

# DESCARTES' RULE OF SIGNS FOR POLYNOMIAL SYSTEMS SUPPORTED ON CIRCUITS

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ABSTRACT. We give the first multivariate version of Descartes' rule of signs to bound the number of positive real roots of a system of polynomial equations in  $n$  variables with  $n+2$  monomials, in terms of the sign variation of a sequence associated both to the exponent vectors and the given coefficients. We show that our bound is sharp and is related to the signature of the circuit.

## 1. INTRODUCTION

The following well-known rule of signs for univariate polynomials was proposed by René Descartes in 1637 in “La Géométrie”, an appendix to his “Discours de la Méthode”, see [14, pp. 96–99]:

**Descartes' rule of signs.** *Given a univariate real polynomial  $f(x) = c_0 + c_1x + \dots + c_r x^r$ , the number of positive real roots of  $f$  (counted with multiplicity) is bounded by the number of sign variations*

$$\text{sgnvar}(c_0, \dots, c_r)$$

*in the ordered sequence of the coefficients of  $f$ .*

The integer  $\text{sgnvar}(c_0, \dots, c_r) \in \{0, \dots, r\}$  is equal to the number of sign changes between two consecutive elements, after removing all zero elements. In other words,  $\text{sgnvar}(c_0, \dots, c_r)$  is the number of distinct pairs  $(i, j)$  of integers,  $0 \leq i < j \leq r$ , which satisfy  $c_i \cdot c_j < 0$  and  $c_\ell = 0$  for any integer  $\ell$  with  $i < \ell < j$ .

No general multivariate generalization is known for this simple rule. A lower bound (of the upper bound) was given by Itenberg and Roy in 1996 [8], by means of a construction based on the associated mixed subdivisions and the signs of the coefficients of the individual polynomials at the vertices of mixed cells. Recently, the first multivariate generalization in case of at most one positive real root, was stated in Theorem 1.5 of [10]. The main change of viewpoint in that article is that the number of positive roots of a square polynomial system (of

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$n$  polynomials in  $n$  variables) is related to the signs of the maximal minors of the matrix of exponents and the matrix of coefficients of the system, that is, to the associated oriented matroids. We take this approach to get the first multivariate version of Descartes' rule of signs for multivariate systems supported on circuits (see Theorem 2.8 below).

We fix an exponent set  $\mathcal{A} = \{w_0, w_1, \dots, w_{n+1}\} \subset \mathbb{Z}^n$  of cardinality  $n + 2$  and for any given real matrix  $C = (c_{i,j}) \in \mathbb{R}^{n \times (n+2)}$  we consider the associated sparse multivariate polynomial system in  $n$  variables  $x = (x_1, \dots, x_n)$  with support  $\mathcal{A}$ :

$$(1.1) \quad f_i(x) = \sum_{j=0}^{n+1} c_{i,j} x^{w_j} = 0, \quad i = 1, \dots, n.$$

We denote by  $n_{\mathcal{A}}(C)$  the number of real positive solutions of (1.1) counted with multiplicity.

We solve the following question:

**Problem.** *When  $n_{\mathcal{A}}(C)$  is finite, find a sharp upper bound in terms of the number of sign variations of an associated sequence of real numbers.*

Note that  $n_{\mathcal{A}}(C)$  is a *linear* invariant of  $C$ , since after multiplying  $C$  on the left by an invertible  $n \times n$  real matrix, the resulting coefficient matrix defines a system with the same solutions. On the other side,  $n_{\mathcal{A}}(C)$  is an *affine* invariant of the configuration  $\mathcal{A}$ , since the number of positive solutions is unaltered if we multiply each equation by a fixed monomial (so, after translation of  $\mathcal{A}$ ) or if we perform a monomial substitution of variables by a matrix with nonzero determinant. It is useful to consider the integer matrix  $A \in \mathbb{Z}^{(n+1) \times (n+2)}$ :

$$(1.2) \quad A = \begin{pmatrix} 1 & \dots & 1 \\ w_0 & \dots & w_{n+1} \end{pmatrix}.$$

Thus,  $n_{\mathcal{A}}(C)$  is also a *linear* invariant of  $A$ .

We make throughout the paper the following natural assumptions. When the convex hull of  $\mathcal{A}$  is of dimension smaller than  $n$  (or equivalently, when the rank  $\text{rk}(A)$  of  $A$  is smaller than  $n + 1$ ), system (1.1) corresponds to an overdetermined system of  $n$  equations in less than  $n$  variables, so in general there are no solutions and in particular cases there might be an infinite number. When the matrix  $C$  does not have rank  $n$ , we have a system of at most  $n - 1$  equations in  $n$  variables. Thus, we will then assume that

$$(1.3) \quad \text{rk}(A) = n + 1, \quad \text{rk}(C) = n.$$

There is a basic necessary condition for the existence of at least one positive solution of a system of  $n$  sparse polynomial equations in  $n$

variables with any number of fixed monomials, in particular for our system (1.1). Let  $C_0, \dots, C_{n+1} \in \mathbb{R}^n$  denote the column vectors of the coefficient matrix  $C$  and call

$$(1.4) \quad \mathcal{C}^\circ = \mathbb{R}_{>0}C_0 + \dots + \mathbb{R}_{>0}C_{n+1}$$

the positive cone generated by these vectors. Given a solution  $x \in \mathbb{R}_{>0}^n$ , the vector  $(x^{w_0}, \dots, x^{w_{n+1}})$  is positive and so the origin  $\mathbf{0} \in \mathbb{R}^n$  belongs to  $\mathcal{C}^\circ$ . So, necessarily

$$(1.5) \quad \mathbf{0} \in \mathcal{C}^\circ.$$

We furthermore give in Proposition 2.9 a necessary and sufficient condition to have  $n_{\mathcal{A}}(C) < \infty$ .

Our main result is Theorem 2.8, where we give a bound for  $n_{\mathcal{A}}(C)$  in case condition (1.5) is satisfied and  $n_{\mathcal{A}}(C)$  is finite. We define an *ordering* on the column vectors of the matrix  $C$  (Definition 2.5), which is determined, up to total reversing, by the signs of the maximal minors of  $C$ . It induces an ordering on the configuration  $\mathcal{A}$ , and the bound for  $n_{\mathcal{A}}(C)$  provided by Theorem 2.8 is the number of sign variations of the corresponding ordered sequence of coefficients in any affine relation of the given exponents  $\mathcal{A}$ . The main ingredients of our proof are a generalization of Descartes' rule of signs for vectors spaces of analytic real-valued functions and the classical Gale duality in linear algebra. Gale duality allows us to reduce the problem of bounding  $n_{\mathcal{A}}(C)$  to the problem of bounding the number of roots of a particular univariate rational function determined by the system.

We show in Theorem 3.3 that the signature of  $\mathcal{A}$  (Definition 2.1) gives a bound for  $n_{\mathcal{A}}(C)$ . For instance, if  $\mathcal{A}$  consists of the vertices of a simplex plus one interior point, then our results imply that  $n_{\mathcal{A}}(C) \leq 2$ , that is, the number of positive real roots of system (1.1) cannot exceed 2, in *any* dimension (see Example 3.4). We also recover the known upper bound  $n_{\mathcal{A}}(C) \leq n + 1$  obtained in [2] and [11], but we show that  $n_{\mathcal{A}}(C) = n + 1$  could only be attained for a particular value of the signature of  $\mathcal{A}$  and *uniform* matrices  $C$ , that is, with all nonzero maximal minors.

An important consequence of classical Descartes' rule of signs is that the number of real roots of a real univariate polynomial can be bounded in terms of the number of monomials (with nonzero coefficient), independently of its degree. In the multivariate case, Khovanskii [9, Corollary 7] proved the remarkable result that the number of nondegenerate solutions in  $\mathbb{R}^n$  of a system of  $n$  real polynomial equations can also be bounded solely in terms of the number of distinct monomials appearing in these equations. In contrast to Descartes' rule, Khovanskii's bound

is far from sharp and the known refinements do not depend on the signs of the coefficients or the particular configuration of the exponents but only on their number (see [13, Chapters 5–6]). Khovanskii’s result, as well as ours, is also valid for real configurations  $\mathcal{A}$ .

The paper is organized as follows. We give all the necessary definitions and state precisely our main results in the next section. In section 3 we present some interesting corollaries of these results. Section 4 contains the proofs of Theorems 2.8 and Proposition 2.9. Finally, we show the optimality of our bounds for  $n_{\mathcal{A}}(C)$  in Section 5.

## 2. ORDERINGS AND STATEMENT OF OUR MAIN RESULT

Let  $\mathcal{A} = \{w_0, w_1, \dots, w_{n+1}\} \subset \mathbb{Z}^n$ ,  $A \in \mathbb{Z}^{(n+1) \times (n+2)}$  and  $C \in \mathbb{R}^{n \times (n+2)}$  as in the Introduction. In particular, we assume that Conditions (1.3) are satisfied. For any natural number  $k$ , denote  $[k] = \{0, \dots, k-1\}$ . In particular  $[n+2] = \{0, \dots, n+1\}$ .

