mathcal{F}(\Omega)$, with consecutive sides $S_1$ and $S_2$, we denote by $\Psi_\varepsilon(\Omega)$, $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, the polygons obtained keeping fixed the other sides and replacing the three sides $(S, S_1, S_2)$ by the new sides $(S^\varepsilon, S_1^\varepsilon, S_2^\varepsilon)$ obtained in the following way (see Figure 3):

- $S^\varepsilon$ lies on the straight-line parallel to $S$ having signed distance $\varepsilon$ from $S$ (precisely, by signed distance $\varepsilon$ from $S$, we mean $+\varepsilon$ or $-\varepsilon$ according to whether, respectively, $S^\varepsilon$ does not intersect or intersects $\Omega$);
- $S_1^\varepsilon$ and $S_2^\varepsilon$ lie on the same straight-line containing respectively $S_1$ and $S_2$;

– the lengths of $S^\varepsilon$, $S_1^\varepsilon$ and $S_2^\varepsilon$, are chosen so that the three sides are consecutive (namely $(S^\varepsilon, S_1^\varepsilon)$ and $(S^\varepsilon, S^2_\varepsilon)$ have one point in common).

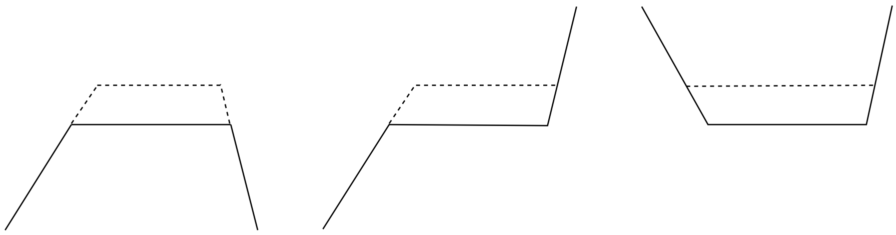

FIGURE 3. Parallel displacement of a side in $\mathcal{F}_{AA}$, $\mathcal{F}_{AO}$, and $\mathcal{F}_{OO}$.

*Remark* 21. If $\Omega \in \overline{\mathcal{P}_N}$ is a a reduced optimal polygon, the arcs of circle contained into its Neumann-Cheeger set cannot meet a side $S \in \mathcal{F}_{CR}(\Omega)$ at a vertex $p$ corresponding to a reflex angle. Otherwise, we choose a point $q$ in the relative interior of $S$ which is closer to $p$ than to the other extreme of $S$: by rotating $S$ around $q$, we get a polygon with smaller area, and the same (or a smaller) Neumann-Cheeger constant.

*Remark* 22. We observe that, if $\Omega \in \overline{\mathcal{P}_N}$ is a a reduced optimal polygon, in view of Proposition 17 and Remark 21, for $\varepsilon$ sufficiently small the Neumann-Cheeger constant of the perturbed polygons $\Phi_\varepsilon(\Omega)$ and $\Psi_\varepsilon(\Omega)$ is still given by formula (14), provided the side $S$ which is rotated as in Definition 19 or displaced as in Definition 20 satisfies one of the following conditions:

– $S \in \mathcal{F}_{CC}(\Omega)$ and, denoting by $\alpha_1, \alpha_2 \in \Theta_C(\Omega)$ the angles formed by $S$ and its two consecutive sides, the two arcs of circumference $\Gamma_{\alpha_1}$ and $\Gamma_{\alpha_2}$ appearing in (13) do not lie on the same circumference:
– $S \in \mathcal{F}_{CR}(\Omega)$;
– $S \in \mathcal{F}_{RR}(\Omega)$.

**Lemma 23.** *Let $\Omega \in \overline{\mathcal{P}_N}$ be a reduced optimal polygon satisfying* (18)*. Let $S \in \mathcal{F}_{CC}(\Omega)$. Denoting by $a$ the length of $S$ and by $\alpha_1, \alpha_2 \in \Theta_C(\Omega)$ the angles formed by $S$ and its two consecutive sides, there holds*

$$\alpha_1 = \alpha_2 \tag{19}$$

$$\frac{\sin(\alpha_1)}{\sin^2\left(\frac{\alpha_1}{2}\right)} \le a\left[\overline{h}(\Omega) - \frac{\Delta(\Omega)}{\overline{h}(\Omega)}\right] \tag{20}$$

*Moreover, if the two arcs of circumference $\Gamma_{\alpha_1}$ and $\Gamma_{\alpha_2}$ in* (13) *do not lie on the same circumference,* (20) *holds as an equality.*

*Proof.* Let us impose the stationarity condition (18) when $\Omega_\varepsilon$ are given respectively by the deformations $\Phi_\varepsilon(\Omega)$ and $\Psi_\varepsilon(\Omega)$ introduced in Definitions 19 and 20. Assume first that the two arcs of circumference $\Gamma_{\alpha_1}$ and $\Gamma_{\alpha_2}$ do not lie on the same circumference. Then, by Remark 22, for $\varepsilon$ sufficiently small the Neumann-Cheeger constants $\overline{h}(\Phi_\varepsilon(\Omega))$ and $\overline{h}(\Psi_\varepsilon(\Omega))$ are still given by formula (14).

– *Rotations around the mid-point.* Let us name the angles $\alpha_1$ and $\alpha_2$ so that, if $\varepsilon > 0$, $\alpha_1$ (resp. $\alpha_2$) is changed into $\alpha_1 - \varepsilon$ (resp. $\alpha_2 + \varepsilon$), whereas, if $\varepsilon < 0$, $\alpha_1$ (resp. $\alpha_2$) is changed into $\alpha_1 + \varepsilon$ (resp. $\alpha_2 - \varepsilon$).
Through elementary geometric arguments, we obtain

$$|\partial(\Phi(\Omega_\varepsilon))| = |\partial\Omega| + \frac{a\,\sin\alpha_1}{2\,\sin(\alpha_1-\varepsilon)} + \frac{a\,\sin\alpha_2}{2\,\sin(\alpha_2+\varepsilon)} + \frac{a\,\sin\varepsilon}{2\,\sin(\alpha_1-\varepsilon)} - \frac{a\,\sin\varepsilon}{2\,\sin(\alpha_2+\varepsilon)}$$
$$|\Phi(\Omega_\varepsilon)| = |\Omega| + o(\varepsilon)$$
$$\tau\big(\Phi(\Omega_\varepsilon)\big) = \tau(\Omega) + \tan\Big(\frac{\pi-\alpha_1+\varepsilon}{2}\Big) + \tan\Big(\frac{\pi-\alpha_2-\varepsilon}{2}\Big) - \sum_{i=1}^{2}\tan\Big(\frac{\pi-\alpha_i}{2}\Big)\,.$$

Then, using (14), some long but straightforward computations lead to write condition (18), when $\Omega_\varepsilon = \Phi_\varepsilon(\Omega)$, as

$$\frac{\sin(\frac{\alpha_1+\alpha_2}{2})\sin(\frac{\alpha_2-\alpha_1}{2})}{\sin\big(\frac{\alpha_1}{2}\big)\sin\big(\frac{\alpha_2}{2}\big)} = a\,\overline{h}(\Omega)\,\sin(\frac{\alpha_2-\alpha_1}{2})\,.$$

