How to prove existence in shape optimization

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Abstract

This paper deals with the existence question in optimal design. We present a general variational technique for proving existence, and give several examples concerning functionals of eigenvalues and of energy type. In particular, we show how the isoperimetric problem for the Dirichlet eigenvalues of an elliptic operator of general order fit into this frame.

1 Introduction

We consider a generic shape optimization problem of the form

$$\min_{\Omega \in \mathcal{U}_{ad}} J(\Omega), \tag{1}$$

where the cost functional $\Omega \to J(\Omega)$ depends on the set Ω, which usually is assumed to be open or quasi-open. The functional J depends on Ω via the solution of a partial differential equation defined on Ω, or the spectrum of a certain operator which is defined on Ω. By $\mathcal{U}_{ad}$ we denote the class of admissible sets. The question we deal with is : *does problem (1) have a solution*?

The answer depends of course both on how J depends on Ω and on the class of admissible sets $\mathcal{U}_{ad}$. A collection of results concerning this topic can be found in [3]. The most classical example of a shape optimization problem which has a solution is the isoperimetric inequality: find an open set Ω which maximizes the volume among all open sets with fixed perimeter. A typical example of a shape optimization problem which does not have solution is to minimize the volume into the same class.

Since the question is not so simple and there is no a standard approach, a general answer can not be given. Being far from giving an exhaustive tool for the treatment of shape optimization problems, we just point out the main difficulties of the existence question and try to give the reader some hints for proving existence for three classes of shape optimization problems: functionals of eigenvalues, minimization of energies and maximization of energies. As concrete examples, notice that the maximization of the torsional rigidity is a minimization of energy, while the minimization of the compliance is a maximization of energy. We do not discuss here the non existence question, which one should consider with different techniques.

An artificial way to obtain existence for a given functional J is to diminish the class $\mathcal{U}_{ad}$. This is usually done by imposing a uniform geometric constraint on the elements of $\mathcal{U}_{ad}$. A typical example is to work in the class of domains which satisfy a uniform cone condition. Since these classes are not stable for deformations by smooth vector fields, from a practical point of view, existence results in these classes may hide what really happens to the shape functional and do not give real information for the numerical computation. The stability by smooth fields is necessary when writing the necessary optimality conditions.

A *good* existence result is a result obtained in a class $\mathcal{U}_{ad}$ which is *stable* for vector field deformations. In particular, this means that $\mathcal{U}_{ad}$ should be large enough and should not involve any boundary smoothness. This is one reason for which we deal only with weak solutions (which are naturally defined on non-smooth domains).

We present a unitary frame and give an abstract method to prove existence for problem (1). In particular we show how to prove existence of an optimal shape for the isoperimetric problem for the eigenvalues of an elliptic operator of arbitrary order with Dirichlet boundary conditions (in the Sobolev space $H^m_0(D)$). For the Dirichlet-Laplacian (and operators in divergence form) this was done by Buttazzo and Dal Maso in [10] by using the relaxed form of the Dirichlet problem and the theory of the Γ-convergence. Without using relaxation, the same result was proved in [4] into the frame of the *weak*-γ convergence. In this paper, we give a simple proof issued from the direct methods of the calculus of variations, which does not require any knowledge of the relaxed forms or Γ or γ convergences. Neumann eigenvalues do not fit into our abstract frame, mainly because the lack of collective compactness of the Sobolev spaces $H^1(\Omega)$ into $L^2(\mathbb{R}^N)$, for varying Ω. We moreover discuss the shape optimization problems for the energy of the system. For minimization problems we give a general existence result (which is valid also for nonlinear pde's), while for maximization problems we only underline the main difficulties.

We point out the fact that the answer to the existence question does not depend "too much" on the structure of the functional J, but only on its continuity properties on moving spaces.