**2.1. The configuration  $\mathcal{A}$  and the matrix  $A$ .** The kernel of the matrix  $A$  has then dimension 1 and can be generated by the following vector  $\lambda \in \mathbb{Z}^n$ . For any  $\ell \in \{0, \dots, n+1\}$ , consider the matrix  $A(\ell) \in \mathbb{Z}^{(n+1) \times (n+1)}$  obtained from the matrix  $A$  in (1.2) by removing the column  $\ell$  and set  $\lambda_\ell = (-1)^{\ell+1} \det(A(\ell))$ . The vector  $\lambda$  gives an affine relation among the elements in  $\mathcal{A}$ :

$$(2.1) \quad \sum_{\ell=0}^{n+1} \lambda_\ell w_\ell = 0, \quad \sum_{\ell=0}^{n+1} \lambda_\ell = 0.$$

Let  $I \in \mathbb{Z}_{>0}$  be the greatest common divisor of  $\lambda_0, \dots, \lambda_{n+1}$  and set

$$(2.2) \quad \tilde{\lambda}_i = \frac{\lambda_i}{I}, \quad i = 0, \dots, n+1.$$

So, this (nonzero) vector  $\tilde{\lambda}$  gives the unique (up to sign) affine relation among the elements of  $\mathcal{A}$  with coprime coefficients. The integer  $I$  is the index of the subgroup  $\mathbb{Z}\mathcal{A}$  generated by the configuration  $\mathcal{A}$  in  $\mathbb{Z}^n$ . The normalized volume  $\text{vol}(\mathcal{A})$  of the configuration  $\mathcal{A}$  with respect to  $\mathbb{Z}^n$  is the Euclidean volume of the convex-hull of  $\mathcal{A}$  scaled so that any basis of  $\mathbb{Z}^n$  defines a simplex having volume one. It is well known that this normalized volume  $\text{vol}(\mathcal{A})$  bounds the number of isolated complex solutions of system (1.1). Moreover, we have  $\text{vol}(\mathcal{A}) = \sum_{\lambda_i > 0} \lambda_i = -\sum_{\lambda_i < 0} \lambda_i$ . The number of isolated positive solutions of system (1.1) is the same if we consider  $\mathcal{A}$  as a configuration in  $\mathbb{Z}\mathcal{A}$  or in  $\mathbb{Z}^n$ . Indeed, we may choose a basis  $(v_1, \dots, v_n)$  for  $\mathbb{Z}\mathcal{A}$ , and see the equations (1.1) as a system depending on new variables  $y_i = x^{v_i}$ . The normalized

volume of our configuration with respect to  $\mathbb{Z}\mathcal{A}$  is precisely  $\text{vol}(\mathcal{A})/I$ . So when  $n_{\mathcal{A}}(C)$  is finite, we have the bound

$$(2.3) \quad n_{\mathcal{A}}(C) \leq \text{vol}(\mathcal{A})/I = \sum_{\lambda_i > 0} \tilde{\lambda}_i = - \sum_{\lambda_i < 0} \tilde{\lambda}_i.$$

The configuration  $\mathcal{A}$  is said to be a *circuit* if  $\lambda_j \neq 0$  for all  $j = 0, 1, \dots, n+1$ , or equivalently, if any subset of  $n+1$  points in  $\mathcal{A}$  is affinely independent (i.e., its convex hull has dimension  $n$ ). In fact, we can always reduce our problem to this case, as we now explain. In general,  $\mathcal{A}$  is an  $(n-m)$ -*pyramid* over a circuit of affine dimension  $m$  (with  $1 \leq m \leq n$ ) given by the subconfiguration  $\mathcal{A}'$  of  $\mathcal{A}$  of  $m+2$  vectors with indices corresponding to nonzero  $\lambda_j$ . Without loss of generality, assume  $\mathcal{A}' = \{w_0, \dots, w_{m+1}\}$ , so that  $\lambda_j \neq 0$  for  $j = 0, \dots, m+1$  and  $\lambda_j = 0$  for any  $j > m+1$ . Then, either we can find a system equivalent to (1.1) of the form:

$$\begin{aligned} f_i(x) &= \sum_{j=0}^{m+1} c'_{i,j} x^{w_j} = 0, & i &= 1, \dots, m, \\ f_i(x) &= \sum_{j=0}^{m+1} c'_{i,j} x^{w_j} + x^{w_{i+1}} = 0, & i &= m+1, \dots, n, \end{aligned}$$

or system (1.1) has either 0 or an infinite number of solutions. In case we can find an equivalent system as above, the first  $m$  equations define essentially a square system of  $m$  equations in  $m$  variables. A positive solution of this smaller system does not necessarily extend to a positive solution  $x \in \mathbb{R}_{>0}^n$ , but if such extension exists, it is unique. So, in this case  $n_{\mathcal{A}}(C)$  is bounded by  $n_{\mathcal{A}'}(C')$ , where  $C' \in \mathbb{R}^{m \times (m+2)}$  is the coefficient matrix of the first  $m$  equations. Therefore, to simplify the notation, we will assume in what follows that  $m = n$ , that is, that  $\mathcal{A}$  is a circuit. If  $\mathcal{A}$  is not a circuit,  $n$  has to be replaced by  $m$  in general in the statements.

We will need the following definition.

**Definition 2.1.** *Given a circuit  $\mathcal{A} = \{w_0, \dots, w_{n+1}\} \subset \mathbb{Z}^n$ , and a nonzero affine relation  $\lambda \in \mathbb{Z}^{n+2}$  among the  $w_j$ . We call  $\aleph_+ = \{j \in [n+2] : \lambda_j > 0\}$ ,  $\aleph_- = \{j \in [n+2] : \lambda_j < 0\}$ , and we denote by  $a_+$  (resp.  $a_-$ ) the cardinality of  $\aleph_+$  (resp.  $\aleph_-$ ). The pair  $\{a_+, a_-\}$  is called the signature of  $\mathcal{A}$ . It is unordered but it may consist of a repeated value.*

Note that as  $\sum_j \lambda_j = 0$ , both  $a_+, a_- \geq 1$ .

**2.2. The matrix  $C$ .** Given  $i, j \in [n+2], i \neq j$ , we call  $C(i)$  the submatrix of  $C$  with columns indexed by the indices in  $[n+2] \setminus \{i\}$ , and by  $C(i, j)$  the submatrix of  $C$  with columns indexed by  $[n+2] \setminus \{i, j\}$

(in the same order). Note that we are not assuming that  $i < j$ , but only that they are different.

Our first lemma points out some easy consequences of Condition (1.5). Note that this condition means that there is a positive vector in the kernel of the coefficient matrix.

**Lemma 2.2.** *Let  $C \in \mathbb{R}^{n \times (n+2)}$  be a matrix of rank  $n$  which satisfies (1.5). Then, the following assertions hold*

- (i)  $C$  is not a pyramid, that is,  $\text{rk}(C(i)) = n$  for any  $i \in [n+2]$ .
- (ii) For any index  $j_1 \in [n+2]$ , there exists another index  $j_2$  such that  $\text{rk}(C(j_1, j_2)) = n$ .
- (iii) Assume  $j_1 \neq j_2$  and  $\text{rk}(C(j_1, j_2)) = n$ . Then, for any other index  $i$ , either  $\text{rk}(C(j_1, i)) = n$  or  $\text{rk}(C(j_2, i)) = n$ .

*Proof.* Assume  $\text{rk}(C(i)) < n$  for some index  $i$ . This means that the column vectors  $C_j, j \neq i$  lie in a proper subspace. Therefore, we can multiply  $C$  by an invertible matrix on the left, to get a matrix  $C'$  with  $i$ -th column equal to the  $i$ -th canonical vector  $e_i$  while the  $i$ -th coordinate of all other columns is equal to 0. So, any vector in  $\text{Ker}(C')$  has its  $i$ -th coordinate equal to 0. But  $\text{Ker}(C) = \text{Ker}(C')$  and this contradicts (1.5), proving (i).

Given an index  $j_1$ , as the rank of  $C(j_1)$  is  $n$  by (i), it has a square submatrix of rank  $n$ . Any such submatrix is of the form  $C(j_1, j_2)$  for an index  $j_1 \neq j_2$ . This proves (ii).

Assume now that the matrix  $C(j_1, j_2)$  has rank  $n$ . We can multiply  $C$  on the left by  $C(j_1, j_2)^{-1}$  and the resulting matrix  $C'$  will have an *identity* matrix in the columns different from  $j_1, j_2$ . For each  $i = 1, \dots, n$ , the  $i$ -th entry of the column  $C'_{j_1}$  (resp.  $C'_{j_2}$ ) equals  $\pm \det(C'(j_2, i))$  (resp.  $\pm \det(C'(j_1, i))$ ). If both  $\det(C'(j_1, i)) = \det(C'(j_2, i)) = 0$ , we deduce that  $C'$  is a pyramid, which contradicts item (i).  $\square$

We now translate condition (1.5) to the *Gale dual* setting. Given a full rank matrix  $C \in \mathbb{R}^{n \times (n+c+1)}$ , let  $B \in \mathbb{R}^{(n+c+1) \times c}$  a matrix whose columns generate  $\text{Ker}(C)$  and denote by  $P_0, \dots, P_{n+c} \in \mathbb{R}^{c+1}$  its row vectors. The configurations of column vectors of  $C$  and row vectors of  $B$  are said to be Gale dual. The following lemma is well known.

