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BETTI NUMBER BOUNDS FOR FEWNOMIAL HYPERSURFACES VIA STRATIFIED MORSE THEORY

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Abstract. We use stratified Morse theory for a manifold with corners to give a new bound for the sum of the Betti numbers of a hypersurface in $\mathbb{R}^n_>$ defined by a polynomial with $n+l+1$ terms.

Let $f(x_1,\ldots,x_n)$ be a Laurent polynomial with $n+l+1$ monomial terms such that $f(x)=0$ defines a smooth manifold X in the positive orthant $\mathbb{R}^n_>$. Khovanskii [5] showed that the sum of the Betti numbers of X is at most

$$(2n^2-n+1)^{n+l}(2n)^{n-1}2^{\binom{n+l}{2}}\,. \tag{1}$$

We use the new upper bound [2] of $\frac{e^2+3}{4}2^{\binom{l}{2}}n^l$ on the number of nondegenerate positive solutions to a system of n polynomials in n variables having $n+l+1$ monomial terms, together with stratified Morse theory for a manifold with corners [4] to give a new bound for the sum of the Betti numbers of X. Fix positive integers $N\geq n$ and l.

Theorem 1. *The sum of the Betti numbers of a hypersurface in $\mathbb{R}^N_>$ defined by a polynomial with $n+l+1$ monomials whose exponent vectors have affine span of dimension n is at most*

$$\frac{e^2+3}{4}2^{\binom{l}{2}}\cdot\sum_{i=0}^{n}\binom{n}{i}i^l\ .$$

The bound of Theorem 1 is bounded above by the simpler expression

$$(e^2+3)2^{\binom{l}{2}}\,n^l\cdot 2^{n-3}\,, \tag{2}$$

which is smaller than the bound on the number of connected components (zeroth Betti number) proven in [1]. The new ingredient here is the use of Morse theory for a manifold with corners.

Khovanskii's bound (1) is a special case of more general bounds which he obtains for fewnomial complete intersections. It remains an important open problem to adapt the new methods of [2] to fewnomial complete intersections.

We remark that our results hold for polynomials with real-number exponents. In fact, our proofs (and the proofs in [2]) involve this added generality.

We first describe Morse theory for a manifold with corners in Section 1 and prove Theorem 1 in Section 2.

1. Morse theory for a manifold with corners

In classical Morse theory, the topology of a compact differentiable manifold X is inferred from critical points of a sufficiently general smooth function $f\colon X\to\mathbb{R}$, called a Morse

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function. For example, the sum of the Betti numbers of X is bounded above by the number of critical points of any Morse function. Goresky and MacPherson [4] develop a version of Morse theory for stratified spaces. This is particularly simple for manifolds with corners, which for us will be the intersection of our fewnomial hypersurface with a large simplex whose every face meets it transversally.

We begin with sketches of classical and of stratified Morse theory, and then explain how stratified Morse theory applies to a manifold with corners. For further discussion and proofs see [4].

A smooth function $f\colon X \to \mathbb{R}$ on a compact differential manifold is a *Morse function* if its critical values (in $\mathbb{R}$) are distinct and each critical point (in X) of f is nondegenerate (the Hessian matrix of second partial derivatives has full rank). This implies that the critical points are discrete and there are finitely many of them. For each $c \in \mathbb{R}$, set $X_c := f^{-1}(-\infty, c])$. If c is smaller than any critical value, then X_c is empty, and if c is greater than all critical values, then $X = X_c$. The Morse Lemmata describe how the topological type of X_c changes as c increases from $-\infty$ to ∞. The first Morse Lemma asserts that the topological type of X_c is constant for all c lying in an open interval that contains no critical values. The second Morse Lemma asserts that if (a, b) contains a unique critical value $c = f(p)$, then the pair (X_b, X_a) is homeomorphic to the pair $(D^\lambda \times D^{n-\lambda}, (\partial D^\lambda) \times D^{n-\lambda})$. Here, X has dimension n, D^m is a closed disc of dimension m, and λ is the number of negative eigenvalues of the Hessian matrix of f at p. The long exact sequence of a pair and induction on the critical values c shows that the sum of the Betti numbers of X is bounded above by the number of critical points of f.