We infer that:

$$\text{either}\quad \alpha_1=\alpha_2,\qquad \text{or}\quad \frac{\sin(\frac{\alpha_1+\alpha_2}{2})}{\sin\big(\frac{\alpha_1}{2}\big)\sin\big(\frac{\alpha_2}{2}\big)} = a\,\overline{h}(\Omega)\,. \tag{21}$$

– *Parallel displacement.* Through elementary geometric arguments, we obtain:

$$|\partial(\Psi(\Omega_\varepsilon))| = |\partial\Omega| + \frac{\varepsilon}{\tan\alpha_1} + \frac{\varepsilon}{\tan\alpha_2} + \frac{\varepsilon}{\sin\alpha_1} + \frac{\varepsilon}{\sin\alpha_2}$$
$$|\Psi(\Omega_\varepsilon)| = |\Omega| + \frac{\varepsilon}{2}\Big(2a + \frac{\varepsilon}{\tan\alpha_1} + \frac{\varepsilon}{\tan\alpha_2}\Big)$$
$$\tau\big(\Psi(\Omega_\varepsilon)\big) = \tau(\Omega)$$

Then, using (14), some long but straightforward computations lead to write condition (18), when $\Omega_\varepsilon = \Psi_\varepsilon(\Omega)$, as

$$\frac{\sin(\frac{\alpha_1+\alpha_2}{2})}{\sin\big(\frac{\alpha_1}{2}\big)\sin\big(\frac{\alpha_2}{2}\big)} = a\left[\overline{h}(\Omega) - \frac{\Delta(\Omega)}{\overline{h}(\Omega)}\right]. \tag{22}$$

By combining (21) and (22), we infer that either $\alpha_1 = \alpha_2$, or $\Delta(\Omega) = 0$. Since the latter possibility is excluded by virtue of Proposition 17, we conclude that $\alpha_1 = \alpha_2$. Moreover, by inserting this condition into (22), we obtain that (20) holds as an equality, and the lemma is proved under the assumption made at the beginning of the proof that $\Gamma_{\alpha_1}$ and $\Gamma_{\alpha_2}$ lie on the same circumference.

Let us now deal with the case when the two arcs of circumference $\Gamma_{\alpha_1}$ and $\Gamma_{\alpha_2}$ lie on the same circumference. Let $p_1$, $p_2$ denote the two vertices of $\partial\Omega$ corresponding to the angles $\alpha_1$, $\alpha_2$, let $o$ denote the center of the circumference $\Gamma$ of radius $(\overline{h}(\Omega))^{-1}$ touching $S$ and its two consecutive sides $S_1, S_2$, ant let $q_1, q_2$ denote the tangency points $\partial C(\Omega)\cap S_1$ and $\partial C(\Omega)\cap S_2$.

Since $\Omega$ is optimal for problem (5), the area of the pentagon with vertices $o, q_1, q_2, p_1, p_2$ must be minimal among all the pentagons $P(x_1, x_2)$ having three vertices fixed at $o, q_1, q_2$, and the other two vertices $x_1, x_2$ free to move so that the three segments $[q_1x_1]$, $[x_1x_2]$, and $[q_2x_2]$ remain tangent to the circumference $\Gamma$. Indeed, all these pentagons have a Neumann-Cheeger constant lower than or equal to $\overline{h}(\Omega)$, so that $\Omega$ needs to minimize the area in order to be a solution to problem (6).

Now, denoting by $\theta_1$ and $\theta_2$ the inner angles of the pentagon $P(x_1, x_2)$ at $x_1$ and $x_2$, and by $\theta_0$ the fixed angle formed by the segments $[oq_1]$ and $[oq_2]$, by summing the inner angles of $P(x_1, x_2)$ we see that $\theta_1$ and $\theta_2$ must obey the linear constraint $\theta_1 + \theta_2 = 2\pi - \theta_0$. Through elementary geometric arguments, we see that the area of $P(x_1, x_2)$ is given by

$$|P(x_1, x_2)| = (\overline{h}(\Omega))^{-2}\Big[\tan\Big(\frac{\pi - \theta_1}{2}\Big) + \tan\Big(\frac{\pi - \theta_2}{2}\Big)\Big],$$

*cf.* Figure 4. Minimizing this function under the constraint $\theta_1 + \theta_2 = 2\pi - \theta_0$, we find $\theta_1 = \theta_2$, which proves (19).

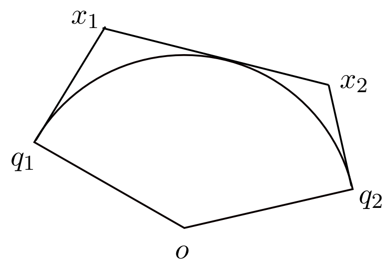


FIGURE 4. A pentagon $P(x_1, x_2)$.

Finally, let us show that also the inequality (20) remains true. We recall it was obtained by considering the parallel displacement deformation $\Psi_\varepsilon(\Omega)$. Now, since $\Gamma_{\alpha_1}$ and $\Gamma_{\alpha_2}$ lie on the same circumference, if the parallel displacement brings $S$ towards the exterior of $\Omega$ (namely if $\varepsilon > 0$, *cf.* Definition 20), the Cheeger constant of $\Psi_\varepsilon(\Omega)$ is no longer given by (14). Nonethless, formula (14) can be applied to $\Psi_\varepsilon(\Omega)$ if the parallell displacement brings $S$ towards the interior $\Omega$ (namely if $\varepsilon < 0$). We conclude that the stationarity condition (18) can be replaced by the following inequality for the left derivative:

$$\frac{d^-}{d\varepsilon}\Big|_{\varepsilon=0}\Big(\overline{h}(\Psi_\varepsilon(\Omega)) + |\Psi_\varepsilon(\Omega)|\Big) \leq 0\,, \tag{23}$$

Then, by arguing as above, we see that (23) is equivalent to the inequality (20).

□

**Lemma 24.** *Let $\Omega \in \overline{\mathcal{P}_N}$ be a reduced optimal polygon satisfying* (18)*. Let $S \in \mathcal{F}_{RR}(\Omega)$. Denoting by $b$ the length of $S$ and by $\beta_1, \beta_2 \in \Theta_R(\Omega)$ the angles formed by $S$ and its two consecutive sides, there holds*

$$\beta_1 = \beta_2 \tag{24}$$

$$\frac{\sin(\beta_1)}{\sin^2\big(\frac{\beta_1}{2}\big)} = b\Big[\overline{h}(\Omega) - \frac{\Delta(\Omega)}{\overline{h}(\Omega)}\Big] \tag{25}$$

*Proof.* We are going to proceed in a similar way as done in the proof of Lemma 23, namely, we impose the stationarity condition (18) when $\Omega_\varepsilon$ are given respectively by the deformations $\Phi_\varepsilon(\Omega)$ and $\Psi_\varepsilon(\Omega)$. Since $S \in \mathcal{F}_{RR}(\Omega)$, by Remark 22 for $\varepsilon$ sufficiently small the Neumann-Cheeger constants $\overline{h}(\Phi_\varepsilon(\Omega))$ and $\overline{h}(\Psi_\varepsilon(\Omega))$ are still given by formula (14).