**Lemma 2.3.** *Let  $C_0, \dots, C_{n+c}$  and  $P_0, \dots, P_{n+c}$  be Gale dual configurations. Then,  $\mathbf{0}$  lies in the open positive cone  $\mathcal{C}^\circ$  generated by  $C_0, \dots, C_{n+c}$  if and only if the vectors  $P_0, \dots, P_{n+c}$  lie in an open half-space through the origin.*

*Proof.* Any vector  $u$  in the kernel of  $C$  is of the form  $B\mu$ , with  $\mu$  in  $\mathbb{R}^{c+1}$ . Thus,

$$u_i = \langle P_i, \mu \rangle, \quad i = 0, \dots, n+c.$$

As we remarked, the condition on the columns of  $C$  means that there is a positive vector  $u$  in the kernel of the matrix  $C$ , and this is clearly equivalent to the fact that the row vectors  $P_i$  lie in the positive halfspace defined by  $\mu$ , which proves the lemma.  $\square$

**Remark 2.4.** *Given a full rank matrix  $C \in \mathbb{R}^{n \times (n+2)}$  which satisfies (1.5) and  $P_0, \dots, P_{n+1}$  a Gale dual of  $C$ , there is in fact a nonempty connected open cone  $\mathcal{U}_P \subset \mathbb{R}^2$  defined by the condition that  $\mu \in \mathcal{U}_P$  if and only if  $\langle P_i, \mu \rangle > 0$  for any  $i \in [n+2]$ . It follows that the open set  $\Delta_P = \{y \in \mathbb{R} : (1, y) \in \mathcal{U}_P\}$  is nonempty.*

**Definition 2.5.** *Let  $C \in \mathbb{R}^{n \times (n+2)}$  be a full rank matrix. A bijection  $\alpha : [n+2] \rightarrow [n+2]$  which verifies that for any choice of Gale dual vectors  $P_0, \dots, P_{n+1} \in \mathbb{R}^2$ ,*

$$\det(P_{\alpha_i}, P_{\alpha_j}) \geq 0, \text{ for any } i < j, \text{ or } \det(P_{\alpha_i}, P_{\alpha_j}) \leq 0, \text{ for any } i < j.$$

*is called an ordering for  $C$ .*

We then have:

**Proposition 2.6.** *Let  $C \in \mathbb{R}^{n \times (n+2)}$  be a full rank matrix satisfying (1.5). Then, there exists an ordering  $\alpha : [n+2] \rightarrow [n+2]$ .*

*Proof.* Let  $P'_0, \dots, P'_{n+1}$  be any choice of Gale dual of  $C$ . We know by Lemma 2.3 that the planar vectors  $P'_0, \dots, P'_{n+1}$  lie in an open halfspace through the origin. We can thus order them according to their arguments, that is, we can find a bijection  $\alpha : [n+2] \rightarrow [n+2]$  which verifies that

$$(2.4) \quad \det(P'_{\alpha_i}, P'_{\alpha_j}) \geq 0, \quad \text{for any } i < j.$$

It remains to prove that one of the two sign conditions in Definition 2.5 holds for any other choice of Gale dual configuration  $P_0, \dots, P_{n+1}$ . But if  $B$  (resp.  $B'$ ) denotes the  $(n+2) \times 2$  matrix with rows  $p_i$  (resp.  $p'_i$ ), the columns of  $B$  (resp.  $B'$ ) give a basis of  $\text{Ker}(C)$ . Then, there is an invertible matrix  $M \in \mathbb{R}^{2 \times 2}$  such that  $B' \cdot M = B$ . Then,  $B^t = M^t B'^t$ , and thus each vector  $P_i$  equals the image of the vector  $P'_i$  by the invertible linear map with matrix  $M^t$  in the canonical bases. Then, if  $\det(M^t) > 0$ ,

$$\det(P_{\alpha_i}, P_{\alpha_j}) \geq 0, \quad \text{for any } i < j,$$

and if  $\det(M^t) < 0$ , we have that

$$\det(P_{\alpha_i}, P_{\alpha_j}) \leq 0, \quad \text{for any } i < j,$$

as wanted.  $\square$

We can translate an ordering  $\alpha$  on the Gale dual vectors of  $C$  to sign conditions on the maximal minors of  $C$ . We first recall the following well known result (stated as Lemma 2.10 in [10], together with several references). Given a subset  $J = \{j_1, j_2\} \subseteq [n+2]$ , let  $J^c = \{i_1, \dots, i_{n-1}\}$  be the complement of  $J$  in  $[n+2]$ . For  $j_1 < j_2$  and  $i_1 < \dots < i_{n-1}$ , let  $\tau(J) \in \{\pm 1\}$  denote the sign of the permutation that sends  $0, 1, \dots, n+1$  to  $j_1, j_2, i_1, \dots, i_{n-1}$  respectively. Then,  $\tau(J) = (-1)^{j_1+j_2+1}$  if  $j_1 < j_2$  (and  $(-1)^{j_1+j_2}$  otherwise). Let  $C \in \mathbb{R}^{n \times (n+2)}$ ,  $B \in \mathbb{R}^{(n+2) \times 2}$  full-rank matrices with  $\text{im}(B) = \ker(C)$ , so that the column vectors of  $C$  and the row vectors  $P_0, \dots, P_{n+1}$  of  $B$  are Gale dual configurations. Then, there exists a nonzero real number  $\delta$  such that

$$(2.5) \quad \delta \det(C(j_1, j_2)) = (-1)^{j_1+j_2} \det(P_{j_1}, P_{j_2}),$$

for all subsets  $J = \{j_1, j_2\} \subseteq [n+2]$  of cardinality 2 such that  $j_1 < j_2$ . We immediately deduce:

**Proposition 2.7.** *Let  $C \in \mathbb{R}^{n \times (n+2)}$  be a full rank matrix satisfying (1.5). A bijection  $\alpha : [n+2] \rightarrow [n+2]$  is an ordering for  $C$  if and only if either  $(-1)^{\alpha_i+\alpha_j} \cdot \det(C(\alpha_i, \alpha_j)) \cdot \frac{\alpha_j-\alpha_i}{j-i} \geq 0$  for any distinct  $i, j \in [n+2]$ , or  $(-1)^{\alpha_i+\alpha_j} \cdot \det(C(\alpha_i, \alpha_j)) \cdot \frac{\alpha_j-\alpha_i}{j-i} \leq 0$  for any distinct  $i, j \in [n+2]$ .*

**2.3. The main result.** Let  $A$  and  $C$  matrices as in the Introduction which satisfy (1.3) and (1.5). We choose a subset  $K \subset [n+2]$  maximal (under containment) such that  $\det(C(i, j)) \neq 0$  for any distinct  $i, j \in K$ . Denote by  $k \geq 2$  the cardinality of  $K$ . Let  $\alpha$  be an ordering for  $C$ . Denote again by  $\alpha$  the bijection  $[k] \rightarrow K$  which is deduced from  $\alpha$ . Thus,  $\det(P_{\alpha_i}, P_{\alpha_j}) \geq 0$  for any  $i < j$  in  $[k]$ , or  $\det(P_{\alpha_i}, P_{\alpha_j}) \leq 0$  for any  $i < j$  in  $[k]$ . We furthermore denote for any  $j \in [k]$

$$(2.6) \quad K_j = \{\ell \in [n+2] : \ell = \alpha_j \text{ or } \det(C(\alpha_j, \ell)) = 0\}.$$

Thus, we get a partition  $\sqcup_{j \in [k]} K_j = [n+2]$ .

We define now the following linear forms in the signed minors  $\lambda_j$  of the matrix  $A$ :

$$(2.7) \quad \bar{\lambda}_j = \sum_{\ell \in K_j} \lambda_\ell = \sum_{\ell \in K_j} (-1)^{\ell+1} \det(A(\ell)).$$

Note that this set of  $k$  linear forms is independent of the choice of subset  $K$ . Observe also that  $k = n+2$  (that is,  $K = [n+2]$ ) if and only if all maximal minors of  $C$  are nonzero, that is,  $C$  is *uniform*.

Moreover, we define the (ordered) sequence:

$$(2.8) \quad s_\alpha = (\bar{\lambda}_{\alpha_0}, \dots, \bar{\lambda}_{\alpha_{k-1}}).$$



We are ready to state our main result:

**Theorem 2.8.** *Let  $\mathcal{A} = \{w_0, w_1, \dots, w_{n+1}\} \subset \mathbb{Z}^n$ ,  $A \in \mathbb{Z}^{(n+1) \times (n+2)}$  and  $C \in \mathbb{R}^{n \times (n+2)}$  as in the Introduction, which satisfy (1.3). Assume moreover that  $C$  satisfies the necessary condition (1.5) and let  $\alpha$  be an ordering for  $C$ . Let  $K, |K| = k \leq n+2$ , and the linear forms  $\bar{\lambda}_j, j \in [k]$ , as above.*

*Then, if  $n_A(C)$  is finite,*

$$(2.9) \quad n_A(C) \leq \text{signvar}(s_\alpha),$$

*where  $s_\alpha$  is the ordered sequence of linear forms in the maximal minors of  $A$  defined in (2.8). Moreover,*

$$(2.10) \quad n_A(C) \leq \min\{\text{sgnvar}(s_\alpha), \text{vol}(\mathcal{A})/I\},$$

*where  $I$  is the index of the subgroup  $\mathbb{Z}\mathcal{A}$  generated by the configuration  $\mathcal{A}$  in  $\mathbb{Z}^n$ .*

We now state the following general result, which gives a necessary and sufficient condition for  $n_A(C)$  to be finite as required in the statement of Theorem 2.8.

