Intersection multiplicity numbers between tropical hypersurfaces

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Abstract. We present several formulas for the intersection multiplicity numbers considered in [**3**]. These numbers are associated to any dimensional cell of the common intersection of the tropical hypersurfaces. The first formula involves a sum of generalized mixed volumes, and turns out to be equal to the absolute value of the Euler characteristic of some complex toric complete intersection. Another formula uses an alternating sum of volumes of Cayley polytopes.

Introduction

Tropical geometry is a recent field of mathematics which has attracted researchers from various other fields like algebraic geometry, real algebraic geometry, combinatorial geometry, to cite only few of them. We refer to [**13**], [**8**], [**17**] and [**6**] for general papers on the subject. Intersection theory is one of the most fundamental theory in classical algebraic geometry and it is natural to try to build such a theory in tropical geometry. Several works have been done in this direction, see for instance, [**13**], [**17**], [**10**], [**18**], [**1**], [**14**], [**15**], [**16**], [**11**].

In the present paper, we consider a finite number of tropical hypersurfaces in $\mathbb{R}^n$ and give new formulas for the intersection multiplicity numbers defined in [**3**]. A tropical polynomial in n variables is a polynomial in the usual sense but with the addition and multiplication of the tropical semiring. The corner locus of the resulting convex piecewise-linear function on $\mathbb{R}^n$ is the tropical hypersurface defined by the tropical polynomial. This is a piecewise-linear polyhedral complex of dimension $n-1$ whose pieces together with the closures of the connected components of the complementary part form a subdivision of the ambient space $\mathbb{R}^n$, whose elements are called cells. This subdivision is in one-to-one correspondence with a convex polyhedral subdivision of the Newton polytope of the tropical hypersurface. This correspondence, also called duality, sends a cell to a polytope of complementary dimension and lying on a orthogonal space in the dual space. It can be seen as a counterpart of the classical duality between the faces of an integral polytope and the cones in its normal fan (see sections 3 and 4). The tropical cycle associated with a tropical hypersurface in $\mathbb{R}^n$ is the weighted $(n-1)$-dimensional piecewise-linear polyhedral complex given by the tropical hypersurface together with weights on its

2010 *Mathematics Subject Classification.* 14T05.

top-dimensional cells which are the integer lenghts of the dual edges. The common intersection of a finite number of tropical hypersurfaces $Z_1, \dots, Z_k \subset \mathbb{R}^n$ is a piecewise-linear polyhedral complex whose cells ξ are common intersections of cells $\xi_1, \dots, \xi_k$ of the tropical hypersurfaces. The associated tropical intersection cycle (also called stable tropical intersection cycle) is the weighted $(n-k)$-dimensional piecewise-linear polyhedral complex given by the union of the $(n-k)$-dimensional cells ξ equipped with weights defined as follows (see Definition 5.2). If the cell is a transversal intersection of top-dimensional cells of the hypersurfaces, then the weight is the product of the weights of these cells scaled by some lattice index. In the general case, the weight is defined as the sum of the weights of all transversal intersection cells emerging from the given cell after small generic translations of the tropical hypersurfaces. This comes from a more general definition of (stable) tropical intersection cycle given in [**13**] (see also [**17**]). In our case, it is well-known that the weight of a $(n-k)$-dimensional cell ξ of the tropical intersection cycle associated with k tropical hypersurfaces in $\mathbb{R}^n$ is equal to the k-dimensional mixed volume $MV_k(\sigma_1, \dots, \sigma_k)$ of the polytopes $\sigma_1, \dots, \sigma_k$ which are dual to $\xi_1, \dots, \xi_k$, respectively.

In [**3**] the previous definition of $w(\xi)$ was extended for cells ξ of any codimension $d \geq k$ by dropping out the conditions on the dimensions of the intersecting cells and equipping any cell of a single tropical hypersurface with the normalized volume of its dual polytope (see Definition 5.8). In the present paper, we give another equivalent definition of $w(\xi)$ as a sum of classical weights of cells of tropical intersection cycles. Namely, $w(\xi)$ is defined as the sum over all collections $\underline{t}$ of positive integers $t_1, \dots, t_k$ suming up to d of the classical weight of ξ seen as a $(n-d)$-cell of the intersection cycle of the d tropical hypersurfaces obtained by taking t_i copies of Z_i for $i = 1, \dots, k$ (see Definition 5.5). In particular, when $k = 1$ there is only one partition and $w(\xi)$ is the classical weight of ξ seen as a cell of the cycle given by the d-fold intersection of the tropical hypersurface. This gives $w(\xi) = MV_d(\sigma, \dots, \sigma) = \mathrm{vol}_d(\sigma)$ (the suitable normalized volume of σ), where σ is the dual d-polytope.

We present three formulas for $w(\xi)$. The following one was obtained in [**3**], it is a direct consequence of Definition 5.5:

$$w(\xi) = \sum_{\underline{t}=(t_1,\dots,t_k)} MV_d(\sigma_1, \dots, \sigma_k; \underline{t}), \tag{0.1}$$

where the sum is over all collections $(t_1, \dots, t_k)$ as above and $MV_d(\sigma_1, \dots, \sigma_k; \underline{t})$ is the mixed volume, with respect to the lattice of rank d formed of all integer vectors of $\mathbb{R}^n$ parallel to σ, of the d polytopes obtained by taking t_i copies of σ_i (see Theorem 5.7). It is well-known that the normalized volume of a polytope P coincides up to a sign with the Euler characteristic of any non degenerate toric complex hypersurface with Newton polytope P. This provides a geometric interpretation for the weights of the cells of a single tropical hypersurface, which has the following generalization:

$$w(\xi) = (-1)^{d-k} \chi(\{f_1 = \dots = f_k = 0\}), \tag{0.2}$$

where $\chi(\{f_1 = \dots = f_k = 0\})$ is the Euler characteristic of a non degenerate toric complex complete intersection with Newton polytopes $\sigma_1, \dots, \sigma_k$ (see Theorem 5.9). We also prove the following formula involving suitable normalized volumes of Cayley polytopes (see Theorem 6.2):

$$w(\xi) = \sum_{\emptyset \neq I \subset \{1,\dots,k\}} (-1)^{k-|I|} \mathrm{vol}_{d+|I|-1}\left(C(\sigma_i, i \in I)\right). \tag{0.3}$$

Here $C(\sigma_i, i \in I)$ is the Cayley polytope of the polytopes $\sigma_i \subset \mathbb{R}^n$ for $i \in I$: this is the convex hull of all the points $(w_i, e_i) \in \mathbb{R}^n \times \mathbb{R}^k$ with $w_i \in \sigma_i$ and where e_i is the i-th vector in the standard basis of $\mathbb{R}^k$.

A complex polynomial is non degenerate if it defines a nonsingular hypersurface in the complex torus, and if all the truncations of the polynomial to the faces of its Newton polytope have the same property. A collection of complex polynomials is non degenerate if the corresponding Cayley polynomial is non degenerate in the previous sense. We show in Proposition 2.2 that this is equivalent to the fact that any admissible sub-collection of polynomials defines a complete intersection in the corresponding torus (see Section 2 for precise definitions). A tropical hypersurface in $\mathbb{R}^n$ is nonsingular if the dual subdivision of its Newton polytope is a so-called unimodular (or primitive) triangulation. This allows us to define the notion of a nondegenerate collection of tropical polynomials as in the complex case. In analogy with Proposition 2.2, we show in Proposition 7.4 that a collection of tropical polynomials is nondegenerate if and only if any admissible sub-collection of tropical polynomials defines tropical hypersurfaces intersecting with multiplicity numbers 1. This justifies to call nondegenerate tropical complete intersection a tropical variety defined by a nondegenerate collection of tropical polynomials.

A part of the present paper comes from [**3**] which appeared on Arxiv in 2007 and in which the principal motivation was to extend a previous result of the first author from the hypersurface case to the case of complete intersection (see [**2**]). Precisely, the main objective in [**3**] was to prove that the Euler characteristic of a real nondegenerate tropical complete intersection is equal to the mixed signature of a corresponding complex non degenerate complete intersection. In order to get Proposition 7.4, we needed to extend the intersection multiplicity numbers given in [**13**] in order to associate intersection multiplicity numbers to any dimensional cell of a common intersection of tropical hypersurfaces. We think that the intersection multiplicity numbers we defined in [**3**] are of independent interest and this has motivated us to write a separate paper. Comparatively to [**3**], we have added Definition 5.5, which gives a new interpretation of our weight as a sum of classical intersection numbers, and proved the new Formulas (0.2) and (0.3).