– *Rotations around the mid-point.* Let us name the angles $\beta_1$ and $\beta_2$ so that, if $\varepsilon > 0$, $\beta_1$ (resp. $\beta_2$) is changed into $\beta_1 - \varepsilon$ (resp. $\beta_2 + \varepsilon$), whereas, if $\varepsilon < 0$, $\beta_1$ (resp. $\beta_2$) is changed into $\beta_1 + \varepsilon$ (resp. $\beta_2 - \varepsilon$).

Through elementary geometric arguments, we obtain

$$\begin{aligned}
&|\partial(\Phi(\Omega_\varepsilon))| = |\partial\Omega| + \frac{a\,\sin\beta_1}{2\,\sin(\beta_1-\varepsilon)} + \frac{a\,\sin\beta_2}{2\,\sin(\beta_2+\varepsilon)} + \frac{a\,\sin\varepsilon}{2\,\sin(\beta_1-\varepsilon)} - \frac{a\,\sin\varepsilon}{2\,\sin(\beta_2+\varepsilon)}\\
&|\Phi(\Omega_\varepsilon)| = |\Omega| + o(\varepsilon)\\
&\tau\big(\Phi(\Omega_\varepsilon)\big) = \tau(\Omega)\,.
\end{aligned}$$

Then, by using as usual (14) and some algebraic computations, we can rewrite condition (18), when $\Omega_\varepsilon = \Phi_\varepsilon(\Omega)$, as

$$\sin\Big(\frac{\beta_2-\beta_1}{2}\Big) = 0\,. \tag{26}$$

– *Parallel displacement.* Through elementary geometric arguments, we obtain:

$$\begin{aligned}
&|\partial(\Psi(\Omega_\varepsilon))| = |\partial\Omega| + \frac{\varepsilon}{\tan\beta_1} + \frac{\varepsilon}{\tan\beta_2} + \frac{\varepsilon}{\sin\beta_1} + \frac{\varepsilon}{\sin\beta_2}\\
&|\Psi(\Omega_\varepsilon)| = |\Omega| + \frac{\varepsilon}{2}\Big(2a + \frac{\varepsilon}{\tan\beta_1} + \frac{\varepsilon}{\tan\beta_2}\Big)\\
&\tau\big(\Psi(\Omega_\varepsilon)\big) = \tau(\Omega)
\end{aligned}$$

Then condition (18), when $\Omega_\varepsilon = \Psi_\varepsilon(\Omega)$, can be rewritten as

$$\frac{\sin(\frac{\beta_1+\beta_2}{2})}{\sin\big(\frac{\beta_1}{2}\big)\sin\big(\frac{\beta_2}{2}\big)} = a\,\Big[\overline{h}(\Omega) - \frac{\Delta(\Omega)}{\overline{h}(\Omega)}\Big]\,. \tag{27}$$

By combining (26) and (27), we obtain (24) and (25). □

**Lemma 25.** *Let $\Omega \in \overline{\mathcal{P}_N}$ be a reduced optimal polygon satisfying* (18)*. Let $S \in \mathcal{F}_{CR}(\Omega)$. Denoting by $c$ the length of $S$ and by $\alpha_0 \in \Theta_C(\Omega)$ and $\beta_0 \in \Theta_R(\Omega)$ the angles formed by $S$ and its two consecutive sides, there holds*

$$\frac{\sin\Big(\frac{\alpha_0+\beta_0}{2}\Big)}{\sin\Big(\frac{\alpha_0}{2}\Big)\sin\Big(\frac{\beta_0}{2}\Big)} = c\Big[\overline{h}(\Omega) - \frac{\Delta(\Omega)}{\overline{h}(\Omega)}\Big] \tag{28}$$

$$\frac{\sin\Big(\frac{\beta_0-\alpha_0}{2}\Big)}{\sin\Big(\frac{\alpha_0}{2}\Big)\sin\Big(\frac{\beta_0}{2}\Big)}\overline{h}(\Omega)\,c = \frac{\cos^2\Big(\frac{\alpha_0}{2}\Big)}{\sin^2\Big(\frac{\alpha_0}{2}\Big)} \tag{29}$$

$$\tan\Big(\frac{\alpha_0}{2}\Big) = -\tan\Big(\frac{\beta_0}{2}\Big)\frac{\sqrt{\Delta(\Omega)}}{\overline{h}(\Omega)}\,. \tag{30}$$

*Proof.* The proof proceeds along the same line of Lemmas 23 and 24. Also in this case, we impose the stationarity condition (18) when $\Omega_\varepsilon$ are given respectively by the deformations $\Phi_\varepsilon(\Omega)$ and $\Psi_\varepsilon(\Omega)$. Again, since $S \in \mathcal{F}_{CR}(\Omega)$, by Remark 22 for $\varepsilon$ sufficiently small the Neumann-Cheeger constants $\overline{h}(\Phi_\varepsilon(\Omega))$ and $\overline{h}(\Psi_\varepsilon(\Omega))$ are still given by formula (14).

– *Rotations around the mid-point.* If $\varepsilon > 0$, we change $\alpha_0$ (resp. $\beta_0$) into $\alpha_0 - \varepsilon$ (resp. $\beta_0 + \varepsilon$), whereas, if $\varepsilon < 0$, we change $\alpha_0$ (resp. $\beta_0$) into $\alpha_0 + \varepsilon$ (resp. $\beta_0 - \varepsilon$).

Through elementary geometric arguments, we obtain

$$\begin{aligned}
&|\partial(\Phi(\Omega_\varepsilon))| = |\partial\Omega| + \frac{a\,\sin\alpha_0}{2\,\sin(\alpha_0-\varepsilon)} + \frac{a\,\sin\beta_0}{2\,\sin(\beta_0+\varepsilon)} + \frac{a\,\sin\varepsilon}{2\,\sin(\alpha_0-\varepsilon)} - \frac{a\,\sin\varepsilon}{2\,\sin(\beta_0+\varepsilon)}\\
&|\Phi(\Omega_\varepsilon)| = |\Omega| + o(\varepsilon)\\
&\tau\big(\Phi(\Omega_\varepsilon)\big) = \tau(\Omega) + \tan\Big(\frac{\pi-(\alpha_0-\varepsilon)}{2}\Big) - \tan\Big(\frac{\pi-\alpha_0}{2}\Big) - \frac{\varepsilon}{2}\,.
\end{aligned}$$

Then by (14) and some algebraic computations one can check that condition (18), when $\Omega_\varepsilon = \Phi_\varepsilon(\Omega)$, is equivalent to (29).