**Proposition 2.9.** *Let  $A, C$  of maximal rank. Assume  $n_A(C) > 0$  and let  $d = (d_0, \dots, d_{n+1})$  be a nonzero vector in  $\ker(C)$ . Then  $n_A(C)$  is finite if and only if there exists  $r \in [k]$  such that  $d_{\alpha_r} \cdot \bar{\lambda}_r \neq 0$ .*

The proofs of Theorem 2.8 and Proposition 2.9 will be given in Section 4.

**2.4. Relation with previously known cases.** At a first glance, one might think that the right member of (2.9) depends only on the exponent vectors  $\mathcal{A}$ , but this is not the case since the bijection  $\alpha$  is determined by the coefficient matrix  $C$ . This is in a sense dual to the classical Descartes' rule of signs for univariate polynomials with respect to a monomial basis, where the exponents determine the ordering, while the sign variation is about the coefficients.

In case  $n = 1$ , Theorem 2.8 is equivalent to the classical Descartes' rule of signs applied to trinomials in one variable. Given  $f(x) = c_0x^{w_0} + c_1x^{w_1} + c_2x^{w_2} \in \mathbb{R}[x]$  a trinomial in one variable, where  $w_0 < w_1 < w_2$ ,  $n_A(C)$  is the number of positive roots of  $f$  (counted with multiplicity). In any affine relation  $\lambda_0 w_0 + \lambda_1 w_1 + \lambda_2 w_2 = 0$ , the coefficients  $\lambda_0$  and  $\lambda_2$  have the same sign, which is opposite to that of  $\lambda_1$ . Note that always  $\text{vol}(\mathcal{A})/I \geq 2 = n + 1$ . Assume  $c_0, c_1, c_2 \neq 0$  (otherwise, it is straightforward). A Gale dual of the coefficient matrix  $C$  is given by the vectors  $P_0 = (c_1, c_2), P_1 = (-c_0, 0), P_2 = (0, -c_0)$ . Considering the possible signs of the coefficients and the corresponding ordering

$\alpha : [3] \rightarrow [3]$ , it is straightforward to check that  $s_\alpha = \text{sgnvar}(c_0, c_1, c_2)$ , which is the classical Descartes' bound.

Given any  $n$ , we may translate  $\mathcal{A}$  so that  $w_0 = (0, \dots, 0)$ . Theorem 1.5 in [10] asserts the following in the particular case of circuits: If  $\det(A(j)) \cdot \det(C(0, j)) \geq 0$  for  $j = 1, \dots, n+1$  and at least one of the inequalities  $\det(A(j)) \cdot \det(C(0, j)) \neq 0$  holds, then  $n_A(C) \leq 1$ .

We can assume that  $C$  has maximal rank and satisfies condition 1.5, since otherwise  $n_A(C) = 0$ , and that  $\mathcal{A}$  is a circuit, that is,  $\lambda_j = (-1)^j \det(A(j)) \neq 0$  for all  $j$ . Consider the vector

$$B^0 = (0, -\det(C(0, 1)), +\det(C(0, 2)), \dots, (-1)^{n+1} \det(C(0, n+1)))^t$$

in  $\ker(C)$ . Consider a choice of Gale dual  $P_0, \dots, P_{n+1}$  of  $C$  for which the coefficients of  $B^0$  give the second coordinates of the  $P_j$  and let  $\alpha$  be an ordering. As the vectors  $P_j$  lie in an open halfspace,  $\alpha$  will indicate first all indices  $j$  with  $B_j^0 < 0$ , then  $j = 0$  and then those with  $B_j^0 > 0$ , if any (or just the opposite). Assume first that all  $P_j$  are linearly independent. It follows that all entries  $j \geq 1$  of  $B^0$  are nonzero and have the same sign as the coefficient  $\lambda_j$ . When we consider the sign variation  $s_\alpha$  we will be then taking the sign variation of those coefficients  $\lambda_j$  which are negative, and then those which are positive, if any. It follows that  $s_\alpha \leq 1$  and we get  $n_A(C) \leq 1$  via Theorem 2.8. When  $P_j, P_\ell$  are linearly dependent, as the Gale configuration lies in an open halfspace, we deduce that  $B_j^0$  has the same sign as  $B_\ell^0$ . So, when we compute the linear forms  $\bar{\lambda}_j$  we add numbers with this same sign and therefore we also get that our bound (2.10) reads  $n_A(C) \leq 1$ .

### 3. BOUNDS AND SIGNATURE OF THE CIRCUIT

Along this section, we keep the previous notations. In particular, we have a configuration  $\mathcal{A} \subset \mathbb{Z}^n$ ,  $A \in \mathbb{Z}^{(n+1) \times (n+2)}$  and  $C \in \mathbb{R}^{n \times (n+2)}$  as in the Introduction. We assume that  $A, C$  satisfy (1.3) and the necessary condition (1.5). Let  $\alpha$  be an ordering for  $C$ .

The first consequence of Theorem 2.8 is the following.

**Corollary 3.1.** *Let  $k$  be the cardinality of a maximal subset  $K \subset [n+2]$  with  $\det(C(i, j)) \neq 0$  for any distinct  $i, j \in K$ . Then*

$$(3.1) \quad n_A(C) \leq k - 1.$$

The proof follows in a straightforward way from (2.9), since  $s_\alpha$  is a sequence of length  $k$  and so its sign variation can be at most equal to  $k - 1$ .

**Remark 3.2.** *In case that one of the columns of  $C$  is identically zero, we have in fact a system with support on a simplex. If, for instance,*

$C_0 = 0$ , then the vector  $(1, 0, \dots, 0)$  lies in  $\text{Ker}(C)$ , and so any Gale dual configuration satisfies that  $p_0$  is linearly independent of the other  $p_i$ , which all have first coordinate equal to 0 and are thus linearly dependent. It follows that  $k = 2$  and we deduce from Corollary 3.1 that if  $n_{\mathcal{A}}(C)$  is finite, there is at most a single positive solution. As we are assuming that  $\text{rk}(C) = n$  and it is not a pyramid, there cannot be two zero columns.

We will now give upper bounds for  $n_{\mathcal{A}}(C)$  for any  $C$  for which it is finite, only in terms of the combinatorics of the points in the configuration  $\mathcal{A}$ . Interestingly, we can also deduce from Theorem 2.8 necessary conditions to achieve the bound  $n_{\mathcal{A}}(C) = n + 1$ .

**Theorem 3.3.** *Let  $\mathcal{A}$  and  $C$  as in the statement of Theorem 2.8 with  $n_{\mathcal{A}}(C) < \infty$ . If  $\mathcal{A}$  has signature  $\{a_+, a_-\}$ ,*

$$(3.2) \quad n_{\mathcal{A}}(C) \leq \begin{cases} 2 \min\{a_+, a_-\} & \text{if } a_+ \neq a_- \\ 2 \min\{a_+, a_-\} - 1 & \text{if } a_+ = a_- \end{cases}$$

Moreover, if  $n_{\mathcal{A}}(C) = n + 1$ , then  $C$  is uniform and the support  $\mathcal{A}$  is a circuit with signature

$$(3.3) \quad \left\{ \lfloor \frac{n+2}{2} \rfloor, n+2 - \lfloor \frac{n+2}{2} \rfloor \right\},$$

that is, there are essentially the same number of positive and negative coefficients  $\lambda_j$ .

*Proof.* Let  $\alpha$  be a bijection certifying the ordering property for  $C$  and let  $K$  be a maximal subset as before, with cardinal  $k \leq n + 2$ . By Theorem 2.8, we have that  $n_{\mathcal{A}}(C)$  is bounded by  $\text{sgnvar}(\bar{\lambda}_{\alpha_0}, \dots, \bar{\lambda}_{\alpha_{k-1}})$ .

For any  $j \in [k]$ , if the value of  $\bar{\lambda}_j$  is nonnegative (resp. nonpositive) then  $K_j \cap \mathfrak{N}_+ \neq \emptyset$  (resp.  $K_j \cap \mathfrak{N}_- \neq \emptyset$ ). Therefore, the number  $n_+$  of positive (resp.  $n_-$  of negative) terms in the sequence  $(\bar{\lambda}_{\alpha_0}, \dots, \bar{\lambda}_{\alpha_{k-1}})$  is at most  $a_+$  (resp.  $a_-$ ). So its sign variation is at most  $2 \min\{n_+, n_-\} \leq 2 \min\{a_+, a_-\}$ . In case  $a_+ = a_- = a$  and  $\min\{n_+, n_-\} < a$ , clearly  $2 \min\{n_+, n_-\} < 2a - 1$ . If  $\min\{n_+, n_-\} = a$ , then  $n_+ = n_- = a$  and then  $\text{sgnvar}(\bar{\lambda}_{\alpha_0}, \dots, \bar{\lambda}_{\alpha_{k-1}}) \leq 2a - 1$ , which concludes the proof of the inequalities.