This paper is organized as follows. In the first sections, we give basic notions of toric geometry and recall known results of tropical geometry. In Section 5, we give the definitions of intersection multiplicity numbers and prove formulas (0.1) and (0.2). Section 6 is devoted to Formula (0.3) while the last section is concerned with nondegenerate tropical complete intersections.

We thank the referee for useful comments which helped us improve the exposition of this paper.

1. Toric hypersurfaces

We fix some notations and recall some standard properties of toric geometry. We refer to [**5**] for more details. Let $N \simeq \mathbb{Z}^n$ be a lattice of rank n and $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ be its dual lattice. The associated complex torus is $\mathbb{T}_N := \mathrm{Spec}(\mathbb{C}[M]) = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{C}^*) = N \otimes_{\mathbb{Z}} \mathbb{C}^* \simeq (\mathbb{C}^*)^n$. Let $f \in \mathbb{C}[M]$ be a Laurent polynomial in the group algebra associated with M

$$f(x) = \sum c_m x^m,$$

where each m belongs to M and only a finite number of c_m are non zero. We will usually have $M = \mathbb{Z}^n$, so that $\mathbb{C}[M] = \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. The *support* of f is the subset of M consisting of all m such that the coefficient c_m is non zero. The convex hull of this support in the real affine space generated by M is called *the Newton polytope* of f. This is a *lattice* polytope, or a polytope *with integer vertices*, which means that all the vertices of Δ belong to M. In this paper all polytopes will be lattice polytopes and the ambient lattice M will be clear from the context. We denote by $M(\Delta)$ the saturated sublattice of M which consists of all integer vectors parallel to Δ and by $N(\Delta)$ the dual lattice. The dimension of Δ is the rank of $M(\Delta)$, or equivalently the dimension of the real vector space $M(\Delta)_{\mathbb{R}}$ generated by Δ. The polynomial f (or rather $x^{-m}f \in \mathbb{C}[M(\Delta)]$ for any choice of m in the support of f) defines an hypersurface Z_f in the torus $\mathbb{T}_{N(\Delta)}$. Let X_Δ denote the projective toric variety associated with Δ. The variety X_Δ contains $\mathbb{T}_{N(\Delta)}$ as a dense Zarisky open subset and we denote by $\bar{Z}_f$ the Zarisky closure of Z_f in X_Δ. Let Γ be any face of Δ. If f^Γ is *the truncation* of f to Γ, that is, the polynomial obtained from f by keeping only those monomials whose exponents belong to Γ, then $\bar{Z}_f \cap \mathbb{T}_{N(\Gamma)} = Z_{f^\Gamma}$ and $\bar{Z}_f \cap X_\Gamma = \bar{Z}_{f^\Gamma}$. We have the classical notion of nondegenerate Laurent polynomial.

Definition 1.1. A polynomial f with Newton polytope Δ is called *nondegenerate* if for any face Γ of Δ of positive dimension (including Δ itself), the toric hypersurface Z_{f^Γ} is a nonsingular hypersurface.

Note that if Γ is a vertex of Δ, then Z_{f^Γ} is empty. In the previous definition, one may equivalently consider f^Γ as a polynomial in $\mathbb{C}[M]$ and thus look at the corresponding hypersurface of the whole torus $\mathbb{T}_N$. Indeed, this hypersurface of $\mathbb{T}_N$ is the product of $Z_{f^\Gamma} \subset \mathbb{T}_{N(\Gamma)}$ with the subtorus of $\mathbb{T}_N$ corresponding to a complement of $M(\Gamma)$ in M. If Δ is the Newton polytope of f, then the projective hypersurface $\bar{Z}_f \subset X_\Delta$ is nonsingular if and only if f is nondegenerate and X_Δ has eventually a finite number of singularities which are zero-dimensional $\mathbb{T}_{N(\Delta)}$-orbits corresponding to vertices of Δ.

2. Intersection of toric hypersurfaces

Consider polynomials $f_1, \ldots, f_k \in \mathbb{C}[M]$ and denote by Δ_i the Newton polytope of f_i. Let Δ be the *Minkowsky sum* of these polytopes

$$\Delta = \Delta_1 + \cdots + \Delta_k.$$

Each polynomial f_i seen as a polynomial in $\mathbb{C}[M(\Delta)]$ defines a toric hypersurface $Z_{f_i,\Delta}$ in $\mathbb{T}_{N(\Delta)}$ and it makes sense to consider the toric intersection

$$Z_{f_1,\Delta} \cap \cdots \cap Z_{f_k,\Delta} \subset \mathbb{T}_{N(\Delta)}. \tag{2.1}$$

Denote by $\bar{Z}_{f_i,\Delta}$ the Zarisky closure in X_Δ of $Z_{f_i,\Delta}$. For each $i = 1, \ldots, k$ there is a toric surjective map $\rho_i : X_\Delta \to X_{\Delta_i}$ such that $Z_{f_i,\Delta} = \rho_i^{-1}(Z_{f_i})$ and $\bar{Z}_{f_i,\Delta} = \rho_i^{-1}(\bar{Z}_{f_i})$. This leads to

$$\bar{Z}_{f_1,\Delta} \cap \cdots \cap \bar{Z}_{f_k,\Delta} \subset X_\Delta. \tag{2.2}$$

Each face Γ of Δ can be uniquely written as a Minkowsky sum

$$\Gamma = \Gamma_1 + \cdots + \Gamma_k \tag{2.3}$$

where Γ_i is a face of Δ_i. Substituting the truncation $g_i := f_i^{\Gamma_i}$ to f_i and Γ_i to Δ_i gives the toric intersection

$$Z_{g_1,\Gamma} \cap \cdots \cap Z_{g_k,\Gamma} \subset \mathbb{T}_{N(\Gamma)}. \tag{2.4}$$

which leads to

$$\bar{Z}_{g_1,\Gamma} \cap \cdots \cap \bar{Z}_{g_k,\Gamma} \subset X_\Gamma. \tag{2.5}$$

Similarly to the hypersurface case the intersection of (2.2) with $\mathbb{T}_{N(\Gamma)}$ (resp., X_Γ) coincides with (2.4) (resp., with (2.5)). Moreover, the intersection (2.2) is the union over all faces Γ of Δ of the toric intersections (2.4).

The *Cayley polynomial* associated with $f_1, \dots, f_k$ is the polynomial $F \in \mathbb{C}[M \oplus \mathbb{Z}^k]$ defined by

$$F(x, y) = \sum_{i=1}^{k} y_i f_i(x). \tag{2.6}$$

Its Newton polytope is the *Cayley polytope* associated with $\Delta_1, \dots, \Delta_k$ and will be denoted by

$$C(\Delta_1, \dots, \Delta_k) \subset M_\mathbb{R} \times \mathbb{R}^k. \tag{2.7}$$

Since F is a homogeneous (of degree 1) with respect to the variable y, the polytope $C(\Delta_1, \dots, \Delta_k)$ lies on a hyperplane and has thus dimension at most $n+k-1$. In fact, the dimension of $C(\Delta_1, \dots, \Delta_k)$ is $\dim(\Delta) + k - 1$. The faces of $C(\Delta_1, \dots, \Delta_k)$ are themselves Cayley polytopes. Namely, the faces of $C(\Delta_1, \dots, \Delta_k)$ are the Newton polytopes of all polynomials

$$\sum_{i \in I} y_i f_i^{\Gamma_i}(x)$$

such that $\emptyset \neq I \subset \{1, \dots, k\}$ and $\Gamma = \sum_{i \in I} \Gamma_i$ is a face of $\sum_{i \in I} \Delta_i$ with Γ_i a face of Δ_i for each i. We will call *admissible* such a collection $(\Gamma_i)_{i \in I}$. Note that by face we do not mean proper face. In particular $(\Delta_i)_{i \in I}$ is admissible for any non empty subset I of $\{1, \dots, k\}$. If $(\Gamma_i)_{i \in I}$ is admissible, we also call admissible the collection of polynomials $(f_i^{\Gamma_i})_{i \in I}$ and the corresponding toric intersection

$$\bigcap_{i \in I} Z_{f_i^{\Gamma_i},\Gamma} \subset \mathbb{T}_{N(\Gamma)}. \tag{2.8}$$

Definition 2.1. The collection $(f_1, \dots, f_k)$ is *nondegenerate* if the associated Cayley polynomial $F(x, y) = \sum_{i=1}^{k} y_i f_i(x)$ is nondegenerate.

The following result is based on the classical *Cayley trick* (see, for example, [**7**]).

Proposition 2.2. The collection $(f_1, \dots, f_k)$ is nondegenerate if and only if any admissible toric intersection (2.8) is a complete intersection.