Suppose now that X is a Whitney stratified space, which we assume is embedded in an ambient manifold M. A smooth function $f\colon X \to \mathbb{R}$ is the restriction to X of a smooth function on M. A *critical point* of f is a critical point of the restriction of f to any stratum. (Each stratum in a Whitney stratified space is a manifold.) A *Morse function* $f\colon X \to \mathbb{R}$ is a smooth function on X whose critical values are distinct, and at each critical point p of F, the restriction of f to the stratum containing p is is nondegenerate at p. There is a third condition that the differential of f at p does not annihilate any limit of tangent spaces to any stratum other than the stratum containing p.

In stratified Morse theory, the first Morse Lemma holds as before and the second Morse Lemma is modified as follows. Let p be a critical point of f lying in a stratum S of X. Then let $D(p)$ be a small disk in M transversal to the stratum S such that $D(p) \cap S = \{p\}$ and call its intersection $N(p)$ with X the *normal slice* to S at p. *Normal Morse data* at p are a pair (A, B), where A is the set of points x in $N(p)$ for which $|f(x) - f(p)| \leq \epsilon$ and B is that part of the boundary of A where $f(x) = f(p) - \epsilon$, and ϵ is any sufficiently small positive number. The tangential Morse data at p are the pair $(D^\lambda \times D^{n-\lambda}, (\partial D^\lambda) \times D^{n-\lambda})$ appearing in the second Morse Lemma for the Morse function f restricted to the stratum S, which is a manifold of dimension n. The second Morse Lemma in stratified Morse theory asserts that if the interval (a, b) contains a unique critical value $c = f(p)$, then the pair (X_b, X_a) is homeomorphic the product of pairs

$$\text{(normal Morse data at } p) \times \text{(tangential Morse data at } p)\,. \tag{3}$$

In general, we must have detailed information about the interaction between the Morse function and the stratification to use stratified Morse theory.

Such detailed information is available for manifolds with corners. Let $\mathbb{R}_\geq := [0, \infty)$ be the non-negative real numbers and $\mathbb{R}^m_\geq$ be the nonnegative orthant in $\mathbb{R}^m$. A *manifold with* *corners* is a compact topological space X with a covering by charts, each homeomorphic to $\mathbb{R}^m_\geq \times \mathbb{R}^n$, where $n+m$ is the dimension of X. Each point of X has a well-defined tangent space isomorphic to $\mathbb{R}^{n+m}$, *and* has a *tangent cone* isomorphic to $\mathbb{R}^m_\geq \times \mathbb{R}^n$. Points with tangent cone isomorphic to $\mathbb{R}^m_\geq \times \mathbb{R}^n$ for m fixed form a submanifold of dimension n. All such submanifolds form the *boundary strata* of X.

Suppose that $f\colon X \to \mathbb{R}$ is a Morse function. The third condition on Morse functions requires that the differential of f at a critical point p in a boundary stratum (locally homeomorphic to $\mathbb{R}^m_\geq \times \mathbb{R}^n$) has rank m—while it annihilates the tangent space to the stratum (the second factor), it is injective on the tangent space to the first factor, which is the normal bundle to the stratum. The normal slice $N(p)$ to such a point is a neighborhood of the origin in the cone $\mathbb{R}^m_\geq$ and there are exactly two possibilities for the normal Morse data (A,B) at p. The first component A may be taken to be the m-simplex

$$\Delta_m \;:=\; \{x \in \mathbb{R}^m_\geq \mid |x| := x_1 + \cdots + x_m \leq 1\} \quad (\simeq\; D^m)$$

and the second component B is either the empty set $\emptyset$ (if $f(p)$ is locally the minimum value of f on $N(p)$) or the $(m-1)$-simplex where $|x| = 1$ (if $f(p)$ is locally the maximum value of f on $N(p)$). We illustrate this in Figure 1.