In case  $n_{\mathcal{A}}(C) = n + 1$ , we deduce that  $n + 1 \leq 2 \min\{a_+, a_-\}$  or moreover that  $n + 1 \leq 2 \min\{a_+, a_-\} - 1$  in case  $a_+ = a_-$ . As  $a_+ + a_- = n + 2$ , it is easy to see that the signature of  $\mathcal{A}$  is as in (3.3). In this case, we get for  $n$  even that  $a_+ = a_- = \frac{n+2}{2}$  and so  $n + 1 = 2 \frac{n+2}{2} - 1$ , and for  $n$  odd we get that the signature of  $\mathcal{A}$  equals  $\{\frac{n+1}{2}, \frac{n+3}{2}\}$  and so  $n + 1 = 2 \frac{n+1}{2}$ . Thus, we get equality in (3.2), and so  $n_{\mathcal{A}}(C) =$

$n + 1 = \text{sgnvar}(\bar{\lambda}_{\alpha_0}, \dots, \bar{\lambda}_{\alpha_{k-1}})$ , from which we deduce that  $k = n + 2$  and therefore  $C$  is uniform.  $\square$

**Example 3.4.** *As we stated in the Introduction, if for instance  $\mathcal{A}$  consists of the vertices of a simplex plus one interior point, then its signature equals  $\{1, n + 1\}$  and so we deduce from Theorem 3.3 that  $n_{\mathcal{A}}(C) \leq 2$  in any dimension (for any choice of  $C$  for which  $n_{\mathcal{A}}(C)$  is finite).*

#### 4. PROOF OF THEOREM 2.8

We first need to recall a generalization of Descartes' rule of signs in the univariate case and apply it in our case via the notion of ordering in Section 4.1. Then, we complete the proof of our main Theorem 2.8 in Section 4.2, which expands some basic facts in [4, 1, 2].

**4.1. A univariate generalization of Descartes' rule of signs and orderings.** Descartes' rule of signs concerns polynomials, that is, analytic functions of a single variable that can be written as real linear combinations of monomials with integer exponents. In fact, the same result holds for monomials with real exponents, which can be evaluated on  $\mathbb{R}_{>0}$ . We recall the generalization of univariate Descartes' rule for vector spaces generated by different choices of analytic functions in [12] and we then use it for our system in a Gale dual formulation.

**Definition 4.1.** *A sequence  $(h_1, h_2, \dots, h_s)$  of real valued analytic functions defined on an open interval  $\Delta \subset \mathbb{R}$  satisfies Descartes' rule of signs on  $\Gamma$  if for any sequence  $a = (a_1, a_2, \dots, a_s)$  of real numbers, the number of roots of  $a_1 h_1 + a_2 h_2 + \dots + a_s h_s$  in  $\Delta$  counted with multiplicity never exceeds  $\text{sgnvar}(a)$ .*

The classical Descartes' rule of signs asserts that monomial bases  $(1, y, \dots, y^{s-1})$  satisfy Descartes' rule of signs on the open interval  $(0, +\infty)$ . We note that if  $(h_1, h_2, \dots, h_s)$  satisfies Descartes' rule of signs on  $\Delta$ , then the same holds true for the sequence  $(h_s, h_{s-1}, \dots, h_1)$ .

Recall that the Wronskian of  $h_1, \dots, h_s$  is the following determinant

$$W(h_1, \dots, h_s) = \det \begin{pmatrix} h_1 & h_2 & \dots & h_s \\ h'_1 & h'_2 & \dots & h'_s \\ \vdots & \vdots & \dots & \vdots \\ h_1^{(s-1)} & h_2^{(s-1)} & \dots & h_s^{(s-1)} \end{pmatrix}.$$

(The  $(i, j)$  coefficient is the  $(i - 1)$ -th derivative of  $h_j$ .) A well-known result asserts that  $h_1, \dots, h_s$  are linearly dependent if and only if their Wronskian is identically zero.

The following result is proven in [12], part 5, items 87 and 90:

**Proposition 4.2** ([12]). *A sequence of functions  $h_1, \dots, h_s$  satisfies Descartes' rule of signs on  $\Delta \subset \mathbb{R}$  if for any collection of integers  $1 \leq j_1 < j_2 \dots < j_\ell \leq s$  we have*

$$W(h_{j_1}, \dots, h_{j_\ell}) \neq 0, \quad (1)$$

and for any collections of integers  $1 \leq j_1 < j_2 \dots < j_\ell \leq k$  and  $1 \leq j'_1 < j'_2 \dots < j'_\ell \leq k$  of the same size, we have

$$W(h_{j_1}, \dots, h_{j_\ell}) \cdot W(h_{j'_1}, \dots, h_{j'_\ell}) > 0. \quad (2)$$

In particular, if a sequence of analytic functions  $h_1, \dots, h_s$  satisfies Descartes' rule of signs on  $\Delta \subset \mathbb{R}$ , then  $h_1, \dots, h_s$  do not vanish, have the same sign on  $\Gamma$  and no two of them are proportional.

We now apply Proposition 4.2 to the Gale dual of a coefficient matrix  $C \in \mathbb{R}^{n \times (n+2)}$  of rank  $n$  satisfying the necessary condition (1.5). We keep the notations and definitions of Section 2. Let  $C \in \mathbb{R}^{n \times (n+2)}$  be a full rank matrix satisfying (1.5) and  $\alpha : [n+2] \rightarrow [n+2]$  an ordering of  $C$ ; moreover, let  $K$  as in Section 2.3 and  $\alpha : [k] \rightarrow K$ , the restricted ordering. Given a Gale dual configuration  $P_0, \dots, P_{n+1}$  of  $C$ , consider the associated linear functions

$$p_j(y) = \langle P_j, (1, y) \rangle,$$

and let  $\Delta_P \neq \emptyset$  as in Remark 2.4. Then,  $p_j$  are positive functions on  $\Delta_P$  for any  $j \in [n+2]$ .

**Proposition 4.3.** *With the previous hypotheses and notations, the collection of nonzero rational functions  $(1/p_{\alpha_0}, 1/p_{\alpha_2}, \dots, 1/p_{\alpha_{k-1}})$  satisfies Descartes' rule of signs on  $\Delta_P$ .*

*Proof.* We need to check the hypotheses of Proposition 4.2. For simplicity, we only compute the Wronskian of  $1/p_1, 1/p_2, \dots, 1/p_\ell$ . Recall that  $p_j(y) = P_{j,1} + P_{j,2}y$  and denote by  $\gamma_\ell$  the product

$$\gamma_\ell = \prod_{i=1}^{\ell} (-1)^{i-1} (i-1)!.$$

We have

$$(1/p_j)^{(i-1)} = (-1)^{i-1} (i-1)! \cdot \frac{P_{j,2}^{i-1}}{p_j^i},$$

and thus by the computation of a Vandermonde determinant we get

$$\begin{aligned} W\left(\frac{1}{p_1}, \frac{1}{p_2}, \dots, \frac{1}{p_\ell}\right) &= \gamma_\ell \cdot \frac{1}{p_1 \cdots p_\ell} \cdot \det \left( \left( \frac{P_{j,2}}{p_j} \right)^{i-1} \right)_{1 \leq i, j \leq \ell} \\ &= \gamma_\ell \cdot \frac{1}{p_1 \cdots p_\ell} \cdot \prod_{1 \leq r_1 < r_2 \leq \ell} \left( \frac{P_{r_2,2}}{p_{r_2}} - \frac{P_{r_1,2}}{p_{r_1}} \right). \end{aligned}$$

Moreover: for any  $i, j \in [k], j < i$

$$\frac{P_{\alpha_i,2}}{p_{\alpha_i}} - \frac{P_{\alpha_j,2}}{p_{\alpha_j}} = \frac{1}{p_{\alpha_i} p_{\alpha_j}} \cdot \det(P_{\alpha_j}, P_{\alpha_i}) > 0 \text{ on } \Delta_P$$

(or  $\frac{P_{\alpha_i,2}}{p_{\alpha_i}} - \frac{P_{\alpha_j,2}}{p_{\alpha_j}} < 0$  on  $\Delta_P$  for any  $i, j \in [k], j < i$ ). Therefore, we verify condition (1) in Proposition 4.2 that no Wronskian of distinct functions taken among  $1/p_{\alpha_i}$  for  $i \in [k]$  vanishes, and moreover, the sign of the Wronskian only depends on the number  $\ell$  of functions we are considering. This implies condition (2) in Proposition 4.2.  $\square$

**4.2. The proof of Theorem 2.8.** We start with a basic result we will need about sign variations.

**Lemma 4.4.** *Let  $s = (c_0, c_1, \dots, c_{k-1})$  be a sequence of real numbers such that the sign variation  $\text{sgnvar}(s)$  is nonzero. Then  $\text{sgnvar}(s)$  is the minimum of the quantities  $s_q, q \in [k]$ , defined by*

$$s_0 = 1 + \text{sgnvar}(c_1, \dots, c_{k-1}), \quad s_{k-1} = 1 + \text{sgnvar}(-c_0, \dots, -c_{k-2}),$$

and  $s_q = 1 + \text{sgnvar}(-c_0, \dots, -c_{q-1}, c_{q+1}, \dots, c_{k-1})$ ,  $q = 1, \dots, k-2$ .