2 A few facts about moving spaces

We begin by giving a general definition for the convergence of spaces (see [1] for more details). Let X be a reflexive Banach space and $\{G_n\}_{n\in\mathbb{N}}$ a sequence of subsets of X. We denote by $w-X$, $s-X$ the weak and the strong topology on X. The weak upper and the strong lower limits in the sense of Kuratowski are defined as follows:

$$w-\limsup_{n\to\infty} G_n = \{u \in X : \exists\{n_k\}_k, \exists u_{n_k} \in G_{n_k} \text{ such that } u_{n_k} \stackrel{w-X}{\rightharpoonup} u\}$$

$$s-\liminf_{n\to\infty} G_n = \{u \in X : \exists u_n \in G_n \text{ such that } u_n \stackrel{s-X}{\longrightarrow} u\}$$

Definition 2.1 *If $\{G_n\}_{n\in\mathbb{N}}$ are closed subspaces in X, it is said that G_n $s-K$ converges to G if $G \subseteq s-\liminf_{n\to\infty} G_n$. It is said that G_n $w-K$ converges to G if $w-\limsup_{n\to\infty} G_n \subseteq G$.*

Note that the $s-K$ and $W-K$ limits of a sequence $(G_n)_n$ are not unique. In particular G_n $w-K$ converges to X and $s-K$ converges to $\{0\}$.

If $\{G_n\}_{n\in\mathbb{N}}$ are closed subspaces in X, it is said that G_n converges in the sense of Mosco to G if G_n converges both in $s-K$ and $w-K$ to G. Since $s-\liminf_{n\to\infty} G_n \subseteq w-\limsup_{n\to\infty} G_n$, the Mosco limit is unique. Note that in general $s-\liminf_{n\to\infty} G_n \subseteq w-\limsup_{n\to\infty} G_n$. Therefore, if G_n converges in the sense of Mosco to G, then

$$s-\liminf_{n\to\infty} G_n = G = w-\limsup_{n\to\infty} G_n.$$

Definition 2.2 *A family of sets $\mathcal{G} \subseteq \mathcal{P}(X)$ is said to be weak Kuratowski compact (simply $w-K$) if for every sequence $(G_n)_n \subseteq \mathcal{G}$ there exists a subsequence $(G_{n_k})_k$ and an element $G \in \mathcal{G}$ such that*

$$w-\limsup_{k\to\infty} G_{n_k} \subseteq G.$$

The family is said to be strong Kuratowski compact (simply $s-K$) if for every sequence $(G_n)_n \subseteq \mathcal{G}$ there exists a subsequence $(G_{n_k})_k$ and an element $G \in \mathcal{G}$ such that

$$G \subseteq s-\liminf_{k\to\infty} G_{n_k}.$$

3 Functionals depending on eigenvalues

The main situations we have in mind concern the Laplacian with Dirichlet or Neumann boundary conditions, and the bi-Laplacian with Dirichlet conditions (the clamped plate or the buckling load). We begin by giving first an abstract frame which can be used to discuss all these situations (the Neumann b.c. fit only partially into the general theory).

Let $\mathcal{V}, \mathcal{H}$ two real Hilbert spaces such that $\mathcal{V} \subseteq \mathcal{H}$. On $\mathcal{V}$ we have the scalar product $(.,.)_{\mathcal{V}}$ and the norm $|.|_{\mathcal{V}}$ and on $\mathcal{H}$ we have the scalar product $(.,.)_{\mathcal{H}}$ and the norm $|.|_{\mathcal{H}}$. In order to cover both the Dirichlet and Neumann Laplacian, we assume for the moment that the injection mapping $\mathcal{V} \hookrightarrow \mathcal{H}$ is continuous, but not necessarily compact. Let also $a : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ be a continuous, symmetric, coercive bilinear form.

Let now consider a sequence of closed Hilbert subspaces of $\mathcal{V}$, denoted $\{V_n\}_{n\in\mathbb{N}}$ and $\{H_n = cl_{\mathcal{H}} V_n\}_{n\in\mathbb{N}}$. We assume that V_n is compactly embedded in H_n. Since by definition V_n is dense in H_n, the dual space V_n' can be identified through H_n, $V_n \overset{compact}{\hookrightarrow} H_n \hookrightarrow V_n'$. Let a_n be the restriction of a to $V_n \times V_n$ and consider the associate operator

$$A_n : V_n \to V_n',$$

defined by $a(u,v) = < A_n u, v >_{V_n' \times V_n}$, for all $u, v \in V_n$. The operator A_n is also an isomorphism from its domain $D(A_n)$ onto H_n and A_n^{-1} is compact on H_n; $A_n^{-1} : H_n \to D(A_n) \subseteq V_n \subseteq H_n$, $D(A_n)$ is dense in H_n. From [16], the spectrum of A_n consists only on eigenvalues, which can be computed by using the usual Rayleigh formula.