– *Parallel displacement.* Through elementary geometric arguments, we obtain:

$$\begin{aligned}
&|\partial(\Psi(\Omega_\varepsilon))| = |\partial\Omega| + \frac{\varepsilon}{\tan\alpha_0} + \frac{\varepsilon}{\tan\beta_0} + \frac{\varepsilon}{\sin\alpha_0} + \frac{\varepsilon}{\sin\beta_0}\\
&|\Psi(\Omega_\varepsilon)| = |\Omega| + \frac{\varepsilon}{2}\Big(2a + \frac{\varepsilon}{\tan\alpha_0} + \frac{\varepsilon}{\tan\beta_0}\Big)\\
&\tau\big(\Psi(\Omega_\varepsilon)\big) = \tau(\Omega)
\end{aligned}$$

Then by (14) and some algebraic computations one can check that condition (18), when $\Omega_\varepsilon = \Psi_\varepsilon(\Omega)$, is equivalent to (28).

Multiplying the two equalities (28) and (29), we see that the length $c$ simplifies and we get the equality

$$\sin\Big(\frac{\alpha_0+\beta_0}{2}\Big)\sin\Big(\frac{\beta_0-\alpha_0}{2}\Big) = \Big[1 - \frac{\Delta(\Omega)}{(\overline{h}(\Omega))^2}\Big]\sin^2\Big(\frac{\beta_0}{2}\Big)\cos^2\Big(\frac{\alpha_0}{2}\Big)\,.$$

Then some immediate trigonometric computations yield

$$\frac{\sin^2\big(\frac{\beta_0}{2}\big)\cos^2\big(\frac{\alpha_0}{2}\big) - \cos^2\big(\frac{\beta_0}{2}\big)\sin^2\big(\frac{\alpha_0}{2}\big)}{\sin^2\big(\frac{\beta_0}{2}\big)\cos^2\big(\frac{\alpha_0}{2}\big)} = \Big[1 - \frac{\Delta(\Omega)}{(\overline{h}(\Omega))^2}\Big]\,,$$

which in turn gives

$$\tan^2\Big(\frac{\alpha_0}{2}\Big) = \tan^2\Big(\frac{\beta_0}{2}\Big)\frac{\Delta(\Omega)}{(\overline{h}(\Omega))^2}\,.$$

The equality (30) follows by recalling that $\frac{\alpha_0}{2} \in (0, \frac{\pi}{2})$ and $\frac{\beta_0}{2} \in (\frac{\pi}{2}, \pi)$.

□

We now turn to the consequences of Lemmas 23, 24, and 25.

**Proposition 26.** *Let $\Omega \in \overline{\mathcal{P}_N}$ be a reduced optimal polygon satisfying* (18)*. Then, it holds*

$$\Big[\overline{h}(\Omega) - \frac{\Delta(\Omega)}{\overline{h}(\Omega)}\Big] > 0\,.$$

*Consequently, the subclass $\mathcal{F}_{RR}(\Omega)$ is empty.*

*Proof.* Thanks to equality (14), we have

$$\begin{aligned}
(\overline{h}(\Omega))^2 - \Delta(\Omega) \;&= \Big(\frac{|\partial\Omega| + \Delta(\Omega)}{2|\Omega|}\Big)^2 - \Delta(\Omega)\\
&= \frac{|\partial\Omega|^2}{4|\Omega|^2} + \frac{\Delta^2(\Omega)}{4|\Omega|^2} + \Big(\frac{|\partial\Omega|}{2|\Omega|^2} - 1\Big)\Delta(\Omega)\\
&> \Big(\frac{|\partial\Omega|}{2|\Omega|^2} - 1\Big)\Delta(\Omega)\,.
\end{aligned}$$

Now, we observe that the term which multiplies $\Delta(\Omega)$ in the last line above is strictly positive. Indeed, by imposing the vanishing of the derivative of $\varepsilon \mapsto \big(\overline{h}(\varepsilon\Omega) + |\varepsilon\Omega|\big)$ at $\varepsilon = 1$, we see that $\overline{h}(\Omega) = 2|\Omega|$. Thus,

$$\frac{|\partial\Omega|}{2|\Omega|^2} = \frac{|\partial\Omega|}{|\Omega|}\frac{1}{2|\Omega|} = \frac{|\partial\Omega|}{|\Omega|}\frac{1}{\overline{h}(\Omega)} > 1\,,$$

where the latter inequality holds by the definition of $\overline{h}(\Omega)$.
The fact that $\mathcal{F}_{RR}(\Omega)$ is empty follows then from equality (25) in Lemma 24. Indeed, we have just proved that the right member of such equality is positive. It follows from (25) that $\sin(\beta_1)$ is positive, against the fact that $\beta_1 \in (\pi, 2\pi)$. □

**Proposition 27.** *Let $\Omega \in \overline{\mathcal{P}_N}$ be a reduced optimal polygon satisfying* (18)*. If $\alpha_0$ and $\beta_0$ are two consecutive angles in $\partial\Omega$ belonging respectively to $\Theta_C(\Omega)$ and $\Theta_R(\Omega)$, it holds*

$$\pi < \alpha_0 + \beta_0 < 2\pi\,.$$

*Proof.* The inequality $\alpha_0 + \beta_0 > \pi$ is trivially satisfied, since $\alpha_0 > 0$ and $\beta_0 > \pi$. The inequality $\alpha_0 + \beta_0 < 2\pi$ is a consequence of equality (28) in Lemma 25. Indeed, from Proposition 26 we know that the right member of such equality is strictly positive. Since $0 < \frac{\alpha_0}{2} < \frac{\pi}{2}$ and $\frac{\pi}{2} < \frac{\beta_0}{2} < \pi$, also the terms $\sin\left(\frac{\alpha_0}{2}\right)$ and $\sin\left(\frac{\beta_0}{2}\right)$ are positive. We infer that $\sin\left(\frac{\alpha_0+\beta_0}{2}\right)$ is positive, whence $\frac{\alpha_0+\beta_0}{2} < \pi$. □

**Proposition 28.** *Let $\Omega \in \overline{\mathcal{P}_N}$ be a reduced optimal polygon satisfying* (18)*. Let $\Gamma \subset \partial\Omega$ be a chain of consecutive free sides. Set $\Gamma_{CC} := \mathcal{F}_{CC}(\Omega) \cap \Gamma$, $\Gamma_{CR} := \mathcal{F}_{CR}(\Omega) \cap \Gamma$, and denote by $\Theta_C(\Gamma)$ (resp. $\Theta_R(\Gamma)$), the family of the angles in $\Theta_C(\Omega)$ (resp. $\Theta_R(\Omega)$) formed by a side of $\Gamma$ and its two consecutive sides.*
*Then there exist angles $\alpha \in (0, \pi)$, $\beta \in (\pi, 2\pi)$, such that*

$$\theta = \alpha \quad \forall \theta \in \Theta_C(\Gamma) \qquad \textit{and} \qquad \theta = \beta \quad \forall \theta \in \Theta_R(\Gamma)\,, \tag{31}$$

*and the values of $\alpha$ and $\beta$ are related by the equality* (30)*.*
*Moreover, all the sides in $\Gamma_{CC}$, resp. $\Gamma_{CR}$, have the same length.*

*Proof.* From equality (19) in Lemma 23 and equality (30) in Lemma 25, we see that there exist a common value $\alpha$ for all the elements of $\Theta_C(\Gamma)$, and a common value $\beta$ for all the elements of $\Theta_R(\Gamma)$, which are mutually determined through the equality (30). This implies in particular that, if there exist two or more sides in $\Gamma_{CC}$ which are tangent to a same circumference of radius $(\overline{h}(\Omega))^{-1}$, all these sides must have the same length $\ell$. On the other hand, by Lemma 23, if there exist two or more sides in $\Gamma_{CC}$ which are not tangent to a same circumference of radius $(\overline{h}(\Omega))^{-1}$, all these sides must have the same length $a$ (obtained by (20) as an equality). Then the inequality (20) in Lemma 23 tells us that $\ell \geq a$. But clearly $\ell \leq a$, since $\ell$ is minimal among the length of sides in $\Gamma_{CC}$ (because there exist at least two consecutive sides of length $\ell$ which are tangent to the same circumference of radius $(\overline{h}(\Omega))^{-1}$). We conclude that all the sides in $\Gamma_{CC}$ have necessarily the same length. The same assertion holds for all the sides in $\Gamma_{CR}$ thanks to Lemma 25, as the value this length $c$ can be obtained from one of the two equations (28) or (29).