*Proof.* For any  $q \in [k]$ , we easily have  $\text{sgnvar}(s) \leq s_q$ . Assume that the sign variation  $\text{sgnvar}(s)$  is not the minimum of these quantities  $s_q$ . Then  $s_q - \text{sgnvar}(s) > 0$  for any  $q \in [k]$ . But  $s_q - \text{sgnvar}(s)$  is equal to  $1 - \text{sgnvar}(c_0, c_1)$  for  $q = 0$ , to  $1 - \text{sgnvar}(c_{q-1}, c_q, c_{q+1})$  for  $1 \leq q \leq k-2$  and to  $1 - \text{sgnvar}(c_{k-2}, c_{k-1})$  for  $q = k-1$ . It follows that if  $s_q - \text{sgnvar}(s) > 0$  for all  $q \in [k]$ , then  $\text{sgnvar}(s) = 0$ .  $\square$

Let  $C$  be a coefficient matrix of maximal rank satisfying condition (1.5). As we saw in item (ii) of Lemma 2.2, for any given  $j_1 \in [n+2]$  there exists  $j_2 \in [n+2] \setminus \{j_1\}$  such that  $\det(C(j_1, j_2)) \neq 0$ . Indeed, for any pair of distinct indices  $i, j \in [k]$ ,  $\det(C(\alpha_i, \alpha_j)) \neq 0$ . We will understand that  $\det(C_{\alpha_i}, C_{\alpha_i}) = 0$ . We define for each  $i \in [k]$  the column vector  $B^i$  with  $\ell$ -th coordinate equal to

$$(4.1) \quad B_\ell^i = \begin{cases} (-1)^\ell \det(C(\alpha_i, \ell)), & \ell \in [n+2], \alpha_i \leq \ell \\ (-1)^{1+\ell} \det(C(\alpha_i, \ell)), & \ell \in [n+2], \alpha_i > \ell \end{cases}$$

Then  $B^i \in \ker(C)$  for any  $i \in [k]$ , and for any pair of different indices  $i, j \in [k]$ , the vectors  $B^i, B^j$  span  $\ker(C)$ . Moreover, assume that  $\alpha_i < \alpha_j$  and consider the basis of  $\ker(C)$  given by the vectors:

$$P^i = (-1)^{\alpha_j} \det(C(\alpha_i, \alpha_j))^{-1} B^i, \quad P^j = (-1)^{1+\alpha_i} \det(C(\alpha_i, \alpha_j))^{-1} B^j.$$

Then,  $P_{\alpha_i}^i = P_{\alpha_j}^j = 0$ , and  $P_{\alpha_j}^i = P_{\alpha_i}^j = 1$ . If  $\alpha_i > \alpha_j$  we consider the basis given by the opposite vectors so that the same conclusions hold.

We are now ready to present the proof of our main result.

*Proof of Theorem 2.8.* Let  $\mathcal{A} = \{w_0, \dots, w_{n+1}\}$ ,  $A$ ,  $C$ ,  $K$  and an ordering  $\alpha$  of  $C$  as in the statement of Theorem 2.8. We can assume that  $n_A(C) > 0$ , since otherwise the bound is obvious.

Fix any  $i \in [k]$  and pick any other index  $j \in [k]$ . Let  $B^{ji} \in \mathbb{R}^{(n+2) \times 2}$  be the rank 2 matrix with columns  $P^j, P^i$  in this order. The Gale dual vectors  $P_0, \dots, P_n$  read in the rows of  $B^{ji}$  satisfy that  $P_{\alpha_i} = (1, 0)$ ,  $P_{\alpha_j} = (0, 1)$ . Thus,  $x \in \mathbb{R}_{>0}^n$  is a solution of (1.1) if and only if there exists  $\mu \in \mathcal{U}_P$  (defined in Remark 2.4) such that  $(x^{w_0}, \dots, x^{w_{n+1}})^t = B^{ji} \mu^t$ , or equivalently:  $\mu_1 = x^{w_{\alpha_i}}$ ,  $\mu_2 = x^{w_{\alpha_j}}$ , and

$$(4.2) \quad x^{w_\ell} = P_{\ell,1} x^{w_{\alpha_i}} + P_{\ell,2} x^{w_{\alpha_j}}, \quad \text{for all } \ell \in [n+2].$$

In particular, both coordinates  $\mu_1, \mu_2 \neq 0$  and the open set  $\Delta_P \subset \mathbb{R}$  is a nonempty interval. Up to a translation of the configuration  $\mathcal{A}$ , we can assume without loss of generality that  $w_{\alpha_i} = 0$ . Then,  $\mu_1 = 1$ . Consider the function  $g : \Delta_P \rightarrow \mathbb{R}$  defined by

$$g(y) = \prod_{\ell \in [n+2]} p_\ell(y)^{\tilde{\lambda}_\ell},$$

where  $\tilde{\lambda}_\ell$  are the coefficients of the coprime affine relation among the  $w_\ell$  defined in (2.2). Then,  $x$  is a positive solution of (1.1) if and only if  $y \in \Delta_P$  satisfies  $g(y) = 1$ . Moreover, this bijection  $x \mapsto y = x^{w_{\alpha_j}}$  between positive solutions of (1.1) and solutions of  $g(y) = 1$  in  $\Delta_P$  preserves the multiplicities [5]. Thus, to bound  $n_A(C)$  we then have to bound the number of solutions of  $g = 1$  over the interval  $\Delta_P$  counted with multiplicities. As we assume that  $n_A(C)$  is finite, we get that  $g \neq 1$ .

The logarithm  $G = \log(g)$  is well defined over  $\Delta_P$  and for any  $y \in \Delta_P$ ,  $g(y) = 1$  if and only if  $G(y) = \sum_{\ell \in [n+2]} \tilde{\lambda}_\ell \log(p_\ell(y)) = 0$ . Remark that  $g(y) = 1$  if and only if  $y$  is a root of the polynomial  $\prod_{\tilde{\lambda}_\ell > 0} p_\ell(y)^{\tilde{\lambda}_\ell} - \prod_{\tilde{\lambda}_\ell < 0} p_\ell(y)^{-\tilde{\lambda}_\ell}$ , which is nonzero because  $g \neq 1$  and has degree bounded by  $\text{vol}(\mathcal{A})/I$ . Recall the partition of  $[n+2]$  defined in (2.6). For any  $r \in [k]$ , an index  $\ell \in K_r$  if and only if there exists a positive constant  $\gamma_{\ell r}$  such that  $P_\ell = \gamma_{\ell r} P_{\alpha_r}$ . The derivative of  $G$  over  $\Delta_P$  equals

$$G'(y) = \sum_{\ell \in [n+2]} \tilde{\lambda}_\ell \frac{P_{\ell,2}}{p_\ell(y)} = \frac{1}{I} \sum_{r \in [k]} \bar{\lambda}_r \frac{P_{\alpha_r,2}}{p_{\alpha_r}(y)}.$$

Thus,

$$G'(y) = 0 \text{ if and only if } \sum_{r \in [k]} \frac{\bar{\lambda}_r P_{\alpha_r,2}}{p_{\alpha_r}(y)} = 0.$$

Now, we can use Proposition 4.3 to deduce that the number of roots of  $G'$  on  $\Delta_P$  counted with multiplicities is at most

$$s_i = \operatorname{sgnvar}(P_{\alpha_0,2}\bar{\lambda}_0, \dots, P_{\alpha_{k-1},2}\bar{\lambda}_{k-1}).$$

Rolle's theorem leads to the bound  $s_i + 1$  for the number of roots of  $g$  contained in  $\Delta_P$  counted with multiplicities, and thus to this bound for the number  $n_A(C)$ . Note that for any  $r \in [k]$ ,

$$P_{\alpha_r,2} = (-1)^{\alpha_r + \alpha_j} \delta_{i,j,r} \det(C(\alpha_r, \alpha_i)) \det(C(\alpha_j, \alpha_i))^{-1},$$

where  $\delta_{i,j,r} = 1$  if  $(\alpha_j - \alpha_i)(\alpha_r - \alpha_i) > 0$  and  $\delta_{i,j,r} = -1$  otherwise. Thus, as suggested by the notation, the number  $s_i$  does not depend on  $j$ . So we get in fact  $k$  Descartes' rule of signs given by the choice of  $i$  in  $[k]$ . Recall that  $P_{\alpha_i} = (1, 0)$ . Thus  $P_{\alpha_r,2} = \det(P_{\alpha_i}, P_{\alpha_r})$  for any  $r \in [k]$ . Since  $\alpha$  is an ordering, we get either  $P_{\alpha_r,2} < 0$  for any  $r < \alpha_i$  and  $P_{\alpha_r,2} > 0$  for any  $r > \alpha_i$ , or  $P_{\alpha_r,2} > 0$  for any  $r < \alpha_i$  and  $P_{\alpha_r,2} < 0$  for any  $r > \alpha_i$ . Therefore,

$$s_i = \operatorname{sgnvar}(-\bar{\lambda}_0, \dots, -\bar{\lambda}_{i-1}, \bar{\lambda}_{i+1}, \dots, \bar{\lambda}_{k-1}) \quad i = 1, \dots, k-2,$$

$s_0 = \operatorname{sgnvar}(\bar{\lambda}_1, \dots, \bar{\lambda}_{k-1})$  and  $s_{k-1} = \operatorname{sgnvar}(-\bar{\lambda}_0, \dots, -\bar{\lambda}_{k-2})$ . So we are in the hypotheses of Lemma 4.4. Then, we can combine the  $k$  obtained bounds to get our bound (2.10).  $\square$

We end this section with the proof of Proposition 2.9.

*Proof of Proposition 2.9.* Since  $n_A(C) > 0$ , condition (1.5) is satisfied. We use the notations of the proof of Theorem 2.8 above.