Let's denote by $\lambda_k(V_n)$ the k-th eigenvalue of the operator A_n counted with its multiplicity. Then

$$\lambda_k(V_n) = \min_{S\in S_k(V_n)} \max_{u\in S\setminus\{0\}} \frac{a(u,u)}{|u|^2_{\mathcal{H}}},$$

where $S_k(V_n)$ is the family of all subspaces of dimension k of V_n.

The following conventions are made. If the number of eigenvalues is finite, we complete the sequence with $\lambda_k(V_n) = +\infty$. If $V_n = \{0\}$ then $\forall k \geq 1$ $\lambda_k(V_n) = +\infty$.

Theorem 3.1 *Let V_n, V be closed subspaces of $\mathcal{V}$. Assume that $V_n \overset{s-\mathcal{V}}{\to} V$. Then*

$$\lambda_k(V) \geq \limsup_{n\to\infty} \lambda_k(V_n).$$

Proof Let $\varepsilon > 0$ and $S \in S_k(V)$ such that

$$\lambda_k(V) + \varepsilon \geq \max_{u\in S\setminus\{0\}} \frac{a(u,u)}{|u|^2}.$$

Let $u_1, \ldots, u_k$ a $\mathcal{H}$-orthonormal basis of S. Let $u_1^n, \ldots, u_k^n$ be k-sequences given by the hypothesis $V \subseteq s - \liminf_{n\to\infty} V_n$, such that $u_i^n \to u_i$ in $\mathcal{V}$-strong. We can assume that $(u_i^n, u_j^n)_{\mathcal{H}} = \delta_{ij}$, otherwise we apply a usual orthonormalisation procedure.

Let us denote

$$S_n = \text{span } (u_1^n, \ldots, u_k^n),$$

and let $u_n \in S_n$ be such that $|u_n|_{\mathcal{H}} = 1$ and

$$\max_{u\in S_n} \frac{a(u,u)}{|u|_{\mathcal{H}}} = \frac{a(u_n,u_n)}{|u_n|_{\mathcal{H}}}.$$

For a subsequence, still denoted using the same index, we have $u_n \to u$, strongly in $\mathcal{V}$ and $\mathcal{H}$. Then

$$\frac{a(u,u)}{|u|_{\mathcal{H}}} = \lim_{n\to\infty} \frac{a(u_n,u_n)}{|u_n|_{\mathcal{H}}}.$$

Consequently

$$\lambda_k(V) + \varepsilon \geq \frac{a(u,u)}{|u|_{\mathcal{H}}} = \lim_{n\to\infty} \frac{a(u_n,u_n)}{|u_n|_{\mathcal{H}}} \geq \limsup_{n\to\infty} \lambda_k(V_n).$$

Taking $\varepsilon \to 0$, we conclude the proof. □

Theorem 3.2 *Let V_n, V be closed subspaces of $\mathcal{V}$. Assume that $V_n \overset{w-\mathcal{V}}{\to} V$, an moreover assume that the injection $\bigcup_n V_n \hookrightarrow \mathcal{H}$ is compact. Then*

$$\lambda_k(V) \leq \liminf_{n\to\infty} \lambda_k(V_n).$$

Proof Let $\varepsilon > 0$ and $S_n \in S_k(V_n)$ such that

$$\lambda_k(V_n) + \varepsilon \geq \max_{u\in S_n\setminus\{0\}} \frac{a(u,u)}{|u|^2_{\mathcal{H}}}.$$

Let $u_n \in V_n$ be a maximizer of $max_{u\in S_n\setminus\{0\}} \frac{a(u,u)}{|u|^2_{\mathcal{H}}}$ such that $|u_n|^2_{\mathcal{H}} = 1$.