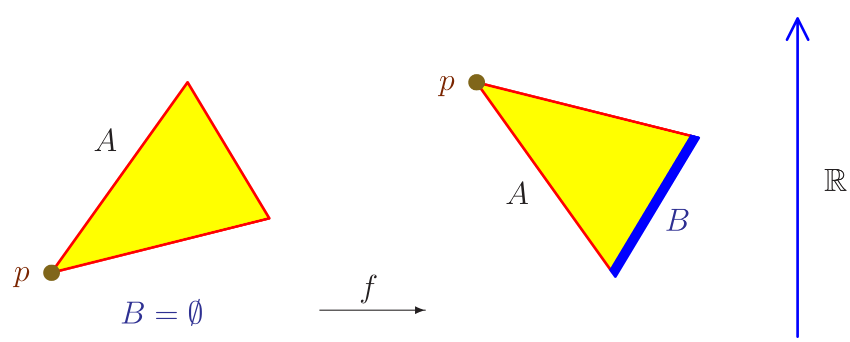

Figure 1. Normal Morse data

Observe that in the case when $B = \emptyset$, we have

$$(A,B) \times (D^\lambda \times D^{n-\lambda}, (\partial D^\lambda) \times D^{n-\lambda}) \;\simeq\; (D^\lambda \times D^{m+n-\lambda}, (\partial D^\lambda) \times D^{m+n-\lambda})$$

Thus this critical point could contribute to the sum of the Betti numbers of X. On the other hand, when $B = \Delta_{n-1}$, the pair (B,B) is a deformation retract of (A,B), and so the critical point p does not contribute to the sum of the Betti numbers of X, as the topology of X_a does not change as a passes $f(p)$.

Proposition 2. *The sum of the Betti numbers of a manifold Z with corners is at most the number of critical points p of a Morse function f for Z where the minimum of f on the normal slice to Z at p is attained at p.*

Example 3. Proposition 2 is illustrated by the Cannoli shell, which is a manifold with corners. It is a cylinder $D^1 \times S^1$ having two boundary strata, each homeomorphic to the circle S^1. The height function of Figure 2 is a Morse function for the Cannoli shell with four critical points. Only the two smallest critical values contribute to its topology and its Betti numbers, by Proposition 2.

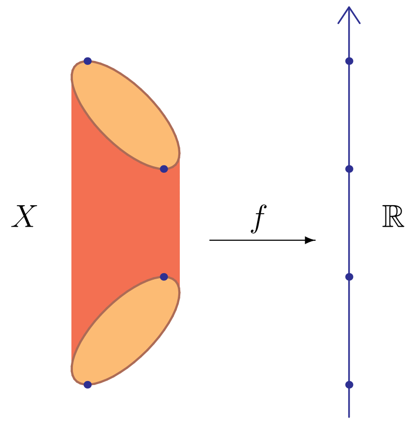

Figure 2. Morse function on the Cannoli shell

2. Proof of Theorem 1

Let $f(x_1,\dots,x_N)$ be a real Laurent polynomial with $n{+}l{+}1$ exponent vectors that span an n-dimensional affine subspace of $\mathbb{R}^N$ such that $X := \mathcal{V}(f) \subset \mathbb{R}^N_{\geq}$ is a smooth hypersurface. Rather than work with polynomials in $\mathbb{R}^N_{\geq}$, we work with exponential sums in $\mathbb{R}^N$. These notions are related via a logarithmic change of coordinates. Consider the isomorphisms of Lie groups.

$$\begin{array}{rcl} \mathrm{Exp}\colon\ \mathbb{R}^N & \longrightarrow & \mathbb{R}^N_{\geq} \\ (z_1,\dots,z_N) & \longmapsto & (e^{z_1},\dots,e^{z_N}) \end{array} \qquad \begin{array}{rcl} \mathrm{Log}\colon\ \mathbb{R}^N_{\geq} & \longrightarrow & \mathbb{R}^N \\ (x_1,\dots,x_N) & \longmapsto & (\log(x_1),\dots,\log(x_N)) \end{array}$$

Under this isomorphism monomials x^α correspond to exponentials $e^{z\cdot\alpha}$, and so the fewnomial $f=\sum_i c_i x^{\alpha_i}$ corresponds to the exponential sum

$$\varphi \ := \ \sum_{i=0}^{n+l} c_i e^{z\cdot\alpha_i} \ .$$

Let $Z := \mathcal{V}(\varphi) \subset \mathbb{R}^N$ be the hypersurface defined by φ, which is homeomorphic to X. For exponential sums, it is quite natural to allow the exponents α_i to be real vectors. We will prove Theorem 1 in these logarithmic coordinates and with real exponents.