Assume that  $d_{\alpha_r} \cdot \bar{\lambda}_r \neq 0$  for some  $r \in [k]$ . Then, since  $P^i$  and  $P^j$  give a basis of  $\ker(C)$ , either  $P_{\alpha_r,1} \cdot \bar{\lambda}_r \neq 0$  or  $P_{\alpha_r,2} \cdot \bar{\lambda}_r \neq 0$ . We may assume  $P_{\alpha_r,2} \cdot \bar{\lambda}_r \neq 0$ . Then the derivative of  $G$  over  $\Delta_P$  is not identically zero, thus  $g$  is not identically equal to 1 on  $\Delta_P$  and  $n_A(C)$  is finite. To show the other implication, assume on the contrary that  $d_{\alpha_r} \cdot \bar{\lambda}_r = 0$  for all  $r \in [k]$ . Then, the derivative of  $G$  over  $\Delta_P$  is identically zero, and thus  $g$  is constant on  $\Delta_P$ . But since  $n_A(C) > 0$ , there exists  $y \in \Delta_P$  such that  $g(y) = 1$ . Thus  $g$  is identically equal to 1 on  $\Delta_P$ , and  $n_A(C)$  is infinite.  $\square$

## 5. OPTIMALITY OF THE BOUNDS

We prove the optimality of our bound (2.10) in Theorem 2.8 in general, by exhibiting particular configurations  $A$  and associated coefficient matrices  $C$  for which the bound is attained. We keep the notations in the previous sections.

In [2], polynomial systems supported on a circuit in  $\mathbb{R}^n$  and having  $n+1$  positive solutions have been obtained with the help of real desins



d'enfants. In [11], the main tool for constructing such systems is the generalization of the Viro's patchworking theorem obtained in [15]. We will recall the construction of [11] since we will only have to slightly modify it in order to prove Theorem 5.1 for any value of the sign variation.

**Theorem 5.1.** *Let  $r$  and  $n$  be any integer numbers such that  $0 \leq r \leq n$  and  $n > 0$ .*

- (1) *There exist matrices  $A, C \in \mathbb{R}^{n \times (n+2)}$  and an ordering  $\alpha : [k] \rightarrow K$  of  $C$  such that*

$$\text{sgnvar}(\bar{\lambda}_{\alpha_0}, \dots, \bar{\lambda}_{\alpha_{k-1}}) = 1 + r \quad \text{and} \quad n_A(C) = 1 + r.$$

- (2) *For any positive integers  $a, b$  with  $a \leq b$  there exist matrices  $A, C \in \mathbb{R}^{n \times (n+2)}$  such that the set  $\mathcal{A} \subset \mathbb{R}^n$  consisting of the column vectors of  $A$  has signature  $(a, b)$  and*

$$n_A(C) = \begin{cases} 2a & \text{if } a \neq b \\ 2a - 1 & \text{if } a = b. \end{cases}$$

*Proof.* We first note that (2) follows easily from (1). In order to prove (1), let  $n \geq 2$  be any integer. The following system is considered in [11] :

$$(5.1) \quad \begin{aligned} x_1 x_2 &= \epsilon + x_1^2, \\ x_i x_{i+1} &= 1 + \epsilon^{2i-3} x_1^2, \quad i = 2, \dots, n-1, \\ x_n &= 1 + \epsilon^{2n-3} x_1^2. \end{aligned}$$

where  $\epsilon$  is a positive real parameter that will be taken small enough. The support of this system is the circuit  $\mathcal{A} = \{w_0, w_1, \dots, w_n, w_{n+1}\}$  with  $w_0 = (0, \dots, 0)$ ,  $w_i = e_i + e_{i+1}$  for  $i = 1, \dots, n-1$ ,  $w_n = e_n$  and  $w_{n+1} = 2e_1$ , where  $(e_1, \dots, e_n)$  stands for the canonical basis of  $\mathbb{R}^n$ . Let  $C$  be the associated matrix. A choice of Gale dual of  $C$  is given by the vectors  $P_0 = (1, 0)$ ,  $P_1 = (\epsilon, 1) = \epsilon(1, \epsilon^{-1})$ ,  $P_i = (1, \epsilon^{2i-3})$  for  $i = 2, \dots, n$  and  $P_{n+1} = (0, 1)$ . These points are depicted in Figure 1. It follows that the bijection  $\alpha : [n+2] \rightarrow [n+2]$  defined by  $\alpha_0 = 0$ ,  $\alpha_1 = n$ ,  $\alpha_2 = n-1, \dots, \alpha_n = 1$  and  $\alpha_{n+1} = n+1$  defines an ordering of  $C$ . We compute the affine relation

$$(5.2) \quad (-1)^n w_0 + 2w_1 - 2w_2 + 2w_3 + \dots + (-1)^{n-1} \cdot 2 \cdot w_n - w_{n+1} = 0,$$

which shows that  $\text{sgnvar}(s_\alpha) = \text{vol}(\mathcal{A}) = n+1$ . Therefore, the bound provided by Theorem 2.8 for the system (5.1) is the maximal one  $n+1$ . It is proved in [11] that this system has precisely  $n+1$  positive solutions for  $\epsilon$  small enough (moreover, they prove that  $\epsilon = 1/4$  is sufficiently small). Let us recall how the proof goes on (see [11] for more details). Denote by  $T_i$  the Newton polytope (a triangle) of

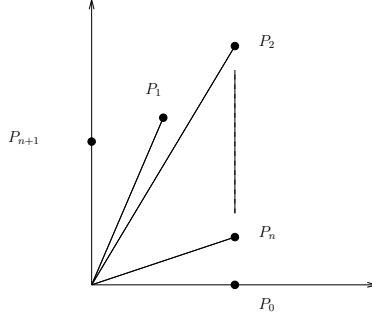


FIGURE 1

the  $i$ -th equation of (5.1) :  $T_i = [0, e_i + e_{i+1}, 2e_1]$  for  $i = 1, \dots, n - 1$  and  $T_n = [0, e_n, 2e_1]$  (see Figure 2). The exponents of  $\epsilon$  in the system (5.1) determine a convex mixed subdivision of the Minkowski sum  $T_1 + \dots + T_n$ . The mixed cells in this mixed subdivision are  $n$ -dimensional zonotopes  $Z_0, Z_1, \dots, Z_n$ . Each zonotope is a Minkowski sum of edges of  $T_1, \dots, T_n$ . For  $i = 1, \dots, n - 1$ , consider the edges of  $T_i$  defined by  $E_{i,0} = [0, e_i + e_{i+1}]$  and  $E_{i,1} = [2e_1, e_i + e_{i+1}]$ . Consider also the edges of  $T_n$  defined by  $E_{n,0} = [0, e_n]$  and  $E_{n,1} = [2e_1, e_n]$  (see Figure 2).



FIGURE 2

For  $j = 1, \dots, n - 1$  the zonotope  $Z_j$  is the Minkowski sum

$$Z_j = E_{1,1} + \dots + E_{j,1} + E_{j+1,0} + \dots + E_{n,0}.$$

Moreover,  $Z_0 = E_{1,0} + \dots + E_{n,0}$  and  $Z_n = E_{1,1} + \dots + E_{n,1}$ . To each zonotope  $Z_j$  corresponds a system of binomial equations obtained from (5.1) by keeping for the  $i$ -th equation its truncation to the  $i$ -th summand of  $Z_j$ , which is the edge  $E_{i,0}$  or  $E_{i,1}$  according as  $j < i$  or not. For instance, for  $j = 1, \dots, n - 1$ , the binomial system corresponding to  $Z_j$  is

$$(5.3) \quad x_1 x_2 = x_1^2, \dots, x_j x_{j+1} = \epsilon^{2j-3} x_1^2, x_{j+1} x_{j+2} = 1, \dots, x_n = 1.$$

The generalization of Viro's patchworking Theorem obtained in [15] gives that the number of positive solutions of the system (5.1) is the total number of positive solutions of the binomial systems corresponding to  $Z_0, \dots, Z_n$ . Moreover, the number of positive solutions of such a binomial system is at most one, and it is equal to one when in each binomial equation  $c_a x^a = c_b x^b$  we have  $c_a c_b > 0$ . Therefore, the total number of positive solutions of (5.1) for  $\epsilon > 0$  small enough can be read off the signs of the coefficients of the binomial systems corresponding to  $Z_0, \dots, Z_n$ . It follows then that (5.1) has exactly  $n + 1$  positive solutions for  $\epsilon$  small enough, which proves the optimality of item (1) of Theorem 5.1 when  $r = n$ .