Let $u_1^n, \ldots, u_k^n$ a $\mathcal{H}$-orthonormal basis of S_n. We can assume that $\liminf_{n\to\infty} \lambda_k(V_n) < \infty$ and for a subsequence (still denoted using the same index) we have

$$\forall i = 1, \ldots, k \;\; u_i^n \rightharpoonup u_i \text{ weakly in } \mathcal{V}.$$

By the collective compactness assumption $\bigcup_n V_n \hookrightarrow \mathcal{H}$, we get $(u_i, u_j)_{\mathcal{H}} = \delta_{ij}$ and $w - \limsup_{n\to\infty} S_n = \text{span } (u_1, \ldots, u_k) := S$, which is a space of dimension k.

For every $u \in S$, there exists a sequence $u_{n_k} \in S_{n_k}$ which converges weakly to u in $\mathcal{V}$. Consequently,

$$\liminf_{n\to\infty} \frac{a(u_{n_k}, u_{n_k})}{|u_{n_k}|_{\mathcal{H}}} \geq \frac{a(u,u)}{|u|_{\mathcal{H}}}.$$

Taking the maximum in the right hand side, we get

$$\liminf_{n\to\infty} \lambda_k(V_n) + \varepsilon \geq \max_{u\in S} \frac{a(u,u)}{|u|_{\mathcal{H}}} \geq \lambda_k(V).$$

□

Remark 3.3 If $\bigcup_n V_n \hookrightarrow \mathcal{H}$ is compact and V_n converges in the sense of Mosco to V, then from Theorems 3.1 and 3.2 we get that

$$\forall k \geq 1 \ \ \lambda_k(V_n) \to \lambda_k(V).$$

In fact, under these hypotheses, even a stronger result can be obtained using the norm convergence of the resolvent operator (see [17]). Since $H_0^m(D)$ is compactly embedded in H^{m-1}, this result covers the stability of the eigenvalues of the Dirichlet-Laplacian and both the clamped plate and the buckling load for the bi-Laplacian. In particular, the collective compactness hypothesis of Theorem 3.2 holds.

It is well known that the spectrum of the Neumann-Laplacian is highly unstable for the geometric domain variation. We point out the fact that this is due mainly to the collective compactness condition of Theorem 3.2 which is very difficult to be satisfied, unless the perturbation of the geometric boundary respects a uniform cone condition, for example.

If the injection $\bigcup_n V_n \hookrightarrow \mathcal{H}$ is not compact, the convergence of the spectrum does not hold in general, even though V_n converges in the sense of Mosco to V. We refer the reader to the classical example of Courant-Hilbert [13].

A set $A \subseteq \mathbb{R}^N$ is said to be quasi-open (see [3] for details) if for every $\varepsilon > 0$ there exists an open set A_ε such that $A \subseteq A_\varepsilon$, and cap $(A_\varepsilon \setminus A) < \varepsilon$. A property is said to hold quasi everywhere (or simply q.e.) if it holds in the complement of a set of zero capacity. A function $u : \mathbb{R}^N \to \mathbb{R}$ is said *quasi continuous* if for every $\epsilon > 0$ there exists an open set A_ϵ such that cap $(A_\epsilon) < \epsilon$ and $u|_{\mathbb{R}^N \setminus A_\epsilon}$ is continuous in $\mathbb{R}^N \setminus A_\epsilon$. Every function $u \in H^1(\mathbb{R}^N)$ has a unique quasi continuous representative (up to a set of zero capacity).

Let D be a smooth bounded open set (called design region) and $H_0^1(D)$ the usual Sobolev space. Let us denote

$$\mathcal{A}_c(D) = \{H_0^1(A) : A \subseteq D,\ A \text{ quasi-open},\ |A| \leq c\}.$$

The Sobolev space $H_0^1(A)$ is seen as closed subspace of $H_0^1(D)$,

$$H_0^1(A) = \{u \in H_0^1(D) : u = 0 \ \text{ q.e. on } D \setminus A\}.$$

In the previous relation, u is a quasi continuous representative. It was proved by Hedberg [18, Theorem 3.1] that this space coincides with the usual Sobolev space as soon as A is open.

We give first the following.

Theorem 3.4 *The family $\mathcal{A}_c(D)$ is compact for the $w-K$ convergence.*

Proof The proof is trivial if $c \geq |D|$. Indeed, in this case, every sequence $H_0^1(A_n)$ $w-K$ converges to $H_0^1(D)$. If $c < |D|$, the proof is related to the fine behaviour of Sobolev functions. In [10] the authors use the relaxation of the Dirichlet problem for the Γ-convergence to prove the assertion. We refer also to [15] for more details concerning this topic.