□

## 4. No self-intersections and no reflex angles

By exploiting the results obtained in the previous section, we are now in a position to prove first that the self-intersection set is of an optimal polygon is actually empty (see

Proposition 29), and then that an optimal polygon is necessarily convex (see Proposition 30).

**Proposition 29.** *Let $\Omega \in \overline{\mathcal{P}_N}$ be a reduced optimal polygon. Then $\Omega \in \mathcal{P}_N$, namely it is a simple polygon.*

*Proof.* We obtain the proposition in two steps, arguing by contradiction.

*Step 1*: We claim that, if $\Omega$ is not a simple polygon, then necessarily there exists a loop $L$ in $\partial\Omega$ which contains only a connected component of the self-intersection set (*i.e.*, either only a self-intersection point, or only a self-intersection segment).

In order to prove the claim, let us choose an oriented parametrization of $\partial\Omega$: for definiteness, assume that $\Omega$ lies on the left side of each edge (recall that, since $\Omega$ is connected, $\partial\Omega$ is a closed lace). Clearly, along $\partial\Omega$ there is a finite number of self-intersections, which may be either points or line segments. For simplicity, assume they all are points; if there are also some line segments, we can still treat them as points from a topological point of view, and apply the same arguments below. Then, if we cover once $\partial\Omega$ according to the chosen parametrization, each intersection point appears at least twice. If it appears just twice, we call it a simple self-intersection point; if it appears more than twice, we call it a multiple self-intersection point.

Let us consider first the case when $p$ is a simple self-intersection point. Then there exist four line segments which meet at $p$ and lie on sides on $\partial\Omega$. We refer to two of these segments as $\gamma_i, \gamma_{i+1}$ for some index $i$, if they are consecutive when covering $\partial\Omega$ according to our parametrization, and we denote by $[\gamma_i, \gamma_{i+1}]$ the path obtained by following in the order $\gamma_i$ and $\gamma_{i+1}$.

Let $B$ be a small ball centered at $p$ (precisely, of radius sufficiently small in order that $\partial B$ meets the segments $\gamma_i$, $\gamma_{i+1}$). We observe that the portion of $\Omega \cap B$ lying on the left side of $[\gamma_i, \gamma_{i+1}]$ (and as well the portion lying on the left side of $[\gamma_j, \gamma_{j+1}]$) is necessarily connected. In other words, among the two configurations represented in Figure 5, only the type (I) represented on the left is possible. Namely, in case of the type (II) represented on the right, by covering the portion of $\partial\Omega$ which starts at $p$, follows $\gamma_{i+1}$ and continues up to arriving back to $p$ along $\gamma_j$, we would find a connected component of $\Omega$ different from $\Omega$ itself, contradiction.

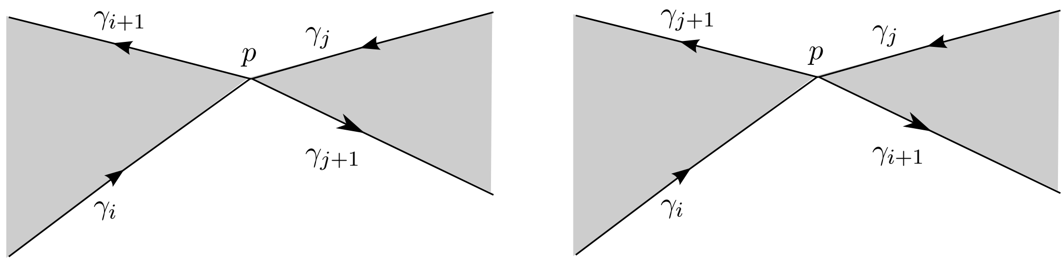


FIGURE 5. The case of a simple self-intersection point: configuration of type (I) on the left, and of type (II) on the right.

In the case when $p$ is a multiple self-intersection point, for the same reason explained above, among all the line segments which meet at $p$ and lie on sides on $\partial\Omega$, there cannot be 4 segments in the configuration of type (II). Thus there exist 4 segments meeting at $p$ in the configuration of type (I).

We claim that, for any other couple of segments $[\gamma_h, \gamma_{h+1}]$ meeting at $p$, if $B$ is a small ball centered at $p$, the portion of $\Omega \cap B$ lying on the left side of $[\gamma_h, \gamma_{h+1}]$ is necessarily connected. Indeed, assume this is not the case. Then we observe firstly that there must be a further pair of segments $\gamma_k$, $\gamma_{k+1}$ meeting at $p$ (otherwise there would be some self-intersection segment contained into two consecutive sides of $\Omega$, which is excluded since we are considering a reduced optimal polygon) and then that the two paths $[\gamma_h, \gamma_{h+1}]$, $[\gamma_k, \gamma_k+1]$ would be in the configuration of type (II) above, hence $\Omega$ would be disconnected (see Figure 6, right).

We have so far obtained that, for every self-intersection point $P$ (simple or multiple it may be), there exist two or more paths $[\gamma_i, \gamma_{i+1}]$ meeting at $p$ in such way that the portion of $\Omega \cap B$ lying on the left side of $[\gamma_i, \gamma_{i+1}]$ is connected (see Figure 6, left).

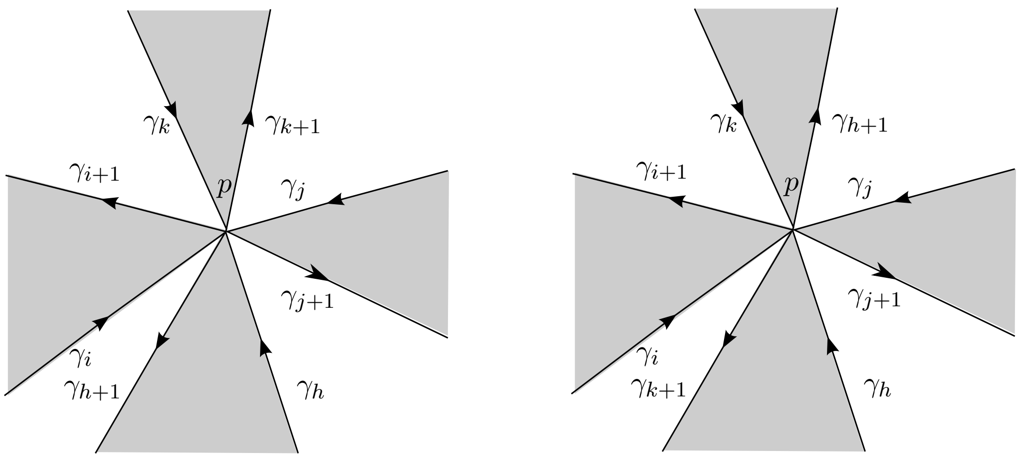


FIGURE 6. The case of a multiple self-intersection point.