Proof. As mentionned earlier, we can consider the polynomials $f_i^{\Gamma_i}$ occurring in (2.8) as polynomials in $\mathbb{C}[M]$ and thus look at the corresponding intersection in the whole torus $\mathbb{T}_N$. An easy computation shows that if hypersurfaces defined by polynomials $g_i \in \mathbb{C}[M]$, $i \in I$, do not intersect transversally at a point $X \in \mathbb{T}_N$, then there exists $\lambda = (\lambda_j)_{j \in J} \in (\mathbb{C}^*)^{|J|}$ with $J \subset I$ so that $\sum_{j \in J} y_j g_j(x)$ defines an hypersurface with a singular point at $(X,\lambda) \in \mathbb{T}_N \times (\mathbb{C}^*)^{|J|}$. Similarly, if a truncation $\sum_{i\in I} y_i g_i(x)$ of F to a face of $C(\Delta_1,\ldots,\Delta_k)$ defines an hypersurface with a singular point (X,λ) in the corresponding torus, then the hypersurfaces defined by g_i for $i \in I$ will not intersect transversally at $X \in \mathbb{T}_N$. $\square$

3. Tropical hypersurfaces

We now review some basic facts of tropical geometry and fix our notations. Useful references are for instance [**13**], [**8**], [**17**], [**6**]. The *tropical semiring* is $\mathbb{R} \cup \{-\infty\}$ endowed with the following tropical operations. The tropical addition of two numbers is the maximum of them; its neutral element is $-\infty$. The tropical multiplication is the ordinary addition with the convention that $x+(-\infty) = -\infty + x = -\infty$. Removing the neutral element $-\infty$ from the tropical semiring, we get the tropical one-dimensional torus $\mathbb{R}$. A tropical polynomial is a polynomial

$$f(z) = \sum_{w\in A} a_w z^w \in \mathbb{R}[z],$$

where $z = (z_1,\ldots,z_n)$, A is a finite set in $\mathbb{Z}^n$, and the addition and multiplication are the tropical ones. Strictly speaking, the coefficients are in the tropical semiring, but as usual, we omit the monomials whose coefficients are the neutral element for the addition. The support of f is A and the Newton polytope of f is the convex-hull of A. We will often denote the newton polytope of a (tropical) polynomial f by Δ. To a tropical polynomial $f(z) = \sum_{w\in A} a_w z^w$ corresponds a convex piecewise-linear (in fact piecewise-affine) function $\mathcal{L} : \mathbb{R}^n \to \mathbb{R}$, $z \mapsto \max_{w\in A}(\langle z,w\rangle + a_w)$. The tropical hypersurface defined by f is the corner locus of $\mathcal{L}$: this is the set of points of $\mathbb{R}^n$ where $\mathcal{L}$ is not linear, or equivalently, where the maximum $\max_{w\in A}(\langle z,w\rangle + a_w)$ is attained at least twice. We will denote by Z_f^{trop} the tropical hypersurface defined by f.

Consider the convex-hull

$$\hat{\Delta} = \text{Conv}\{(w,-a_w) \mid w \in A\} \subset \mathbb{R}^n \times \mathbb{R}.$$

A *lower face* of $\hat{\Delta}$ is a face having an outward normal vector with negative last coordinate (equivalently, the corresponding cone in the normal fan is not contained in the half space $\mathbb{R}^n \times \mathbb{R}_{\geq 0}$ of vectors with non negative last coordinate). The *lower part* of $\hat{\Delta}$ is the union of all lower faces of Δ and is denoted by $\hat{\Delta}_{<0}$. It is the graph of a convex piecewise-affine function $\nu : \Delta \to \mathbb{R}$. We have $\mathcal{L}(z) = \max_{w\in\Delta}(\langle z,w\rangle - \nu(w))$ for all $z \in \mathbb{R}$, which means that $\mathcal{L}$ is the Legendre transform of ν. Indeed, $\mathcal{L}(z) = M(z,-1)$, where M is the map

$$\begin{array}{cccc} M & : & \mathbb{R}^n \times \mathbb{R}_{<0} & \to & \mathbb{R} \\ & & (z,t) & \mapsto & \max_{w\in A}(\langle (z,t),(w,-a_w)\rangle, \end{array}$$

and $\mathbb{R}_{<0}$ stands for the set of negative real numbers. But the maximum of the linear map $\langle (z,t),\cdot\rangle$ on the set of all $(w,-a_w)$ with $w \in A$ is attained on the convex hull $\hat{\Delta}$ of this set, and actually on the lower part of $\hat{\Delta}$ if $t<0$. We have thus $M(z,t) = \max_{w\in\Delta}\langle (z,t),(w,\nu(w))\rangle$, and therefore $\mathcal{L}(z) = M(z,-1) = \max_{w\in\Delta}(\langle z,w\rangle - \nu(w))$ as claimed.

Note that the Legendre transform $A \to \mathbb{R}$, $w \mapsto \max_{z\in\mathbb{R}^n}(\langle w,z\rangle - \mathcal{L}(z))$, of $\mathcal{L}$ is equal to ν since ν is convex. Hence ν and $\mathcal{L}$ are dual to each other. Projecting the faces of $\hat{\Delta}_{<0}$ onto Δ via the projection $\mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, we get a polyhedral subdivision $\mathcal{S}$ of Δ. Polyhedral subdivision of polytopes of this type (obtained by projecting lower faces of polytopes) are called *convex* (or coherent). The polytopes of maximal dimension in $\mathcal{S}$ are the maximal domains of linearity of ν. This gives a one-to-one corresponce between polytopes of $\mathcal{S}$ and faces of $\hat{\Delta}_{<0}$. Similarly, we may project the linear faces of the graph of the convex piecewise-affine map $\mathcal{L}$ via the projection $\mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ in order to obtain a subdivision Ξ of $\mathbb{R}^n$. We will call *cells* the elements of Ξ. By definition Z_f^{trop} is the union of all $(n-1)$ dimensional cells of Ξ, while the n-dimensional cells of Ξ are the maximal domains of linearity of $\mathcal{L}$.

The subdivisions $\mathcal{S}$ and Ξ are in one-to-one correspondence via a correspondence $\sigma \in \mathcal{S} \mapsto \xi \in \Xi$ which reverses the inclusions and with the following properties

(1) $\dim \xi + \dim \sigma = n$,
(2) the cell ξ and the polytope σ span orthonogonal real affine spaces,
(3) the cell ξ is unbounded if and only if σ lies on a proper face of Δ.

This is a straightforward consequence of the classical one-to-one correspondence between the faces of a polytope and the cones in its normal fan, when applied to the lower part of $\hat{\Delta}$.

Recall that if P is a rational polytope in $\mathbb{R}^d$, then its normal fan $\mathcal{N}(P)$ is the complete fan in the dual space whose cones are the cones $\mathcal{C}_F$ over all faces F of Δ defined as follows. For each face F of Δ, the cone $\mathcal{C}_F$ is the set of all vectors v such that the restriction of the linear form $\langle v, \cdot \rangle$ on P attains its maximum at some point in the relative interior of F. This map $F \mapsto \mathcal{C}_F$ defines a one-to-one correspondence between the faces of P and the cones in $\mathcal{N}(P)$. The inverse map sends a cone $\mathcal{C}$ to the face F of P which is the maximal subset of P where $\langle v, \cdot \rangle$ attains its maximum on P for some (and in fact any) $v \in \mathcal{C}$ not contained in a sub-cone of $\mathcal{C}$. This correspondence reverses the inclusion relation, sends a k-dimensional face to a $(d-k)$-dimensional cone. Moreover, a cone and its corresponding face span orthogonal affine spaces. If P has maximal dimension d, then the one-dimensional cones in $\mathcal{N}(P)$ are generated by the outward normal vectors to the facets of P, and the cone corresponding to a face F is the cone generated by the the outward normal vectors to the facets of P of which F is the common face.