Let $w_{A_n} \in H_0^1(A_n)$ be the function which satisfies $-\Delta w_{A_n} = 1$ in $H_0^1(A_n)$ (in the sense given by the bilinear form $a(u,v) = \int_{A_n} \nabla u \nabla v dx$). Since D is bounded, the sequence $(w_{A_n})_n$ is bounded in $H_0^1(D)$.

For a subsequence, still denoted using with the same indices, we have $w_{A_n} \overset{H_0^1(D)}{\rightharpoonup} w$. The convergence being strong in $L^2(D)$ and since $|\{w_{A_n} > 0\}| \leq c$ we get $|\{w > 0\}| \leq c$. We set $A = \{w > 0\}$ and prove that $H_0^1(A_n) \overset{w-K}{\longrightarrow} H_0^1(A)$ in $H_0^1(D)$.

Let $u_n \in H_0^1(A_n)$, such that for a subsequence $u_{n_k} \overset{H_0^1(D)}{\rightharpoonup} u$. We have to prove that $u \in H_0^1(A)$. Let $f_{n_k} = -\Delta u_{n_k} \in H^{-1}(D)$. Then $f_{n_k} \overset{H^{-1}(D)}{\rightharpoonup} f = -\Delta u$. Consequently, if $v_{n_k} \in H_0^1(A_{n_k})$ satisfies in $H_0^1(A_{n_k})$ $-\Delta v_{n_k} = f$, then $u_n - v_{n_k} \rightharpoonup 0$ in $H_0^1(D)$, hence $v_{n_k} \rightharpoonup u$ in $H_0^1(D)$. For every $\varepsilon > 0$, we consider $f_\varepsilon \in L^\infty(D)$ such that $|f_\varepsilon - f|_{H^{-1}(D)} \leq \varepsilon$. If we denote $v_{n_k}^\varepsilon$ the solution in $H_0^1(A_{n_k})$ of $-\Delta v_{n_k}^\varepsilon = f_\varepsilon$, then we get from the maximum principle

$$0 \leq |v_n^\varepsilon| \leq |f_\varepsilon|_\infty w_{A_n}.$$

Any weak limit of $v_{n_k}^\varepsilon$ will vanish quasi everywhere on $\{w = 0\}$ hence it will belong to $H_0^1(A)$. Making $\varepsilon \to 0$, we get that $u \in H_0^1(A)$. □

Let $m \in \mathbb{N}^*$ and $H_0^m(D)$ be the usual Sobolev space on D. For a quasi open set $A \subseteq D$, we define by induction the following space (which by abuse of notation is still denoted with H_0^m)

$$H_0^m(A) = \{u \in H_0^m(D) : u \in H_0^1(A), \nabla u \in H_0^{m-1}(A)\}.$$

Note that for an open set A, the space defined above coincides with the usual Sobolev space, provided that A satisfies a Keldysh like stability property (see [19]) : $u = 0$ q.e. on $D \setminus A$ implies that $u = 0$ m-q.e. on $D \setminus A$ (i.e. in the sense of the m-capacity). Here, we do not develop this point; we just notice that if A is slightly smooth, this property holds from the Hedberg theorem.

Let us denote

$$\mathcal{A}_c^m(D) = \{H_0^m(A) : A \text{ quasi open } A \subseteq D, |A| \leq c\}.$$

Theorem 3.5 *The family $\mathcal{A}_c^m(D)$ is compact for the $w-K$ convergence in $H_0^m(D)$.*

Proof We prove this theorem by induction, namely that $H_0^m(A_n)$ $w-K$ converges to $H_0^m(A)$, where $A = \{w > 0\}$ is defined in the proof of Theorem 3.4. Suppose the assertion true up to $m-1$ and let us prove it for m. Let $(A_n)_n$ be a sequence of quasi open sets from which we extract a subsequence (still denoted using the same indices) such that $H_0^{m-1}(A_n) \overset{w-K}{\longrightarrow} H_0^{m-1}(A)$ and $H_0^1(A_n) \overset{w-K}{\longrightarrow} H_0^1(A)$.