For any of these paths, say $[\gamma_i, \gamma_{i+1}]$, since $\partial\Omega$ is a lace, there exists another one, say $[\gamma_j, \gamma_{j+1}]$, such that $\gamma_{i+1}$ and $\gamma_j$ lie on a same loop $L_p$ contained into $\partial\Omega$. Then two cases may occur: either such loop $L_p$ does not contain any self-intersection point other than $p$ (and in this case our claim is proved), or there is some other self intersection point $q$ lying on $L_p$. In this case, by applying the same arguments above to the point $q$, we infer that there exist two or more paths $[\xi_i, \xi_{i+1}]$ meeting at $q$ so that that the portion of $\Omega \cap B$ lying on the left side of $[\xi_i, \xi_{i+1}]$ is connected. Moreover, for any of these paths, say $[\xi_i, \xi_{i+1}]$, there exists another one, say $[\xi_j, \xi_{j+1}]$, such that $\xi_{i+1}$ and $\xi_j$ lie on a same loop $L_q$ (with $L_q \neq L_p$) contained into $\partial\Omega$.

Again, two cases may occur: either $L_q$ does not contain any self-intersection point other than $q$ (and in this case our claim is proved), or there is some other self intersection point $r$ lying on $L_q$.

We go on proceeding in this way: since the number of self-intersections is finite, either at some moment we find a loop as claimed in Step 1, or it happens that every loop contained in $\partial\Omega$ touches some other loop (see Figure 7). In the former case, the proof of Step 1 is achieved. In the latter case $\Omega$ would be disconnected, contradiction.

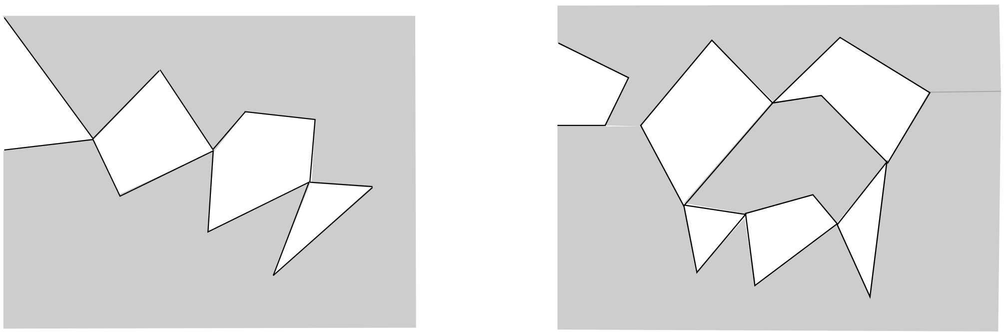

FIGURE 7. Chains of consecutive loops.

*Step 2*: If $\Omega$ is not a simple polygon, the existence of a loop $L$ as in Step 1 allows to reach a contradiction.
Namely, let $\Gamma$ be the chain of consecutive free sides of $\Omega$ contained into $L$. With the same notation as in Proposition 28, denote by $k_C$ and $k_R$ are the numbers of vertices in the loop which correspond respectively to angles in $\Theta_C(\Gamma)$ and in $\Theta_R(\Gamma)$. By applying Proposition 28 to the chain $\Gamma$, we obtain that there exist $\alpha, \beta$ belonging respectively to $(0, \pi)$ and $(\pi, 2\pi)$ such that (31) holds. We denote by $\theta_i$ the inner angles of $\Omega$ formed by two consecutive non-free sides contained into $L$ (so that $i = 1, 2$ if $L$ contains a self-intersection segment, whereas $i = 1$ if $L$ contains a self-intersection point). The sum of all inner angles of the loop is given by

$$
\begin{aligned}
& \textstyle\sum_i (2\pi - \theta_i) + k_R(2\pi - \beta) + k_C(2\pi - \alpha) \\
= \; & \textstyle\sum_i (2\pi - \theta_i) + k_R(2\pi - \beta) + [k_R + (k_C - k_R)](2\pi - \alpha) \\
= \; & \textstyle\sum_i (2\pi - \theta_i) + k_R[4\pi - (\alpha + \beta)] + (k_C - k_R)(2\pi - \alpha) \\
> \; & 2\pi k_R + \pi(k_C - k_R) = \pi(k_C + k_R)
\end{aligned}
\tag{32}
$$

where the strict inequality follows from the fact that $\theta_i < 2\pi$, $k_C \geq k_R$ (by Proposition 26), $\alpha + \beta < 2\pi$ (by Proposition 27), and $\alpha < \pi$.
Next we observe that, since the loop $L$ is chosen as in Step 1, denoting by $k$ the number of vertices on $L$, it holds

$$
k_C + k_R = \begin{cases} k - 2 & \text{if } L \text{ contains only a self-intersection segment} \\ k - 1 & \text{if } L \text{ contains only a self-intersection point.} \end{cases} \tag{33}
$$

Then, by combining (32) and (33) we reach a contradiction since the sum of the inner angles of the loop is equal to $\pi(k - 2)$ (as $k$ is the number of vertices of the loop). $\square$

**Proposition 30.** *Let $\Omega \in \overline{\mathcal{P}_N}$ be a solution to problem* (5) *satisfying* (18). *Then $\Theta_R(\Omega)$ is empty, namely $\Omega$ is a convex polygon.*

*Proof.* Assume by contradiction that there exists $\beta \in \Theta_R(\Omega)$. Denote by $p$ the corresponding vertex of $\partial\Omega$, by $p_1$ and $p_2$ its two consecutive vertices, and by $S_1 = pp_1$ and $S_2 = pp_2$ the two sides of $\partial\Omega$ which form the angle $\beta$.
By Proposition 29, $\Omega$ is a simple polygon. Moreover, by Proposition 26, both sides $S_1$ and $S_2$ belong to $\mathcal{F}_{CR}(\Omega)$. Hence, by Proposition 28, $S_1$ and $S_2$ have the same length (say $c$), and the two (convex) inner angles of $\partial\Omega$ at $p_1$ and $p_2$ are equal to the same angle (say $\alpha$). The geometry is illustrated in Figure 8 below.
Denote by $\eta$ be the straight line passing through $p$ which which bisects the angle $\beta$. Set $T := c\cos\left(\pi - \frac{\beta}{2}\right)$ and, for $t \in [0, T]$, let $\gamma^t$ be the straight line perpendicular to $\eta$, such that the intersection point between $\eta$ and $\gamma^t$ lies outside $\Omega$ at distance $t$ from $p$. Then, $\gamma^t$ meets $S_1$ and $S_2$; we set

$$q_1^t := \gamma^t \cap S_1 \qquad \text{and} \qquad q_2^t := \gamma^t \cap S_2 \,.$$