Theorem 1′. *The sum of the Betti numbers of a hypersurface in $\mathbb{R}^N$ defined by an exponential sum with $n{+}l{+}1$ terms whose exponent vectors have affine span of dimension n is at most*

$$\frac{e^2+3}{4} 2^{\binom{l}{2}} \cdot \sum_{i=0}^{n} \tbinom{n}{i} i^l \ .$$

Multiplying φ by $e^{-z\cdot\alpha_0}$, we may assume that $\alpha_0=0$. Then $\alpha_1,\dots,\alpha_{n+l}$ span an n-dimensional linear subspace of $\mathbb{R}^N$. After a linear change of coordinates, we may assume that φ only involves the first n variables, and thus the hypersurface Z becomes a cylinder

$$Z \ \simeq \ \{z\in\mathbb{R}^n \mid \varphi(z)=0\} \times \mathbb{R}^{N-n} \, .$$

Thus it suffices to prove Theorem 1′ when $N=n$.

Since the exponents $\alpha_1,\dots,\alpha_{n+l}$ span $\mathbb{R}^n$, we may assume that the first n are the standard unit basis vectors in $\mathbb{R}^n$, and thus φ includes the coordinate exponentials e^{z_i} for

$i = 1, \ldots, n$. Let $M := (M_0, M_1, \ldots, M_n)$ be positive numbers and set

$$\Delta_M \ := \ \{z \in \mathbb{R}^n \mid z_i \geq -M_i,\ i = 1, \ldots, n \quad \text{and} \quad \sum_i z_i \leq M_0\}\,,$$

which is a non-empty simplex. We will use stratified Morse Theory to bound the Betti numbers of $Y := Z \cap \Delta_M$ when M is general.

Theorem 4. *For M general, the sum of the Betti numbers of Y is at most*

$$\frac{e^2+3}{4} 2^{\binom{l}{2}} \cdot \sum_{i=0}^{n} \tbinom{n}{i} i^l\ .$$

Proof of Theorem 1′. For any $r > 0$, set

$$Z_r \ := \ \{z \in Z \mid \|z\| < r\}\,.$$

By [3, corollary 9.3.7], there is some $R > 0$ such that if $r \geq R$ then Z_r is deformation retract of Z and Z_R is a deformation retract of Z_r.

Choose $r > R$ so that Δ_M is sandwiched between the balls of radius R and r centered at the origin. Let $\rho\colon Z_r \to Z_R$ be the retraction. We have the maps

$$Z_R \ \hookrightarrow \ Y = Z \cap \Delta_M \ \hookrightarrow \ Z_r \ \xrightarrow{\ \rho\ } \ Z_R\,,$$

whose composition is the identity. The induced maps on the ith homology groups,

$$H_i(Z_R) \ \longrightarrow \ H_i(Y) \ \longrightarrow \ H_i(Z_r) \ \xrightarrow{\ \rho_*\ } \ H_i(Z_R)\,,$$

have composition the identity. This gives the inequality

$$\dim H_i(Y) \ \geq \ \dim H_i(Z_R) \ = \ \dim H_i(Z)\,.$$

Summing over i shows that Theorem 1′ is a consequence of Theorem 4. □

Proof of Theorem 4. Given positive numbers $M = (M_0, M_1, \ldots, M_n)$, define affine hyperplanes in $\mathbb{R}^n$

$$H_0 \ := \ \{z \mid \textstyle\sum_i z_i = M_0\} \qquad \text{and} \qquad H_i \ := \ \{z \mid z_i = -M_i\}\ i = 1, \ldots, n\,.$$

For each proper subset $S \subset \{0, \ldots, n\}$, define an affine linear subspace

$$H_S \ := \ \bigcap_{i \in S} H_i\,.$$

Since each $M_i > 0$, this has dimension $n - |S|$, and these subspaces are the affine linear subspaces supporting the faces of the simplex Δ_M.

Choose M generic so that for all S the subspace H_S meets Z transversally. For each S, set $Z_S := Z \cap H_S$. This is a smooth hypersurface in H_S and therefore has dimension $n - |S| - 1$. The boundary stratum Y_S of $Y = Z \cap \Delta_M$ lying in the face supported by H_S is an open subset of Z_S.

For a non-zero vector $u \in \mathbb{R}^n$, the directional derivative $D_u\varphi$ is

$$\sum_{i=1}^{n+l} (u \cdot \alpha_i) c_i u e^{z \cdot \alpha_i}\ ,$$

which is an exponential sum having the same exponents as φ. Let L_u be the linear function on $\mathbb{R}^n$ defined by $z \mapsto u \cdot z$.