From the definition of the Sobolev spaces H_0^m on quasi open sets, we directly get $H_0^m(A_n) \stackrel{w-K}{\longrightarrow} H_0^m(A)$ in $H_0^m(D)$.

□

Remark 3.6 Following the same lines, one can prove that for every $m \in \mathbb{N}$ and $1 < p < +\infty$, the family

$$\{W_0^{m,p}(A) : A \text{ quasi open } A \subseteq D, |A| \leq c\}$$

is $w - K$ compact in $W_0^{m,p}(D)$.

In the sequel we give an existence theorem concerning the eigenvalues of general operators issued from symmetric, continuous and coercive bilinear forms on $H_0^m(D)$. For example, this is the case of the m-Laplacian $(-\Delta)^m$ with Dirichlet boundary conditions. This result was obtained by Buttazzo and Dal Maso in [10]. Their proof relay on the relaxed form and on the Γ-convergence theory. Here, we give a simple proof issued from the direct methods of the calculus of variations.

Let $a(\cdot, \cdot)$ be a symmetric, continuous and coercive bilinear form on $H_0^m(D)$. For every quasi open set $A \subseteq D$, let us denote by $\lambda_1(A), \ldots, \lambda_k(A)$ the first k eigenvalues of the bilinear form restricted to $H_0^m(A)$.

Theorem 3.7 *Let $\Phi : \overline{\mathbb{R}}_+^k \to \overline{\mathbb{R}}$ be a lower semicontinuous function, increasing in each variable. Then problem*

$$\min\{\Phi(\lambda_1(A), \ldots, \lambda_k(A)) : A \textit{ quasi open } A \subseteq D, |A| \leq c\}$$

has at least one solution.

Proof Let $H_0^m(A_n)$ be a minimizing sequence. From the compactness result of Theorem 3.5, there exists a subsequence (still denoted using the same indices) such that $H_0^m(A_n) \stackrel{w-k}{\longrightarrow} H_0^m(A)$, and $H_0^m(A) \in \mathcal{A}_c^m(D)$.

Following Theorem 3.2 we have that $\forall i = 1, \ldots, k$

$$\lambda_i(A) \leq \liminf_{n\to\infty} \lambda_i(A_n).$$

For a subsequence (still denoted using the same indices) there exists $x_1, \cdots, x_k \in \overline{\mathbb{R}}$ such that $\lambda_i(A) \to x_i$. The lower semicontinuity property of Φ gives that

$$\Phi(x_1, \cdots, x_k) \leq \liminf_{n\to\infty} \Phi(\lambda_1(A_n), \ldots, \lambda_k(A_n)).$$

The monotonicity of ϕ in each variable together with the inequalities $\lambda_i(A) \leq x_i$ give

$$\Phi(\lambda_1(A), \ldots, \lambda_k(A)) \leq \liminf_{n\to\infty} \Phi(\lambda_1(A_n), \ldots, \lambda_k(A_n)).$$

Using the monotonicity of the functional Φ, we conclude the proof by pointing out that $A \neq \emptyset$. □

Remark 3.8 Notice that, contrary to the proof of Buttazzo and Dal Maso for elliptic operators in divergence form in the case $m = 1$, we do not know whether or not $x_1, \cdots, x_k$ are eigenvalues of a certain operator. In [10], a precise description of the relaxed operator having $x_1, \ldots, x_k$ as first k eigenvalues was given.

Remark 3.9 Notice also, that Theorem 3.7 does not have a natural extension to Neumann boundary conditions, i.e. to the Sobolev spaces $H^m(A)$, mainly because the collective compactness required by Theorem 3.2 does not hold. Indeed, the collective compactness hypothesis should be satisfied by the minimizing sequence of spaces, consequently no hypothesis can be made *a priori*.

Remark 3.10 In [4], the authors introduced the concept of *weak*-γ convergence which is somehow related to the $w-K$ convergence. In order to prove existence results for monotonous shape functionals, a fundamental property had to be satisfied by the couple γ and *weak*- γ-convergences. Note that this property is not required here, and it is not at all clear that such a property could be verified, since the spaces we deal with (for $m \geq 2$) are not reticular. We also refer to [11] for a review of the role of the monotonicity in shape optimization. In this paper, existence results are proved in the frame of γ and *weak*- γ- convergences (which can not be applied here).