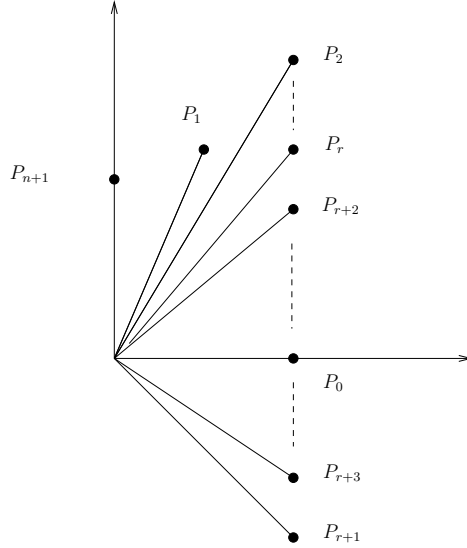


FIGURE 3

Assume now that  $0 \leq r \leq n - 1$ . To get a system with  $1 + r$  positive solutions from (5.1), we may multiply the term  $\epsilon^{2r-1} x_1^2$  of the  $(r + 1)$ -th equation by  $-1$  and keep unchanged the other equations. So, for  $r \leq n - 2$  this gives the system

$$(5.4) \quad \begin{aligned} x_1 x_2 &= \epsilon + x_1^2, \\ x_i x_{i+1} &= 1 + \epsilon^{2i-3} x_1^2, \quad i = 2, \dots, n-1, i \neq r+1, \\ x_{r+1} x_{r+2} &= 1 - \epsilon^{2r-1} x_1^2, \\ x_n &= 1 + \epsilon^{2n-3} x_1^2. \end{aligned}$$

while for  $r = n - 1$  it gives

$$(5.5) \quad \begin{aligned} x_1 x_2 &= \epsilon + x_1^2, \\ x_i x_{i+1} &= 1 + \epsilon^{2i-3} x_1^2, \quad i = 2, \dots, n-1 \\ x_n &= 1 - \epsilon^{2n-3} x_1^2. \end{aligned}$$

Then, for each zonotope  $Z_j$  having the edge  $E_{r+1,1}$  as a summand, the corresponding binomial system has no positive solution. Indeed, the  $(r+1)$ -th equation of this binomial system is  $x_{r+1} x_{r+2} = -\epsilon^{2r-1} x_1^2$  if  $r \leq n-2$ , or  $x_n = -\epsilon^{2n-3} x_1^2$  if  $r = n-1$ . On the other hand, binomial systems corresponding to zonotopes without the edge  $E_{r+1,1}$  as a summand are unchanged, and thus have still one positive solution. The zonotopes having  $E_{r+1,1}$  as a summand are all  $Z_j$  with  $j \geq r+1$ . Therefore, by Viro patchworking Theorem, for  $\epsilon > 0$  small enough the system (5.4) has  $r+1$  positive solutions (where  $r < n-1$ ) while the system (5.5) has  $n$  positive solutions. However, the bound  $\text{sgnvar}(s_\alpha)$  given by Theorem 2.8 for these systems will be bigger than  $1+r$  in general. To get a system with  $1+r$  positive solutions and for which  $\text{sgnvar}(s_\alpha) = 1+r$ , we proceed as follows. First, we note that if we multiply in (5.1) the term  $\epsilon^{2i-3} x_1^2$  of the  $i$ -th equation by  $-1$  for  $i = r+1$  and *any* other arbitrary values of  $i \geq r+1$ , then we still get a system with  $r+1$  positive solutions for  $\epsilon > 0$  small enough (by the Viro patchworking Theorem). Indeed, for such a system, we again find that binomial systems corresponding to zonotopes  $Z_j$  with  $j < r+1$  have each a positive solution while the other binomial systems have no positive solutions.

In order to get a system with  $r+1 = \text{sgnvar}(s_\alpha)$  positive solutions, we multiply in (5.1) the term  $\epsilon^{2i-3} x_1^2$  of the  $i$ -th equation (for  $i = 1$ , this term is  $x_1^2$ ) by  $-1$  for any  $i \geq r+1$  such that  $i$  and  $r+1$  have the same parity (and  $i \leq n$ ). If  $r \geq 1$ , this gives the system

$$(5.6) \quad \begin{aligned} x_1 x_2 &= \epsilon + x_1^2, \\ x_i x_{i+1} &= 1 + \epsilon^{2i-3} x_1^2, \quad i = 2, \dots, r, \\ x_i x_{i+1} &= 1 + (-1)^{i-r} \epsilon^{2i-3} x_1^2, \quad i = r+1, \dots, n-1, \\ x_n &= 1 + (-1)^{n-r} \epsilon^{2n-3} x_1^2. \end{aligned}$$

while for  $r = 0$  this gives

$$(5.7) \quad \begin{aligned} x_1 x_2 &= \epsilon - x_1^2, \\ x_i x_{i+1} &= 1 + (-1)^i \epsilon^{2i-3} x_1^2, \quad i = 2, \dots, n-1, \\ x_n &= 1 + (-1)^n \epsilon^{2n-3} x_1^2. \end{aligned}$$

Figure 3 depicts a choice of Gale dual  $P_0, \dots, P_{n+1}$  for the resulting matrix of coefficients  $C$ , from which we can read an associated ordering  $\alpha : [n+2] \rightarrow [n+2]$ , and it is straightforward to check that  $\text{sgnvar}(s_\alpha) = r+1$ .  $\square$

For a univariate real polynomial  $f(x) = c_0 + c_1x + \dots + c_r x^r$  with any number of monomials, the sign variation  $\text{sgnvar}(c_0, \dots, c_r)$  is always bounded above by the degree of the polynomial divided by the index of the subgroup of  $\mathbb{Z}$  generated by the exponents. This is no longer true in arbitrary dimensions, in particular in the case where the exponent set  $\mathcal{A}$  is a circuit. We present two examples of particular configurations.

**Example 5.2.** *When  $\text{vol}(\mathcal{A}) \geq n+1$ , the bound we give in 2.9 reduces to the sign variation of the ordered sequence of  $\lambda_j$ 's. But, for instance in the case in which  $\mathcal{A} = \{(0,0), (1,0), (0,1), (1,1)\} \subset \mathbb{Z}^2$ , the normalized volume equals 2 (and  $I = 1$ ), while we can get matrices  $C$  with  $k = 4$  and with an ordering such that the sign variation equals 3. The sign variation of the corresponding ordered sequence of  $\lambda_j$ 's is an upper bound, which is not sharp in this case, but the minimum is indeed a sharp upper bound, since the two complex roots can be real positive.*

**Example 5.3.** *Let  $n = 4$  and let  $\mathcal{A}$  be a circuit with affine relation  $(1, -1, 3, -3, 1, -1)$ . Then, the maximum possible bound (2.10) in Theorem 2.8 for a coefficient matrix  $C$  and support  $\mathcal{A}$  is equal to  $5 = \text{vol}(\mathcal{A}) = n+1$ . However, Theorem 0.2 in [2] shows that there cannot be  $\text{vol}(\mathcal{A})$  positive real solutions to this system, so the bound we give is not sharp in that case.*

We end by mentioning work in progress inspired by the present paper. Theorem 3.3 gives a necessary condition on a circuit  $\mathcal{A} \subset \mathbb{Z}^n$  for the existence of a coefficient matrix  $C$  such that  $n_A(C)$  reaches the maximal possible value  $n+1$ . Recently, Boulos El Hilany [6] has obtained a sufficient condition on  $\mathcal{A}$  for this to hold. His characterization of circuits supporting systems having the maximal number of positive solutions involves inequalities between sums of coefficients of any given affine relation on  $\mathcal{A}$ . The first author of the present paper has obtained a partial generalization of Descartes' rule of signs for any polynomial system [3]. The obtained bound is partial in that it is a sum of two terms: a sign variation plus a quantity depending on the number of variables and the total number of monomials of the system.

## REFERENCES

- [1] F. Bihan, *Polynomial systems supported on circuits and dessins d'enfant*, J. Lond. Math. Soc. **75**, no. 1 (2007), 116–132.

- [2] F. Bihan, *Maximally positive polynomial systems supported on circuits*, J. Symbolic Comput. **68** (2015) 61–74.
- [3] F. Bihan, *Towards a generalized Descartes’ rule of signs for multivariate polynomial systems*, in preparation.
- [4] B. Bertrand, F. Bihan, and F. Sottile, *Polynomial systems with few real zeroes*, Math. Z. **253**, no. 2 (2006), 361–385.
- [5] F. Bihan and F. Sottile, *Gale duality for complete intersection*, Ann. Inst. Fourier **58**, no. 3 (2008), 877–891.
- [6] B. El Hilany, *Characterization of circuits supporting polynomial systems with the maximal number of positive solutions*, in preparation.
- [7] I. M. Gelfand, M. M. Kapranov, A. V. Zelevinsky, *Discriminants, resultants, and multidimensional determinants*, Mathematics: Theory & Applications. Birkhauser Boston, Inc., Boston, MA, (1994).
- [8] I. Itenberg and M. F. Roy, *Multivariate Descartes’ rule*, Beitr. Algebra Geom. **37**, no. 2 (1996), 337–346.
- [9] A. G. Khovanskii, *Fewnomials*, Translations of Mathematical Monographs, vol. 88, American Mathematical Society, Providence, RI, 1991.
- [10] S. Mueller, E. Feliu, G. Regensburger, C. Conradi, A. Shiu, A. Dickenstein, *Sign conditions for injectivity of generalized polynomial maps with applications to chemical reaction networks and real algebraic geometry*, to appear: FoCM Journal, 2016.
- [11] K. Phillipson, J.-M. Rojas, *Fewnomial systems with Many Roots, and an Adelic Tau Conjecture*, Contemp. Math. **605** (2013), 45–71.
- [12] G. Pólya and G. Szegő, *Problems and theorems in analysis, Vol. II*, Springer-Verlag, New York, 1976.
- [13] Frank Sottile, *Real solutions to equations from geometry*, University Lecture Series, vol. 57, American Mathematical Society, Providence, RI, 2011.
- [14] D. J. Struik (ed.), *A source book in mathematics, 1200-1800*, Source Books in the History of the Sciences. Cambridge, Mass.: Harvard University Press, XIV, 427 p., 1969.
- [15] B. Sturmfels, *Viro’s theorem for complete intersections*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **4**, vol. 21, no. 3 (1994), 377–386.

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