The critical points of the function L_u restricted to Z are the zeroes of the system

$$\varphi(z) \ = \ 0 \qquad \text{and} \qquad D_v\varphi(z) \ = \ 0 \quad \text{for } v \in u^\perp\,.$$

When u is general and we choose a basis for $u^\perp$, this is a system of n exponential sums in n variables, all involving the same $n+l+1$ exponents. By the fewnomial bound in [2], the number of solutions is at most

$$\frac{e^2+3}{4}2^{\binom{l}{2}}n^l\,.$$

We use this to estimate the number of critical points of the function L_u restricted to Z_S. The restriction of φ to H_S defines Z_S as a hypersurface in H_S. We determine this restriction. If $i\in S$ with $i>0$ then we may use the equation $z_i=-M_i$ to eliminate the variable z_i and the exponential e^{z_i} from φ. If $0\in S$, then we choose $j\notin S$ and use the equation $\sum_i z_i=M_0$ to eliminate z_j from φ. Let φ_S be the result of this elimination. It is an exponential sum in $n-|S|$ variables and its number of terms is at most

$$\begin{aligned}(n-|S|)+l+1 &\quad \text{if } 0\notin S\\ (n-|S|)+(l+1)+1 &\quad \text{if } 0\in S\end{aligned}$$

Thus if u is a general vector in $\mathbb{R}^n$, then the number of critical points of the linear function $L_u|_{H_S}$ on Z_S is at most

$$\begin{aligned}\tfrac{e^2+3}{4}2^{\binom{l}{2}}(n-|S|)^l &\quad \text{if } 0\notin S\\ \tfrac{e^2+3}{4}2^{\binom{l+1}{2}}(n-|S|)^{l+1} &\quad \text{if } 0\in S\end{aligned}$$

We use this estimate and stratified Morse theory to bound the Betti numbers of Y.

Let u be a general vector in $\mathbb{R}^n$ such that L_u is a Morse function for the stratified space Y. By Proposition 2, the sum of the Betti numbers of Y is bounded by the number of critical points p of L_u for which L_u achieves its minimum on the normal slice $N(p)$ at p. Since the strata Y_S of Y are open subsets of the manifolds Z_S, this number is bounded above by the number of such critical points of L_u on the manifolds Z_S. We argue that we can alter u so that no critical point in any Z_S with $0\in S$ contributes.

Suppose that p is a critical point of L_u on a stratum Z_S with $0\in S$. Then p lies in H_0, and so the linear function $L_{(1,\ldots,1)}$ restricted to the normal slice $N(p)$ at p takes its maximum value M_0 at p. If we replace u by $u+\lambda(1,\ldots,1)$ we change L_u on H_S by the constant λM_0 and p will still be a critical point for $L_{u+\lambda(1,\ldots,1)}$ on Z_S. If λ is sufficiently large, then $L_{u+\lambda(1,\ldots,1)}$ will achieve its maximum value on $N(p)$ at p.

There are finitely many critical points p of L_u on strata Z_S with $0\in S$. Hence, there is a positive number λ so that at each of these critical points p, $L_{u+\lambda(1,\ldots,1)}$ will achieve its maximum value on $N(p)$ at p. We may further choose λ so that $L_{u+\lambda(1,\ldots,1)}$ is a Morse function for the stratified space Y. Since only the critical points on strata Z_S with $0\notin S$ can contribute to the Betti numbers of Y, we see that its sum of Betti numbers is bounded above by

$$\frac{e^2+3}{4}2^{\binom{l}{2}}\sum_{S\subset\{1,\ldots,n\}}(n-|S|)^l \;=\; \frac{e^2+3}{4}2^{\binom{l}{2}}\sum_{i=0}^{n}\binom{n}{i}(n-i)^l\,.$$

Since $\binom{n}{i}=\binom{n}{n-i}$, we replace i by $n-i$ to complete the proof of Theorem 4. □

We deduce the formula (2) from Theorem 1. Observe that

$$\sum_{i=0}^{n}\binom{n}{i}i^l \;=\; n^l\,\sum_{i=0}^{n}\binom{n}{i}\left(\frac{i}{n}\right)^l\,,$$

and thus the sum is a decreasing function of l. When $l = 1$, this sum is

$$\sum_{i=0}^{n} \binom{n}{i} \frac{i}{n} \;=\; \sum_{i=1}^{n} \binom{n-1}{i-1} \;=\; 2^{n-1}\,.$$

Substituting this into the formula of Theorem 1 gives (2).

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