4 Energy type functionals

We give here into an abstract frame, two semicontinuity results of energy type functionals for the Kuratowski convergences. From a practical point of view, these cases will cover the shape optimization problems where the functional $\Omega \to J(\Omega)$ is precisely the energy.

Let X be a reflexive Banach space and

$$E : X \mapsto \overline{\mathbb{R}}$$

be a functional satisfying the following coerciveness property:

$$\exists\, \alpha, \beta > 0, \ \ \forall u \in X \quad E(u) \geq \alpha |u|_X - \beta.$$

Theorem 4.1 *If the functional E is weakly lower semicontinuous then the functional*

$$J : \mathcal{P}(X) \to \overline{\mathbb{R}}$$

defined by

$$J(v) = \inf_{u \in V} E(u)$$

is $w - K$ l.s.c.

Proof Let $(V_n)_n, V$ be elements of $\mathcal{P}(X)$ such that

$$w - \limsup_{n \to \infty} V_n \subseteq V.$$

Then, for every $\varepsilon > 0$, we take $v_n \in V_n$ such that

$$E(v_n) \leq \inf_{u \in V_n} E(u) + \varepsilon.$$

If $\liminf_{n\to\infty} J(V_n) < +\infty$, then the coerciveness of E gives that the sequence $(v_n)_n$ is bounded in X. Without loss of generality we can assume that $v_n \rightharpoonup v$ in X-weak. The Kuratowski convergence gives that $v \in V$ and the weak-l.s.c. of E gives

$$E(v) \leq \liminf_{n\to\infty} E(v_n).$$

Hence

$$J(V) \leq E(v) \leq \liminf_{n\to\infty} J(V_n) + \varepsilon.$$

Taking $\varepsilon \to 0$, we get that J is w-K l.s.c. □

Theorem 4.2 *Let E be a weakly l.s.c., coercive functional on $H_0^m(D)$. Then the shape optimization problem*

$$\min_{H_0^m(A) \in \mathcal{A}_c^m(D)} \min_{u \in H_0^m(A)} E(u)$$

has at least one solution.

Proof Apply Theorems 4.1 and 3.5. □

Following Remark 3.6, this theorem is also valid in $W_0^{m,p}(D)$ for $1 < p < +\infty$.

Example 4.3 Let, for simplicity fix $m = 1$, and a bounded design region D. The energy functional associated to the Dirichlet-Laplacian on variable domains, fits under the hypotheses of the previous theorem. We set $X = H_0^1(D)$. We also set $f \in L^2(D)$. Then $E := H_0^1(D) \to \mathbb{R}$ is defined by

$$E(u) = \frac{1}{2}\int_D |\nabla u|^2 dx - \int_D f u dx.$$

Following Theorem 4.1, the mapping $\Omega \to \min_{u \in H_0^1(\Omega)} E(u)$ is $w - K$ lower semicontinuous.

Several shape optimization problems, such as the maximization of the torsional rigidity for example, fit into this frame. Indeed, the problem reads:

$$\max_{A \subseteq D} \int_A |\nabla u_A|^2 dx, \tag{2}$$

where u_A is the minimizer in $H_0^1(A)$ of E. Since $\int_A |\nabla u_A|^2 dx = -2E(u_A)$, problem (2) becomes

$$\min_{A \subseteq D} E(u_A),$$

or

$$\min_{A \subseteq D} \min_{u \in H_0^1(A)} \frac{1}{2}\int_D |\nabla u|^2 dx - \int_D f u dx.$$

Using the $w - K$ compactness of $\mathcal{A}_c(D)$ together with the result of Theorem 4.1, we get via Theorem 4.2 the existence of a solution for the shape optimization problem (2).

Theorem 4.4 *If E is strongly upper semicontinuous on X, then the functional*

$$J : \mathcal{P}(X) \to \overline{\mathbb{R}}$$

defined by

$$J(v) = \inf_{u \in V} E(u)$$

is $s - K$ upper semicontinuous.