The correspondence between faces of $\hat{\Delta}$ and the cones in its normal fan $\mathcal{N}(\hat{\Delta})$, restricts to a one-to-one correspondence between the faces of $\hat{\Delta}_{<0}$ and the cones in the (non complete) fan $\mathcal{N}_{<0}(\hat{\Delta})$ formed of all cones of $\mathcal{N}(\hat{\Delta})$ not contained in $\mathbb{R}^n \times \mathbb{R}_{\geq 0}$. This correspondence reverses the inclusions and sends a k-dimensional face of $\hat{\Delta}$ to a $(n+1-k)$-dimensional cone of $\mathcal{N}_{<0}(\hat{\Delta})$ lying on a orthogonal affine space. We note that the map M considered above is linear on each cone of $\mathcal{N}_{<0}(\hat{\Delta})$. Moreover, its maximal domains of linearity are the cones of maximal dimension of this fan. A maximal dimensional cone of $\mathcal{N}_{<0}(\hat{\Delta})$ corresponds to a vertex $(w, \nu(w))$ of $\hat{\Delta}_{<0}$, and the restriction of M to this cone is the linear map $(z,t) \mapsto \langle (z,t), (w, \nu(w)) \rangle$. It follows that the subdivision Ξ, whose n-cells are the maximal domains of linearity of $\mathcal{L}$, can be obtained by taking the images under the projection $p : \mathbb{R}^n \times \{-1\} \to \mathbb{R}^n$ of the intersections of the cones of $\mathcal{N}_{<0}(\hat{\Delta})$ with $\mathbb{R}^n \times \{-1\}$. On the other hand, polytopes of $\mathcal{S}$ are obtained by taking the images of the faces of $\hat{\Delta}_{<0}$ by the projection $\pi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$. Thus, we get a one-to-one correspondence between $\mathcal{S}$ and Ξ in the following commutative diagram

$$\begin{array}{ccc} \text{faces of } \hat{\Delta}_{<0} & \longrightarrow & \text{cones of } \mathcal{N}_{<0}(\hat{\Delta}) \\ \downarrow \pi & & \downarrow p \\ \mathcal{S} & \longrightarrow & \Xi \end{array} \tag{3.1}$$

Here, the first horizontal map is the restriction to $\hat{\Delta}_{<0}$ of the usual duality between the faces of a polytope and the cones in its normal fan, the left vertical map is the bijection (respecting the dimensions and inclusions) induced by $\pi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ between the faces of $\hat{\Delta}_{<0}$ and the polytopes in $\mathcal{S}$, the right vertical map is the bijection between $\mathcal{N}_{<0}(\hat{\Delta})$ and Ξ which consists in taking the intersection of a cone with $\mathbb{R}^n \times \{-1\}$ and then project it onto $\mathbb{R}^n$ using $p : \mathbb{R}^n \times \{-1\} \to \mathbb{R}^n$ (this last bijection respects the inclusions but lowers the dimension by 1). It may be useful to note that the polytope $\sigma \in \mathcal{S}$ corresponding to $\xi \in \Xi$ is the convex-hull of $\{w \in A \,|\, \langle x, w\rangle - \nu(w) = \mathcal{L}(x)\}$, where x is any point in the relative interior of ξ.

4. Intersection of tropical hypersurfaces

Consider now tropical polynomials $f_1, \dots, f_k$ in $\mathbb{R}[z]$. For $i = 1, \dots, k$ write

$$f_i(z) = \sum_{w_i \in A_i} a_{w_i} z^{w_i},$$

so that $A_i \subset \mathbb{Z}^n$ is the support of f_i. Denote by Δ_i the Newton polytope of f_i. Consider the tropical polynomial

$$f(z) = \prod_{i=1}^{k} f_i(z).$$

We retain the notations used in the previous section for the tropical polynomial f, and use the same notations with a subscript i for each polynomial f_i. The support of f is the sum $A = A_1 + \cdots + A_k$ and its Newton polytope is the Minkowsky sum $\Delta = \Delta_1 + \cdots + \Delta_k$. The coefficient a_w in the expansion $f(z) = \sum_{w \in A} a_w z^w$ is

$$a_w = \operatorname{Max}\left(\sum_{i=1}^{k} a_{w_i}\right),$$

where the maximum is taken over all $(w_1, \dots, w_k) \in A_1 \times \cdots \times A_k$ such that $w = w_1 + \cdots + w_k$. We have thus

$$-a_w = \operatorname{Min}\left(\sum_{i=1}^{k} (-a_{w_i})\right),$$

which immediately shows that the lower part $\hat{\Delta}_{<0}$ of $\hat{\Delta} = \operatorname{Conv}\{(w, -a_w) \,|\, w \in A\}$ verifies

$$\hat{\Delta}_{<0} = \hat{\Delta}_{1,<0} + \cdots + \hat{\Delta}_{k,<0},$$

where $\hat{\Delta}_{i,<0}$ is the lower part of $\hat{\Delta}_i = \operatorname{Conv}\{(w_i, -a_{w_i}) \,|\, w_i \in A_i\}$. The normal fan $\mathcal{N}(P)$ of a Minkowsky sum $P = P_1 + \cdots + P_k$ is the set of cones $\mathcal{C} = \cap_{i=1}^{k} \mathcal{C}_i$ with $\mathcal{C}_i \in \mathcal{N}(P_i)$ for $i = 1, \dots, k$. Note that cones $\mathcal{C}_1, \dots, \mathcal{C}_k$ are not determined by their common intersection $\mathcal{C}$, unless they are minimal in that $\mathcal{C} \neq \cap_{i=1}^{k} \mathcal{C}'_i$ for any proper collection $\mathcal{C}'_i$, $i = 1, \dots, k$, of sub-cones of $\mathcal{C}_i$. It follows then from the diagram (3.1) that the cells of the subdivision Ξ of $\mathbb{R}^n$ induced by the tropical polynomial f are the non-empty common intersections

$$\xi = \bigcap_{i=1}^{k} \xi_i \tag{4.1}$$

of cells $\xi_i \in \Xi_i$, $i = 1, \ldots, k$. Moreover, the representation (4.1) of a cell of Ξ is unique if we impose that ξ lies *in the relative interior* of each ξ_i. We shall always refer to this unique form when writing (4.1). Note that n-cells of Ξ are intersections of n-cells of the subdivisions Ξ_i. Therefore,

$$Z_f^{\text{trop}} = \bigcup_{i=1}^{k} Z_{f_i}^{\text{trop}}$$

as expected. Consider a cell $\xi = \bigcap_{i=1}^{k} \xi_i$ of Ξ. Each ξ_i corresponds to a polytope σ_i in the convex polyhedral subdivision $\mathcal{S}_i$ of Δ_i induced by f_i. On the other hand, the cell ξ corresponds to a polytope σ in the subdivision $\mathcal{S}$ of $\Delta = \Delta_1 + \cdots + \Delta_k$ induced by f.

Proposition 4.1. Let ξ be a cell of Ξ with presentation $\xi = \bigcap_{i=1}^{k} \xi_i$ with $\xi_i \in \Xi_i$ for all $i = 1, \ldots, k$. If σ is the polytope of $\mathcal{S}$ corresponding to ξ and σ_i is the polytope of $\mathcal{S}_i$ corresponding to ξ_i for $i = 1, \ldots, k$, then

$$\sigma = \sigma_1 + \cdots + \sigma_k. \tag{4.2}$$

Proof. We use the diagram (3.1). The cell $\xi = \bigcap_{i=1}^{k} \xi_i$ corresponds (via the right vertical map) to a cone $\mathcal{C} = \cap_{i=1}^{k} \mathcal{C}_i$ of $\mathcal{N}(\Delta)$, where $\mathcal{C}_i$ is the cone of $\mathcal{N}(\Delta_i)$ corresponding to ξ_i. Let v be any vector with negative last coordinate and properly contained in $\mathcal{C}$ (recall that here properly contained means not contained in a subcone). Then v is properly contained in each cone $\mathcal{C}_i$ since ξ lies in the relative interior of each ξ_i. The face of $\hat{\Delta}$ which corresponds to $\mathcal{C}$ is the maximal face where the linear map $\langle v, \cdot \rangle$ attains its maximum. This face is a face of $\hat{\Delta}_{<0}$ since v has negative last coordinate and it is the Minkowsky sum of the maximal faces of $\hat{\Delta}_1, \ldots, \hat{\Delta}_k$ where the same linear map attains its maximum. These faces are the faces of $\hat{\Delta}_{1,<0}, \ldots, \hat{\Delta}_{k,<0}$ which correspond to the cones $\mathcal{C}_1, \ldots, \mathcal{C}_k$ and which project to $\sigma_1, \ldots, \sigma_k$. □

Note that there can be different ways to write a polytope of $\mathcal{S}$ as a Minkowsky sum of polytopes of $\mathcal{S}_1, \ldots, \mathcal{S}_k$. When writing (4.2), we shall always refer to the Minkowsky sum induced by the tropical polynomials $f_1, \ldots, f_k$ (obtained by projecting a lower face of $\hat{\Delta}_{<0} = \hat{\Delta}_{1,<0} + \cdots + \hat{\Delta}_{k,<0}$). Convex polyhedral subdivisions like $\mathcal{S}$ are called *convex mixed subdivisions*. Pictures illustrating intersections of tropical hypersurfaces can easily be found in the litterature, see for instance [**6**] (Figure 9) or [**17**].