Notice in particular that $q_1^0 = q_2^0 = p$, whereas $q_1^T = p_1$ and $q_2^T = p_2$.
For $t \in (0, T)$, we denote by $\Pi^t$ the half-plane determined by $\gamma^t$ not containing $p$, or equivalently containing $p_1$ and $p_2$. We extend this definition also for $t = 0$ and for $t = T$, setting $\Pi^0$ and $\Pi^T$ respectively the half-plane determined by $\gamma^0$ containing $p_1$ and $p_2$, and the half-plane determined by $\gamma^T$ not containing $p$.
Now, for $t \in [0, T]$, we define:

$$\begin{aligned} \Delta^t &:= \text{ the triangle with vertices } p,\ q_1^t \text{ and } q_2^t \\ A_1^t &:= \text{ the connected component of } \Omega \cap \Pi^t \text{ containing } p_1 \text{ in its boundary} \\ A_2^t &:= \text{ the connected component of } \Omega \cap \Pi^t \text{ containing } p_2 \text{ in its boundary.} \end{aligned}$$

Notice that, though in Figure 8 the sets $A_1^t$ and $A_2^t$ are represented for simplicity as triangles, they might be more general polygons.

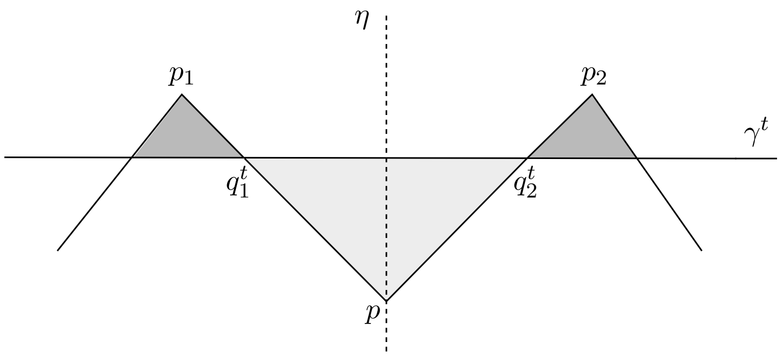


FIGURE 8. The triangle $\Delta^t$ (in light grey) and the set $A_1^t \cup A_2^t$ (in grey).

We claim that there exists $\hat{t} \in (0, T)$ such that

$$|A_1^{\hat{t}} \cup A_2^{\hat{t}}| = |\Delta^{\hat{t}}| \,. \tag{34}$$

Indeed, the function $\psi(t) := |A_1^t \cup A_2^t| - |\Delta^t|$ is clearly continuous in $[0, T]$. Moreover, it satisfies

$$\psi(0) > 0 \qquad \text{and} \qquad \psi(T) < 0 \,.$$

Namely, the condition $\psi(0) > 0$ follows immediately from the fact that the triangle $\Delta^0$ is degenerated into the point $p$ whereas the sets $A_1^0$ and $A_2^0$ have positive area. The condition $\psi(T) < 0$ follows from the fact that the triangle $\Delta^T$ coincides with the triangle $pp_1p_2$ (in particular it has positive area), whereas the sets $A_1^T$ and $A_2^T$ are degenerated respectively into the points $p_1$ and $p_2$. We emphasize that the last assertion is due to the fact that the angle $\alpha$ is convex and satisfies the condition $\alpha + \beta < 2\pi$ (thanks to Propositions 26 and 27).

Then, we define the modified polygon

$$\widehat{\Omega} := \big(\Omega \setminus (A_1^{\hat{t}} \cup A_2^{\hat{t}})\big) \cup \Delta^{\hat{t}} .$$

By the equality (34), $\widehat{\Omega}$ has the same area as $\Omega$; moreover, by construction, it has at least one vertex less than $\Omega$: with respect to $\Omega$, it has gained (at most) two vertices (lying on $\gamma^t$), and lost (at least) three vertices (that is, $p, p_1$, and $p_2$).

We are now going to obtain a contradiction by considering the Cheeger set of $\Omega$. We distinguish two cases.

*Case 1: $C(\Omega)$ is contained into $\widehat{\Omega}$.*

In this case, we have $h(\widehat{\Omega}) \leq h(\Omega)$; since $|\Omega| = |\widehat{\Omega}|$, we infer that $\widehat{\Omega}$ is as well a solution to problem (6). But since we know $\widehat{\Omega}$ has at least one vertex less than $\Omega$, this contradicts the fact that any optimal domain for problem (5) (and hence also for problem (6)) has exactly $N$ vertices (*cf.* Proposition 9).

*Case 2: $C(\Omega)$ is not contained into $\widehat{\Omega}$.*

In this case, we denote by $H$ the connected component of the set $\big(\mathbb{R}^2 \setminus \Pi^{\hat{t}}\big) \setminus C(\Omega)$ which contains $p$ in its boundary, and we consider the subset $E$ of $\widehat{\Omega}$ given by

$$E := \big(C(\Omega) \cap \widehat{\Omega}\big) \cup H ,$$

see Figure 9.

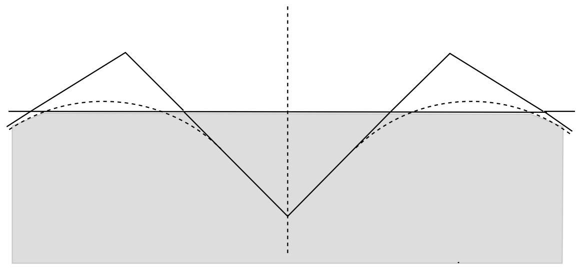

FIGURE 9. The set $E$ (in grey) locally near $p$

It follows by construction that

$$\mathrm{Per}(E; \mathbb{R}^2) \leq \mathrm{Per}(C(\Omega); \mathbb{R}^2) \qquad \text{and} \qquad |E| \geq |C(\Omega)| .$$

In particular, the latter inequality comes from the fact that

$$|H| \geq |\Delta^{\hat{t}}| = |A_1^{\hat{t}} \cup A_2^{\hat{t}}| \geq |C(\Omega) \setminus \widehat{\Omega}| .$$

so that

$$|C(\Omega)| = |C(\Omega) \cap \widehat{\Omega}| + |C(\Omega) \setminus \widehat{\Omega}| \leq |C(\Omega) \cap \widehat{\Omega}| + |H| = |E| .$$

We conclude that

$$h(\widehat{\Omega}) \leq \frac{\mathrm{Per}(E;\mathbb{R}^2)}{|E|} \leq \frac{\mathrm{Per}(C(\Omega);\mathbb{R}^2)}{|C(\Omega)|} = h(\Omega)$$

Thus, as in Case 1, it turns out that $\widehat{\Omega}$ is optimal for problem (5), against the fact that any optimal domain has exactly $N$ vertices. □

## 5. Conclusion and further remarks

*Proof of Theorem 1.* The statement follows by exploiting the results contained in Section 4. Namely, let $\Omega \in \overline{\mathcal{P}_N}$ be a solution to problem (5). Up to an homothety, and using Proposition 11 (iii), we may assume that $\Omega$ is a reduced optimal polygon which satisfies condition (18). Then, by Propositions 29 and 30, $\Omega$ is a simple convex polygon. Finally, by Proposition 28, we deduce that $\Omega$ is the regular $N$-gon. □

*Remark* 31. (*Stronger version of Theorem 1*) Note that the proof above yields a statement stronger than Theorem 1, as we have shown that the regular $N$ gon minimizes the Neumann-Cheeger constant $\overline{h}$ under a volume constraint over the class $\overline{\mathcal{P}_N}$.