Proof Let $(V_n)_n, V$ be elements of $\mathcal{P}(X)$ such that

$$V \subseteq s - \liminf_{n \to \infty} V_n.$$

Let $\varepsilon > 0$ ant $u \in V$ be such that

$$E(v) \leq \inf_{u \in V} E(u) + \varepsilon.$$

There exists $v_n \in V_n$ such that $v_n \to v$ strongly in X. Hence

$$E(v) \geq \limsup_{n \to \infty} E(v_n),$$

consequently

$$J(V) + \varepsilon \geq \limsup_{n \to \infty} J(V_n).$$

Making $\varepsilon \to$ we conclude the proof. □

Remark 4.5 A similar result as Theorem 4.2 can not be stated for maximization problems, since there are no suitable $s - K$ compact classes. Moreover, a new difficulty appear when applying the abstract frame to concrete shape optimization problems, because the space $\{0\}$ is an abstract solution which does not correspond to any shape. In Example 4.7 below, we make a short analysis of the Cantilever problem, and point out the main difficulties.

Example 4.6 The energy functional defined in Example 4.3 satisfies the hypotheses of Theorem 4.4.

Example 4.7 A significant example of a shape optimization problem where the energy is to be maximized is the Cantilever problem. The main feature of this problem is that on the unknown part of the boundary the natural condition is of Neumann type.

The first difficulty is that the Sobolev spaces corresponding to Neumann boundary conditions do not embed naturally into a fixed space. Indeed, for a non-smooth set Ω, there is no injection of $H^1(\Omega)$ into $H^1(D)$. For this reason, $H^1(\Omega)$ is seen as a subspace of $L^2(D) \times L^2(D, \mathbb{R}^N)$ via the following injection

$$H^1(\Omega) \ni u \to (1_\Omega u, 1_\Omega \nabla u) \in L^2(D) \times L^2(D, \mathbb{R}^N).$$

The second difficulty is that compactness results for the Kuratowski convergences are more difficult to obtain. Up to our knowledge, there is no suitable general compactness result for the $s-K$ convergence (this is the one which is important into the Cantilever problem, for example). In the last section we refer to some compactness results which can be used to obtain existence for the Cantilever problem in the class of domains with a prescribed number of holes.

The third difficulty is that from the abstract setting the space $\{0\}$ could be solution since is $s-K$ limit of any sequence of spaces. This degenerated situation has to be eliminated, since there is no shape supporting this space (unless the empty-set)! In practice, this is done by a careful study of the minimizing sequence (see [5, 12]).

5 Further remarks

A quite large class of functionals involved in the optimal design fit into one of the frames introduced into the previous sections. The very difficult question does not concern the structure of the shape functional, but is precisely related to find compact classes for the Kuratowski convergences. The compactness is exclusively related to the functional spaces! This fact supports the idea that in a shape optimization problem the existence question is not *so much* related to the shape functional and to the PDE, but only to the functional space which, in our considerations, is of Sobolev type.

There are very few reliable compact classes of domains for the Kuratowski or Mosco convergences in Sobolev spaces, which are also stable by vector field transformations.

Given $l \in \mathbb{N}^*$, for every $N \geq 2$ and every $p > N-1$, the class

$$\{W_0^{1,p}(\Omega) : \Omega \text{ open } \Omega \subseteq D,\ \sharp\Omega^c \leq l\}$$

is compact for the Mosco convergence in $W_0^{1,p}(D)$. Here D is a bounded open set of $\mathbb{R}^N$ and $\sharp\Omega^c$ denotes the number of the connected components of $\mathbb{R}^N \setminus \Omega$. For $p = N = 2$ this was proved by Sverak [20] and for arbitrary values of p and N by Bucur and Trebeschi [6].

Concerning the $s-K$ compactness for problems involving Neumann boundary conditions, we refer to the result of [7] concerning the Dirichlet spaces $L^{1,2}(\Omega)$. These are the natural spaces which replace $H^1(\Omega)$ for shape functionals without zero order term (see [5], [12]). This space is embedded into $L^2(D, \mathbb{R}^N)$ by

$$L^{1,2}(\Omega) \ni u \to 1_\Omega \nabla u \in L^2(D, \mathbb{R}^N),$$

and a somehow similar result to the one of Sverak is proved for the family

$$\{L^{1,2}(\Omega) : \Omega \subseteq D \subseteq \mathbb{R}^2,\ \sharp\Omega^c \leq l\}.$$

We refer to [14] for an extension to nonlinear spaces and to [12] to the elasticity one.

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