Convex mixed subdivisions of a polytope $\Delta_1 + \cdots + \Delta_k \subset \mathbb{R}^n$ are in one-to-one correspondence with convex polyhedral subdivisions of the Cayley polytope $C(\Delta_1, \ldots, \Delta_k) \subset \mathbb{R}^{n+k}$. This is the so-called *the combinatorial Cayley trick* that we recall now. Let (a, b) be coordinates on $\mathbb{R}^{n+k} = \mathbb{R}^n \times \mathbb{R}^k$. Consider the subspace B of $\mathbb{R}^{n+k}$ defined by $b_1 = b_2 = \cdots = b_k = 1/k$ and identify it with $\mathbb{R}^n$ via the projection $(a, b) \mapsto a$. This identifies $B \cap C(\Delta_1, \ldots, \Delta_k)$ with $\Delta = \Delta_1 + \cdots + \Delta_k$ dilated by $1/k$. Note that the space defined by $b_i = 1$ and $b_j = 0$ for $j \neq i$ intersects

$C(\Delta_1, \ldots, \Delta_k)$ along a face which can be identified with Δ_i via the projection. Consider a polyhedral subdivision of $C(\Delta_1, \ldots, \Delta_k)$. If F is a polytope of maximal dimension $\dim \Delta + k - 1$ in this subdivision, then it intersects the space defined by $b_i = 1$ and $b_j = 0$ for $j \neq i$ along a nonempty face F_i, which projects to a (nonempty) subpolytope Γ_i of Δ_i. Then $F \cap B$ is identified via the projection with the polytope $\Gamma = \Gamma_1 + \cdots + \Gamma_k \subset \Delta$ dilated by $1/k$. This gives a correspondence between polyhedral subdivisions of $C(\Delta_1, \ldots, \Delta_k)$ and convex mixed subdivisions of $\Delta = \Delta_1 + \cdots + \Delta_k$. The following result can be found, for example, in [**20**].

Proposition 4.2. The correspondence described above is a bijection between the set of convex polyhedral subdivisions of $C(\Delta_1, \ldots, \Delta_k)$ and the set of convex mixed subdivisions of $\Delta = \Delta_1 + \cdots + \Delta_k$. Precisely, let $\mu : C(\Delta_1, \ldots, \Delta_k) \to \mathbb{R}$ be any convex piecewise-linear function and let ν_i denote its restriction to Δ_i identified with a face of $C(\Delta_1, \ldots, \Delta_k)$ via the projection $(a, b) \mapsto a$. Then the correspondence described above sends the coherent polyhedral subdivision of $C(\Delta_1, \ldots, \Delta_k)$ defined by μ to the convex mixed subdivision of Δ defined by $(\nu_1, \ldots, \nu_k)$.

5. Intersection multiplicity numbers between tropical hypersurfaces

Recall that all polytopes under consideration have vertices in the underlying lattice $M \simeq \mathbb{Z}^n$. A k-dimensional simplex σ with vertices $m_0, m_1, \ldots, m_k$ is called *primitive* if the vectors $m_1 - m_0, \ldots, m_k - m_0$ form a basis of the lattice $M(\sigma)$, or equivalently, if these vectors can be completed to form a basis of M. Obviously, the faces of a primitive simplex are themselves primitives simplices.

Consider a k-dimensional vector subspace of $M_{\mathbb{R}}$ with rational slopes. It intersects M in a saturated subgroup γ of rank k and coincides with the real vector space $\gamma_{\mathbb{R}}$ generated by γ. Any basis of γ produces an isomorphism between γ and $\mathbb{Z}^k$, and then by extension an isomorphism between $\gamma_{\mathbb{R}}$ and $\mathbb{R}^k$. Let Vol_γ be the volume form on $\gamma_{\mathbb{R}}$ obtained as the pull-back via such an isomorphism of the usual Euclidian k-volume on $\mathbb{R}^k$. For simplicity, we will write Vol_k instead of Vol_γ since the lattice γ will be clear from the context. Note that Vol_k does not depend on the isomorphism $\gamma \simeq \mathbb{Z}^k$ since two basis of γ are obtained from each other by integer invertible linear map which has determinant ± 1. Any basis $(\gamma_1, \ldots, \gamma_k)$ of γ generate a k-dimensional parallelotope $P \subset \gamma_{\mathbb{R}}$ (isomorphic to the cube $[0,1]^k \subset \mathbb{R}^k$) called *fundamental parallelotope* of γ and which verifies $\mathrm{Vol}_k(P) = 1$. Two primitive k-simplices on $\gamma_{\mathbb{R}}$ have the same volume under Vol_k (they are interchanged by an invertible integer linear map), and this volume is $\frac{1}{k!}$ since a fundamental parallelotope of γ can be subdivided into $k!$ primitive k-simplices. We will often use the normalized volume

$$\mathrm{vol}_k(\,\cdot\,) := k! \cdot \mathrm{Vol}_k(\,\cdot\,)$$

on $\gamma_{\mathbb{R}}$. This normalized volume takes all nonnegative integer values on polytopes (with vertices in γ), and we have $\mathrm{vol}_k(\sigma) = 1$ for a polytope σ if and only if σ is a k-dimensional primitive simplex. We will use the following elementary fact.

Remark 5.1. Let γ be a subgroup of a free abelian group Λ of finite rank. Assume that Λ and γ have the same rank k, so that the index $[\Lambda : \gamma]$ of γ in Λ is well-defined. Then, for any basis $(\gamma_1, \ldots, \gamma_k)$ of γ and any basis $e = (e_1, \ldots, e_k)$ of Λ we have

$$[\Lambda : \gamma] = \mathrm{Vol}_k(G) = \mathrm{vol}_k(g) = |\det(G_{ij})|,$$

where G (resp., g) is the k-dimensional parallelotope (resp., k-dimensional simplex) generated by $\gamma_1, \ldots, \gamma_k$ and (G_{ij}) is the $k \times k$-matrix whose j-th column is the vector of coordinates of γ_j with respect to $(e_1, \ldots, e_k)$.

Let $P_1, \ldots, P_\ell$ be polytopes with vertices in a saturated lattice γ of rank ℓ. The map $(\lambda_1, \ldots, \lambda_\ell) \mapsto \mathrm{Vol}_\ell(\lambda_1 P_1 + \cdots + \lambda_\ell P_\ell)$ is a homogeneous polynomial map of degree ℓ. The coefficient of the monomial $\lambda_1 \cdots \lambda_\ell$ is called the *mixed volume* of $P_1, \ldots, P_\ell$ and is denoted by

$$MV_\ell(P_1, \ldots, P_\ell).$$

A famous theorem due to Bernstein states that this mixed volume is the number of solutions in the torus associated with the lattice γ of a generic polynomial system $f_1 = \ldots = f_\ell = 0$ where each f_i has P_i as Newton polytope. Note that $MV_\ell(P_1, \ldots, P_\ell) = 0$ if $P = P_1 + \cdots + P_\ell$ does not have full dimension ℓ or if at least one P_i has dimension zero. If $P_1 = \cdots = P_\ell = P$, then we have $MV_\ell(P_1, \ldots, P_\ell) = \mathrm{vol}_\ell(P)$.

The tropical cycle associated to any tropical hypersurface in $\mathbb{R}^n$ is the tropical hypersurface equipped with weights on its top-dimensional cells (cells of dimension $n-1$) which are the integer lenghts of the dual edges. The intersection cycle (also called stable intersection cycle) associated to k tropical hypersurfaces in $\mathbb{R}^n$ is the $(n-k)$-dimensional polyhedral complex given by the union of the $(n-k)$-dimensional cells of the common intersection of these tropical hypersurfaces and equipped with weights on these cells as defined below. Write any cell ξ of the common intersection of k tropical hypersurfaces $Z_i = Z_{f_i}^{\mathrm{trop}}$, $i = 1, \ldots, k$, in $\mathbb{R}^n$ as

$$\xi = \bigcap_{i=1}^{k} \xi_i$$

where $\xi_i \in \Xi_i$ for $i = 1, \ldots, k$ (and ξ lies in the relative interior of each ξ_i). Let $\sigma_i \in \mathcal{S}_i$ be the polytope corresponding to ξ_i. Set $d_i := \operatorname{codim} \xi_i = \dim \sigma_i$ and $d := \operatorname{codim} \xi = \dim \sigma$. Note that $d_i \geq 1$ for $i = 1, \ldots, k$ since ξ is a cell of the common intersection of the k tropical hypersurfaces.

Definition 5.2. ([**13**]) The weight $w(\xi)$ of the $(n-k)$-cell ξ of the intersection cycle $Z_1 \ldots Z_k$ is defined in the following way.

- (*Tranversal case.*) If $d_1 + \cdots + d_k = d$, then

  $$w(\xi) \quad = \quad \left(\prod_{i=1}^{k} w(\xi_i)\right) \cdot [M(\sigma) : M(\sigma_1) + \cdots + M(\sigma_k)]$$

- (*General case.*) Translate the tropical hypersurfaces by small generic vectors so that all intersections emerging from ξ are transversal intersections of top dimensional cells. Define $w(\xi)$ as the sum of the weights at the transversal intersections emerging from ξ and which are cells of codimension d.