*Remark* 32. (*The case of simple convex polygons*) As mentioned in the Introduction, the proof of Theorem 1 would become straightforward if one would restrict attention to the case of simple *convex* polygons. Actually let us show that, if a simple convex polygon $\Omega$ solves problem (5), then necessarily $\Omega$ is the regular $N$-gon of area $c$. (For a weaker form of such statement, see [3, Theorem 3]). Denote by $\Omega^*$ the regular $N$-gon with $|\Omega^*| = c$, and assume by contradiction that $\Omega \neq \Omega^*$. By Proposition 17 and Remark 16, there holds

$$h(\Omega) = \frac{|\partial\Omega| + \sqrt{|\partial\Omega|^2 - 4\tau(\Omega)|\Omega|}}{2|\Omega|} = \frac{|\partial\Omega| + \sqrt{|\partial\Omega|^2 - 4\Big(\sum\limits_{\alpha\in\Theta(\Omega)}\Big[\tan\Big(\frac{\pi-\alpha}{2}\Big)\Big] - \pi\Big)|\Omega|}}{2|\Omega|}.$$

By the isoperimetric inequality for convex polygons (see [3, Lemma 5] and [12, Theorem 2]), we have

$$\frac{|\partial\Omega|^2}{4|\Omega|} \geq \sum_{\alpha\in\Theta(\Omega)} \tan\Big(\frac{\pi-\alpha}{2}\Big),$$

with equality sign if and only if $\Omega$ is a circumscribed polygon (meaning that it contains a ball which is tangent to every side of $\partial\Omega$). Then we obtain

$$h(\Omega) \geq \frac{|\partial\Omega| + \sqrt{4\pi|\Omega|}}{2|\Omega|}.$$

Now we observe that $|\Omega| = |\Omega^*|$ (since both are equal to $c$) and $|\partial\Omega^*| < |\partial\Omega|$ (since the regular $N$-gon is the unique minimizer of perimeter among simple polygons with $N$ sides under volume constraint (see *e.g.* [7]). Hence

$$h(\Omega) > \frac{|\partial\Omega^*| + \sqrt{4\pi|\Omega^*|}}{2|\Omega^*|}.$$

Finally, we observe that the right hand side of the above inequality coincides with $h(\Omega^*)$ (see [16, Section 4] or [3, Theorem 3]), and we conclude that $h(\Omega) > h(\Omega^*)$, contradiction.

*Remark* 33. (*Possible extensions*) It would be interesting to extend the validity of Theorem 1 to more general classes of polygons. Indeed, one could study for instance the non-simply connected case, namely work over the class of open sets homeomorphic to an annulus, whose boundary consists of two polygonal lines with a total number of sides less than or equal to $N$. It would also be intriguing to consider general crossed polygons, which cannot by approximated in the $H^c$ topology by simple polygons: in this case the Cheeger constant (and the eigenvalues in general) has not a clear definition, since the index of every point of the plane with respect to the boundary has somehow to be counted.

*Remark* 34. (*Faber-Krahn inequalities for Dirichlet eigenvalues on polygons*) Let $\lambda_p$ denote the first Dirichlet eigenvalue of the $p$-Laplacian. For $\Omega \in \mathcal{P}_N$, let $\Omega_N^*$ denote the regular polygon with $N$ sides having the same area. With the help of Theorem 1, it is possible to prove a lower bound of the form

$$\lambda_p(\Omega) \geq \gamma_{p,N} \lambda_p(\Omega_N^*) \qquad \forall\, \Omega \in \mathcal{P}_N \,,$$

for a constant $\gamma_{p,N}$ less than 1, which can be explicitly determined. Indeed, one may argue that

$$\lambda_p^{1/p}(\Omega) \geq c_p h(\Omega) \geq c_p h(\Omega_N^*) \geq c_p k_{p,N} \lambda_p^{1/p}(\Omega_N^*) \,, \tag{35}$$

where the first inequality is known to hold with $c_p = 1/p$ (see [16]), the second one is due to Theorem 1, and the last one can be easily proved by taking the distance from the boundary of $\Omega_N^*$ as a trial function. Indeed, set $d(x) := \mathrm{dist}(x, \partial\Omega_N^*)$, $L$ the perimeter of $\Omega_N^*$, and $\rho$ its in-radius; by using the identity $|\nabla d(x)| = 1$, the coarea formula, and the equality $\mathcal{H}^1\big(\{d(x) = t\}\big) = L\big(1 - \frac{t}{\rho}\big)$, we get :

$$\lambda_p(\Omega_N^*) \leq \frac{\int_{\Omega_N^*} |\nabla d(x)|^p \, dx}{\int_{\Omega_N^*} |d(x)|^p \, dx} = \frac{\int_0^\rho L\big(1 - \frac{t}{\rho}\big) \, dt}{\int_0^\rho t^p L\big(1 - \frac{t}{\rho}\big) \, dt} = \frac{1}{\rho^p} \frac{(p+1)(p+2)}{2} \,,$$

hence

$$|\Omega_N^*|^{\frac{1}{2}} \lambda_p^{1/p}(\Omega_N^*) \leq \sqrt{N \tan \frac{\pi}{N}} \Big[\frac{(p+1)(p+2)}{2}\Big]^{\frac{1}{p}} .$$

On the other hand, since $\Omega_N^*$ is circumscribed, it holds

$$h(\Omega_N^*) = \frac{|\partial\Omega_N^*| + \sqrt{4\pi|\Omega_N^*|}}{2|\Omega_N^*|}$$

and so

$$|\Omega_N^*|^{\frac{1}{2}} h(\Omega_N^*) = \frac{2N \sin \frac{\pi}{N} + \sqrt{2\pi N \sin \frac{2\pi}{N}}}{\sqrt{2N \sin \frac{2\pi}{N}}} \,.$$

Hence,

$$h(\Omega_N^*) \geq k_{p,N} \lambda_p^{1/p}(\Omega_N^*) \qquad \text{with} \quad k_{p,N} := \Big[1 + \frac{\sqrt{2\pi N \sin \frac{2\pi}{N}}}{2N \sin \frac{\pi}{N}}\Big] \Big[\frac{2}{(p+1)(p+2)}\Big]^{1/p} .$$

We point out that both the values of $c_p$ and $k_{p,N}$ determined as above are far from being optimal, and we address the open problem of replacing them by larger constants in order to refine the estimate (35).

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