Note that in Definition 5.2 one has $d = k$, and thus $d_1 = d_2 = \cdots = d_k = 1$ in the transversal case so that each $w(\xi_i)$ is well-defined as the integer lenght of the edge σ_i. Recall also that for a polytope $P \subset M_{\mathbb{R}}$, we denote by $M(P)$ the subgroup of M consisting of all integer vectors which are parallel to P. We may express the lattice index in Definition 5.2 as an index lattice in the dual lattice with the help of the following result (see [**3**] for a proof).

Lemma 5.3. Let γ_1 and γ_2 be saturated subgroups of a free group N such that $\gamma_1+\gamma_2$ and N have same rank. Then the index of $\gamma_1+\gamma_2$ in N satisfies to

$$[N:\gamma_1+\gamma_2]=[(\gamma_1\cap\gamma_2)^{\perp}:\gamma_1^{\perp}+\gamma_2^{\perp}],$$

where $\gamma^{\perp}$ denotes the subgroup of the dual lattice $M=\mathrm{Hom}_{\mathbb{Z}}(N,\mathbb{Z})$ consisting of all elements of M which vanish on a subgroup γ of N.

A Minkowsky sum $Q_1+\cdots+Q_\ell$ of polytopes such that $\dim(Q_1+\cdots+Q_\ell)=\dim Q_1+\cdots+\dim Q_\ell$ is called *a direct Minkowsky sum* and is denoted by $Q_1\oplus\cdots\oplus Q_\ell$. A convex mixed subdivision $\mathcal{S}$ of a polytope $P=P_1+\cdots+P_\ell$ is called *pure* if for any polytope $Q\in\mathcal{S}$ with representation $Q=Q_1+\cdots+Q_\ell$ we have $Q=Q_1\oplus\cdots\oplus Q_\ell$. In the transversal case of Definition 5.2, the polytope σ is a direct sum of edges. Such polytopes are often called *zonotopes.* It follows from Remark 5.1 that if σ is a zonotope, then $w(\xi)=\mathrm{Vol}_k(\sigma)$. In the general case of Definition 5.2, one has to take any pure convex mixed subdivision of $\sigma=\sigma_1+\cdots+\sigma_k$ and sum up the weights of the cells dual to the zonotopes in this mixed subdivision. It is well-known (see, for example, [**4**], Ch. 7, Theorem 6.7) that the volumes of the zonotopes in a given pure mixed subdivision of polytopes sum up to the mixed volume of these polytopes, and this leads to the following well-known result.

Lemma 5.4. The weight of the $(n-k)$-cell ξ of the intersection cycle $Z_1\dots Z_k$ verifies $w(\xi)=MV_k(\sigma_1,\dots,\sigma_k)$.

Note that Definition 5.2 makes sense for cells ξ of codimension $d<k$: in that case there is no transversal intersections of top dimensional cells of tropical hypersurfaces and it is natural to set $w(\xi)=0$. One can generalize Definition 5.2 for cells $\xi=\bigcap_{i=1}^k\xi_i$ of any codimension $d\geq k$ in the following way. A *partition of lenght* k of d is a collection $\underline{t}=(t_1,\dots,t_k)$ of positive integers suming up to d (in other words, $t_1,\dots,t_k>0$ and $t_1+\cdots+t_k=d$). For any such partition $\underline{t}$, we may see ξ as a cell of the common intersection of the d tropical hypersurfaces obtained by taking t_i copies of Z_i for $i=1,\dots,k$, and consider the corresponding classical weight of ξ. Suming up these weights over all possible partitions of lenght k of d gives a weight on ξ (which does not depend on any choice of partition). Roughly speaking, one considers the classical weights of ξ seen as a $(n-d)$-cell of the intersection cycle of d tropical hypersurfaces among $Z_1,\dots,Z_k$ allowing repetitions (and imposing that each Z_i appears at least once), and sum up all these weights.

Definition 5.5. The weight of the $(n-d)$-cell ξ is the sum over all partitions of lenght k of d

$$w(\xi)=\sum_{\underline{t}}w(\xi,\underline{t}), \tag{5.1}$$

where $w(\xi,\underline{t})$ is the classical weight (given by Definition 5.2) of ξ seen as a $(n-d)$-cell of the intersection cycle of the d tropical hypersurfaces $\underbrace{Z_1,\dots,Z_1}_{t_1},\dots,\underbrace{Z_k,\dots,Z_k}_{t_k}$.

These weights are the intersection multiplicity numbers considered in [**3**].

Example 5.6.

- If $k = 1$ (ξ is a cell of a single tropical hypersurface), there is only one partition of lenght 1 of $d = \operatorname{codim}\xi$, and the intersection number $w(\xi)$ is the classical weight of ξ seen as a cell of the d-fold intersection cycle of the tropical hypersurface. Thus, by Lemma 5.4, one has $w(\xi) = MV_d(\sigma, \dots, \sigma) = \mathrm{vol}_d(\sigma)$, where σ is the polytope dual to ξ.
- If $k = d-1$, there are k partitions of lenght $d-1$ of d, namely $(2, 1, \dots, 1)$, $(1, 2, 1, \dots, 1)$, $\dots$, $(1, \dots, 1, 2)$, and thus the intersection number $w(\xi)$ is the sum of the k classical weights of ξ seen as a cell of the common intersection of the tropical hypersurfaces where one and only one appears twice.

For any partition $\underline{t} = (t_1, \dots, t_k)$ of d and any polytopes $P_1, \dots, P_k$ with vertices in a lattice of rank d we may consider the mixed volume

$$MV_d(P_1, \dots, P_k\,;\,\underline{t}) := MV_d(\underbrace{P_1, \dots, P_1}_{t_1}, \dots\dots, \underbrace{P_k, \dots, P_k}_{t_k})$$

(on the right, each P_i is repeated t_i times). Note that $\dfrac{1}{t_1! \cdots t_k!} \cdot MV_d(P_1, \dots, P_k\,;\,\underline{t})$ is the coefficient of $\lambda_1^{t_1} \cdots \lambda_k^{t_k}$ in the homogeneous degree d polynomial map sending $(\lambda_1, \dots, \lambda_k)$ to $\mathrm{Vol}_d(\lambda_1 P_1 + \cdots + \lambda_k P_k)$ (see [**4**], page 327). Lemma 5.4 gives directly

$$w(\xi, \underline{t}) = MV_d(\sigma_1, \dots, \sigma_k; \underline{t}). \tag{5.2}$$

If ξ is a transversal intersection cell, then it is easy to see that $w(\xi, \underline{t})$ vanishes unless $\underline{t}$ is the partition $\underline{d} = (d_1, \dots, d_k)$ given by the codimensions of $\xi_1, \dots, \xi_k$ and that

$$MV_d(\sigma_1, \dots, \sigma_k; \underline{d}) = \left(\prod_{i=1}^{k} \mathrm{vol}_{d_i}(\sigma_i)\right) \cdot [M(\sigma) : M(\sigma_1) + \cdots + M(\sigma_k)]. \tag{5.3}$$

A proof of (5.3) using Bernstein's theorem is given in [**3**] (see the proof of Theorem 4.5, page 13). One may also first consider the self intersection numbers $\mathrm{vol}_{t_i}(\sigma_i)$ of the tropical hypersurfaces dual to $\sigma_1, \dots, \sigma_k$ (in particular, if $\underline{t} \neq \underline{d}$, then at least one t_i is bigger than d_i, and thus $\mathrm{vol}_{t_i}(\sigma_i) = 0$), and then multiply the product of these intersection numbers by the index lattice as in Definition 5.2. This corresponds, via the duality between tropical hypersurfaces and convex subdivisions, to considering any convex pure mixed subdivision of $(\sigma_1 + \cdots + \sigma_1) \oplus \cdots \oplus (\sigma_k + \cdots + \sigma_k)$ (each σ_i appears d_i times) and sum up the volumes of its zonotopes. Each such zonotope is a direct sum of zonotopes contained in $d_1\sigma_1, \dots, d_k\sigma_k$, respectively, thus its volume is the product of the volumes of these smaller zonotopes by the index lattice $[M(\sigma) : M(\sigma_1) + \cdots + M(\sigma_k)]$. Formulas (5.2) and (5.3) lead immediately to the following result.

Theorem 5.7. *([**3**])*

- *If the tropical hypersurfaces intersect transversally along ξ, which means that $d = d_1 + \cdots + d_k$, then letting $\underline{d} := (d_1, \dots, d_k)$ we have*

$$w(\xi) \;=\; MV_d(\sigma_1, \dots, \sigma_k; \underline{d}) \tag{5.4}$$

- *In the general case, we have $d \leq d_1 + \cdots + d_k$ and*

$$w(\xi) = \sum_{\underline{t}=(t_1,\ldots,t_k)} MV_d(\sigma_1, \ldots, \sigma_k; \underline{t}) \tag{5.5}$$

where the sum is over all partitions of lenght k of d.

It turns out that $w(\xi)$ may be defined exactly as in Definition 5.2 by dropping out the conditions on the dimensions of the cells. Assume that any cell (of any dimension) of a single tropical hypersurface is equipped with the normalized volume of its dual polytope as in Example 5.6.

Definition 5.8. ([**3**]) The weight of the $(n-d)$-cell ξ is defined in the following way.

- (*Tranversal case.*) If $d_1 + \cdots + d_k = d$, then

$$\begin{aligned} w(\xi) &= \left(\prod_{i=1}^k w(\xi_i)\right) \cdot [M(\sigma) : M(\sigma_1) + \cdots + M(\sigma_k)] \\ &= \left(\prod_{i=1}^k \mathrm{vol}_{d_i}(\sigma_i)\right) \cdot [M(\sigma) : M(\sigma_1) + \cdots + M(\sigma_k)] \end{aligned}$$

- (*General case.*) Translate the tropical hypersurfaces by small generic vectors so that all intersections emerging from ξ are transversal intersections. Define $w(\xi)$ as the sum of the weights at the transversal intersections emerging from ξ and which are cells of codimension d.

This is the original definition used in [**3**], where a detailed proof of Theorem 5.7 starting from Definition 5.8 is given. This proof can be used to show the equivalence of both definitions 5.8 and 5.5. Let us explain briefly how one can show this equivalence. Assume that $w(\xi) = \sum_{\underline{t}} w(\xi, \underline{t})$ as in Definition 5.5. The transversal case $d = d_1 + \cdots + d_k$ follows immediately from (5.3). Consider now the general case. It is not difficult to see that for any partition $\underline{t}$ of d we have

$$w(\xi, \underline{t}) = \sum_{\gamma} w(\gamma, \underline{t}),$$

where the sum is over all transversal intersection cells $\gamma = \cap_{i=1}^k \gamma_i$ which emerge from ξ and have codimension d. By the transversal case, we have $w(\gamma, \underline{t}) = 0$ unless $\underline{t}$ is the partition given by the codimensions of $\gamma_1, \ldots, \gamma_k$ and in that case $w(\gamma, \underline{t}) = w(\gamma)$. Thus,

$$w(\xi) = \sum_{\underline{t}} w(\xi, \underline{t}) = \sum_{\underline{t}} \sum_{\gamma} w(\gamma, \underline{t}) = \sum_{\gamma} w(\gamma),$$

which is what we wanted to show.

It is well-known that the normalized volume $\mathrm{vol}_n(P)$ of any n-dimensional polytope $P \subset \mathbb{R}^n$ is equal to $(-1)^{n-1}$ times the Euler characteristic of any hypersurface of the complex n-dimensional torus defined by a non degenerate polynomial with Newton polytope P. This provides a geometric interpretation of our intersection multiplicity numbers in the hypersurface case, which has the following generalization for intersections of tropical hypersurfaces, thanks to a theorem of A. Khovanskii [**12**].

Theorem 5.9. *We have*

$$w(\xi) = (-1)^{d-k}\chi(\{f_1 = \cdots = f_k = 0\}),$$

where $\chi(\{f_1 = \cdots = f_k = 0\})$ *is the Euler characteristic of any complete intersection of the complex d-dimensional torus defined by nondegenerate collection of polynomials* $(f_1, \ldots, f_k)$ *with Newton polytopes* $\sigma_1, \ldots, \sigma_k$*, respectively.*

Proof. This is a direct application of Theorem 1 (Section 3) in [**12**]. □

Note that in the classical case $k = d$, this reduces to the well-known formula $w(\xi) = MV_k(\sigma_1, \ldots, \sigma_k)$ (see Lemma 5.4) with the help of Bernstein's theorem.

6. Tropical intersection numbers via volumes of Cayley polytopes

It turns out that our intersection multiplicity numbers can also be expressed in term of volumes of Cayley polytopes. This can be shown using the combinatorial Cayley trick relating convex subdivisions of Cayley polytopes to mixed subdivisions (see Proposition 4.2) and the following basic observation.

Lemma 6.1. Suppose that $S = S_1 \oplus \cdots \oplus S_m$ is a direct sum of simplices in $M_{\mathbb{R}}$, where M is a lattice of rank ℓ. Set $\ell_i = \dim S_i$ for $i = 1, \ldots, m$ and $\underline{\ell} = (\ell_1, \ldots, \ell_m)$. For any partition $\underline{t} = (t_1, \ldots, t_m)$ of ℓ, we have $MV_\ell(S_1, \ldots, S_m; \underline{t}) = 0$ if $\underline{t} \neq \underline{\ell}$ and

$$MV_\ell(S_1, \ldots, S_m; \underline{\ell}) = \mathrm{vol}_{\ell+m-1}\left(C(S_1, \ldots, S_m)\right)$$

otherwise.

Proof. According to (5.3), it suffices to show that if $\underline{t} = \underline{\ell}$ then

$$\mathrm{vol}_{\ell+m-1}\left(C(S_1, \ldots, S_m)\right) = \left(\prod_{i=1}^{m} \mathrm{vol}_{\ell_i}(S_i)\right) \cdot [M(S) : M(S_1) + \cdots + M(S_m)]. \tag{6.1}$$

Assume that $\underline{t} = \underline{\ell}$. The Cayley polytope of a direct sum of simplices is a simplex. Thus $\mathrm{vol}_{\ell+m-1}\left(C(S_1, \ldots, S_m)\right)$ equals the absolute value of a $(\ell + m - 1)$-determinant D whose columns are the coordinates with respect to a basis of $M(C(S_1, \ldots, S_m)) = M(S) \times \mathbb{Z}^m$ of vectors spanning $C(S_1, \ldots, S_m)$. The corresponding determinant taken with respect to a basis of $(M(S_1) + \cdots + M(S_m)) \times \mathbb{Z}^m$ is a determinant $\tilde{D}$ which factors into a product of m determinants $D_1, \ldots, D_m$. Each factor D_i has size d_i and is a determinant whose columns are the coordinates with respect to a basis of $M(S_i)$ of vectors spanning the simplex S_i. The absolute value of D_i is just $\mathrm{vol}_{\ell_i}(S_i)$. Formula (6.1) follows now from Remark 5.1. □

Let $\xi = \bigcap_{i=1}^k \xi_i$ be a cell of the common intersection of k tropical hypersurfaces. With the notations of Section 5, we get the following formula for the weight of ξ using Cayley polytopes of dual polytopes.

Theorem 6.2.

$$w(\xi) = \sum_{\emptyset \neq I \subset \{1, \ldots, k\}} (-1)^{k-|I|} vol_{d+|I|-1}\left(C(\sigma_i, i \in I)\right). \tag{6.2}$$

Proof. Take any tight pure convex mixed subdivision $\mathcal{S}$ of σ. Each $S \in \mathcal{S}$ is a direct sum of simplices $S_1 \oplus \cdots \oplus S_k$. According to Definition 5.8, the weight $w(\xi)$ is the sum of the weights of the cells γ corresponding to d-dimensional $S \in \mathcal{S}$ with $\dim S_i \geq 1$ for $i = 1, \ldots, k$ (if S_i is a vertex, then γ is not a cell of the common intersection of the perturbed tropical hypersurfaces). According to Theorem 5.7 and Lemma 6.1, we have $w(\gamma) = \mathrm{vol}_{d+k-1}\left(C(S_1,\ldots,S_k)\right)$. We thus get

$$w(\xi) = \mathrm{vol}_{d+k-1}\left(C(\sigma_1,\ldots,\sigma_k)\right) - \sum_{S\in\mathcal{S}\,:\,\exists i\ \dim S_i=0} \mathrm{vol}_{d+k-1}\left(C(S_1,\ldots,S_k)\right),$$

which leads to

$$w(\xi) = \sum_{\emptyset\neq I\subset\{1,\ldots,k\}} (-1)^{k-|I|} \sum_{S\in\mathcal{S}\,:\,\forall i\notin I\ \dim S_i=0} \mathrm{vol}_{d+k-1}\left(C(S_1,\ldots,S_k\right).$$

Now if $\dim S_i = 0$ for all $i \notin I$, then an easy computation shows that

$$\mathrm{vol}_{d+k-1}\left(C(S_1,\ldots,S_k)\right) = \mathrm{vol}_{d+|I|-1}\left(C(S_i\,,i\in I)\right),$$

where $C(S_i\,,i\in I)$ stands for the Cayley polytope of S_i for $i \in I$. It remains to remark that

$$\sum_{S\in\mathcal{S}(\sigma)\,:\,\forall i\notin I\ \dim S_i=0} \mathrm{vol}_{d+|I|-1}\left(C(S_i\,,i\in I)\right) = \mathrm{vol}_{d+|I|-1}\left(C(\sigma_i\,,i\in I)\right).$$

□

7. Non degenenerate tropical complete intersections

In this section we define non degenenerate tropical complete intersections following the classical definitions in the complex setting, see Sections 1 and 2. We show that our tropical intersection multiplicity numbers behave like classical intersection multiplicity numbers with respect to non degenerate complete intersections. First we start from the well-established definition of a nonsingular tropical hypersurface.

Definition 7.1. A tropical hypersurface is nonsingular if its dual polyhedral subdivision is a primitive (convex) triangulation, that is, a triangulation whose all simplices are primitive.

This definition can be motivated by the fact that around a vertex corresponding to a primitive n-simplex, a tropical hypersurface coincides with a tropical hypersurface with Newton polytope this simplex. But such a simplex is given by a basis of the ambient lattice M, and identifying M with $\mathbb{Z}^n$ via this basis identifies the simplex with the standard unit simplex in $\mathbb{Z}^n$. Hence, up to a basis change of the ambient lattice, a non singular tropical hypersurface coincides around each vertex with a tropical linear hyperplane. Nonsingular tropical hypersurfaces with a given Newton polytope do not always exist. The simplest example is given by the non primitive tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ and $(1,1,2)$ in $\mathbb{R}^3$ which meets the lattice $\mathbb{Z}^3$ at its vertices and has thus no primitive triangulation (see [?]). Recall that a tropical hypersurface lies in $N_{\mathbb{R}} \simeq \mathbb{R}^n$, which is the tropical torus associated with some lattice N. Hence, at this point, a tropical hypersurface is in fact a toric tropical hypersurface. A primitive (convex) triangulation of a polytope induces a primitive (convex) triangulation of each of its faces. Recall that the truncation f^{Γ} of a tropical polynomial f to a face Γ of its Newton polytope also defines a tropical hypersurface in the corresponding tropical torus $N(\Gamma)_{\mathbb{R}}$. Hence, in contrast to the complex case, if f defines a nonsingular tropical hypersurface in the corresponding tropical torus, then so do automatically all its truncations. Comparing with the classical definition 1.1 of a nondegenerate polynomial, this leads to the following definition.

Definition 7.2. A tropical polynomial is nondegenerate if all its truncations define nonsingular tropical hypersurfaces in the corresponding tropical tori, or equivalently, if its dual polyhedral subdivision is a primitive triangulation.

Consider now a collection $(f_1, \dots, f_k)$ of tropical polynomials in $\mathbb{R}[x_1, \dots, x_n]$, or more generally in $\mathbb{R}[M]$ with $M \simeq \mathbb{Z}^n$. Let Δ_i be the Newton polytope of f_i. Define the associated tropical Cayley polynomial $F \in \mathbb{R}[M \oplus \mathbb{Z}^k]$ by

$$F(x, y) = \sum_{i=1}^{k} y_i f_i(x). \tag{7.1}$$

where the operation are the tropical ones. Its Newton polytope is the associated Cayley polytope $C(\Delta_1, \dots, \Delta_k)$. We have the following analogue of the classical definition 2.1.

Definition 7.3. The collection $(f_1, \dots, f_k)$ of tropical polynomials is *nondegenerate* if the associated Cayley polynomial F is nondegenerate which means that the dual polyhedral subdivision of $C(\Delta_1, \dots, \Delta_k)$ is a primitive triangulation.

Recall that a collection $(\Gamma_i)_{i \in I}$ of faces of $\Delta_1, \dots, \Delta_k$ (where Γ_i is a face of Δ_i) is called admissible if $I \subset \{1, \dots, k\}$ and $\Gamma_I = \sum_{i \in I} \Gamma_i$ is face of $\Delta_I = \sum_{i \in I} \Delta_i$. The faces of $C(\Delta_1, \dots, \Delta_k)$ are exactly the Cayley polytopes of the admissible collections $(\Gamma_i)_{i \in I}$. Since a primitive triangulation of a polytope induces primitive triangulations of its faces, it follows that if $(f_1, \dots, f_k)$ is nondegenerate, then for any admissible collection $(\Gamma_i)_{i \in I}$ of faces of $\Delta_1, \dots, \Delta_k$, the collection of tropical polynomials $(f_i^{\Gamma_i})_{i \in I}$ is also nondegenerate. Denote by Z_i the hypersurface defined by f_i. If Γ_i is a face of Δ_i, we will denote by Z_{i,Γ_i} the tropical hypersurface in $N(\Gamma_i)_{\mathbb{R}}$, or in $N_{\mathbb{R}}$, defined by the truncation of f_i to Γ_i. The next result is the tropical analogue of Proposition 2.2.

Proposition 7.4. The collection $(f_1, \dots, f_k)$ of tropical polynomials is nondegenerate if and only if for any admissible collection $(\Gamma_i)_{i \in I}$ of faces of $\Delta_1, \dots, \Delta_k$ the hypersurfaces Z_{i,Γ_i} have only transversal intersections each with intersection multiplicity number 1.

Proof. If $(f_1, \dots, f_k)$ is nondegenerate, then the corresponding convex polyhedral subdivision of the Cayley polytope $C(\Delta_1, \dots, \Delta_k)$ is a primitive triangulation, and thus the corresponding convex mixed subdivision $\mathcal{S}$ of $\Delta = \Delta_1 + \cdots + \Delta_k$ is tight. In particular, the hypersurfaces $Z_1, \dots, Z_k$ have only transversal intersections at cells ξ dual to the direct sums of simplices

$$\sigma = \sigma_1 \oplus \cdots \oplus \sigma_k \in \mathcal{S}$$

such that $d_i := \dim \sigma_i \geq 1$ for $i = 1, \dots, k$. According to Lemma 6.1, the intersection multiplicity number $w(\xi)$ of $Z_1, \dots, Z_k$ along such a cell ξ verifies $w(\xi) = \mathrm{vol}_{d+k-1}\left(C(\sigma_1, \dots, \sigma_k)\right)$, where $d = \dim \sigma = d_1 + \cdots + d_k$. But $C(\sigma_1, \dots, \sigma_k)$ is a primitive simplex, thus $w(\xi) = 1$. The same arguments work for any admissible $(\Gamma_i)_{i \in I}$ since if $(f_1, \dots, f_k)$ is nondegenerate then $(f_i^{\Gamma_i})_{i \in I}$ is nondegenerate too. This show one implication of Proposition 7.4, let us show the reverse one.

Clearly, if for any admissible collection $(\Gamma_i)_{i \in I}$ of faces of $\Delta_1, \dots, \Delta_k$ the hypersurfaces Z_{i,Γ_i} have only transversal intersections, then the mixed subdivision $\mathcal{S}$ of $\Delta = \Delta_1 + \cdots + \Delta_k$ is pure. Consider a full dimensional polytope in the polyhedral subdivision of $C(\Delta_1, \ldots, \Delta_k)$. It may be written as a Cayley polytope $C(\sigma_1, \ldots, \sigma_k)$ for some $\sigma = \sigma_1 \oplus \cdots \oplus \sigma_k \in \mathcal{S}$ with $\dim \sigma = \dim \Delta$. Set as above $d_i = \dim \sigma_i$ and $d = \dim \sigma = \dim \Delta$. Set $I = \{i \in \{1, \ldots, k\}, d_i \neq 0\}$. Then σ is dual to a cell ξ of the common intersection of the hypersurfaces Z_i for $i \in I$. Moreover, the intersection multiplicity number between these hypersurfaces along ξ is $\mathrm{vol}_{d_I+k-1}(C(\sigma_i, i \in I))$, where $C(\sigma_i, i \in I)$ is the Cayley polytope associated with σ_i for $i \in I$ and d_I is the dimension of $\sum_{i \in I} \sigma_i$. This Cayley polytope lies on the face $C(\Delta_i, i \in I)$ of $C(\Delta_1, \ldots, \Delta_k)$. One can check that $\mathrm{vol}_{d+k-1}(C(\sigma_1, \ldots, \sigma_k)) = \mathrm{vol}_{d_I+k-1}(C(\sigma_i, i \in I))$. Thus both members are equal to 1 and it follows that $C(\sigma_1, \ldots, \sigma_k)$ is a primitive simplex. □

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