MathAbs: A Representational Language for Mathematics

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Outline

1. **Background and Context**
2. MathAbs
3. Final Remarks

Background and Context

Wide **gap** between **the language of mathematics** used in textbooks and the mathematics **formalized** in various theorem provers

The language of mathematics

- Universally accepted by mathematicians.
- Mixture of natural language, symbolic expressions and notations.
- Flexible, structured and semantically well understood.
- In principle, formalization of mathematics is possible.

Theorem (Pythagoras' Theorem). $\sqrt{2}$ *is irrational.*

Proof. If $\sqrt{2}$ is rational, then the equation $a^2 = 2b^2$ is soluble in integers a, b with $gcd(a, b) = 1$. Hence a^2 is even, and therefore a is even. If $a = 2c$, then $4c^2 = 2b^2$, $2c^2 = b^2$, and b is also even, contrary to the hypothesis that $gcd(a, b) = 1$.

Introduction to the Theory of Numbers. Hardy & Wright

Current State of art of Formal Mathematics

The mathematical texts are manually formalized in very precise and accurate systems

Formalization with Proof Assistants

In Coq:

```
Theorem irrationalRsqrt2:(irrational (sqrt (S (S O )))).
Proof. Intros p q H; Red; Intros H0; Case H. Apply (main_thm p). Replace
(Div2.double (mult q q)) with (mult (S (S O)) (mult q q));[Idtac | Unfold
Div2.double; Ring].
```

In Isabelle:

corollary `sqrt (real (2::nat))` $\notin Q$ by `(rule sqrt-prime-irrational) (rule two-is-prime)` ......

theorem `sqrt-prime-irrational`: assumes `prime p` shows `sqrt (real p)` $\notin Q$

proof from ⟨`prime p`⟩ have $p : 1 < p$ by `(simp add:prime-def)` ......

- Doesn't reflect mathematical style of writing
- Difficult to understand and learn
- Mathematicians are not interested in such assistants and formalizations

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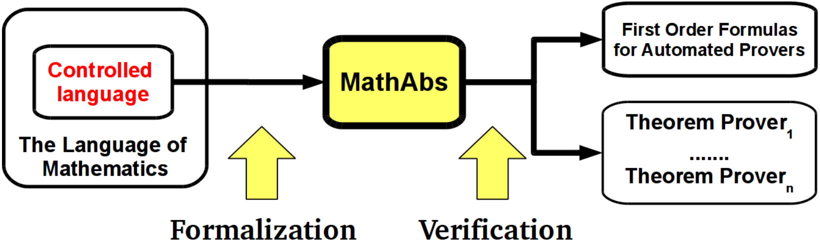
- – Doesn't reflect mathematical style of writing
- – Difficult to understand and learn
- – Mathematicians are not interested in such assistants and formalizations
- + A proof can be trusted
- + Can help to solve very difficult theorems, requiring a lot of steps and computation e.g. four color theorem

The BIG Question

Can we build a program that automatically formalizes textbook mathematics and can we mechanically verify the correctness?

MathNat Project

- Define a small subset of mathematical English with some rich linguistic features.
- **A formal language MathAbs for mathematical texts that can keep the structure of the natural language text.**
- A software translating the first to the second.
- A proof-checker for MathAbs.



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MathAbs: Abstract Mathematical Language

- Attempts to faithfully represent mathematical texts:
  - Preserving its line of reasoning, proof steps and logical structure.
- MathAbs≡Mathematical texts without linguistic features/problems.
- Not based on any particular logic, theory or prover.
- Intermediary between the language of mathematics and the formal language of theorem provers.
- Intended only for machine manipulation.

Why MathAbs?

- Reasoning gaps in mathematical proofs
  - To be concise, obvious parts can are exempted
    - Many reasoning steps can be performed in fewer steps
  - For the purpose of explanation, fewer steps can be performed in many steps
- Logical systems (sequent calculus, natural deduction,...) are not so natural for natural language proofs.
- Mathematicians use these rules freely but implicitly.

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- Logical systems (sequent calculus, natural deduction,...) are not so natural for natural language proofs.
- Mathematicians use these rules freely but implicitly.
- As mathematicians do, MathAbs allows to do:
  - Many steps of reasoning in one step.
  - One step of reasoning in many steps for explanation.

The MathAbs

MathAbs can represent **theorems** and their **proofs** along with supporting **axioms** and **definitions**.

The MathAbs for Describing Proofs

- A quite small grammar: `let`, `assume`, `show`, `deduce`, `trivial`, `{Proof; ...}`, •

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Adding hypotheses in the context:

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  "let *x* and *y* be integers" ⇒ `let x,y:Z`

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  "suppose that *x* is positive" ⇒ `assume pos(x)`

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  "By the definition of rational numbers, we can assume that $\sqrt{2} = \frac{a}{b}$ **where *a* and *b* are non-zero integers**"
  $\Rightarrow$ `let a,b:Z assume positive(a)` $\wedge$ `positive(b)`
  `assume` $\sqrt{2}$`=a/b by def rational_Number`

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- `{Sub-Proof; Sub-Proof;...}`

Adding a deduction in the context

- `deduce formula (hint) proof`
  "we conclude that $x > 10$" $\Rightarrow$ `deduce x>0`
  `{show x>0 trivial; assume x>0 ...}`

  "a^2 is even because it is a multiple of 2"
  $\Rightarrow$ `deduce even(`a^2`) by form multiple_of(`a^2`,2)`

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- `trivial`, •
- Goal: $A \rightarrow B \Rightarrow$ `assume A show B`

MathAbs: example

- Describes a proof as tree of logical (meta) rules
- Each proof step changes the sequent

Theorem. If x and y are two even integers then $x + y$ is even.
Proof. By the definition of even numbers, $x + y = 2a + 2b$ holds.
We deduce that $x + y = 2(a + b)$ by the last statement.
We conclude that $x + y$ is an even integer because it is a multiple of 2.
This concludes the proof.

Theorem. let `x,y:Integer` assume even(x) ∧ even(y) show even(x+y)•
Proof. let `a,b:NoType` deduce x+y=2a+2b by def `Even_Numbers`•
deduce x+y=2(a+b) by Form x+y=2a+2b•
show even(x+y) by Form `multiple_of`([x+y],2)•
trivial•

$$\dfrac{\dfrac{\dfrac{}{\Gamma_2 \vdash x+y=2a+2b} \qquad \dfrac{\dfrac{}{\Gamma_3 \vdash x+y=2(a+b)} \qquad \dfrac{\dfrac{}{\Gamma_4 \vdash \text{multiple\_of}([x+y],2)} \qquad \dfrac{\vdots}{\Gamma_5 \equiv \Gamma_4, \ldots}}{\Gamma_4 \equiv (\Gamma_3,\ x+y=2(a+b)) \vdash \text{even}(x+y)}}{\Gamma_3 \equiv (\Gamma_2,\ x+y=2a+2b) \vdash \text{even}(x+y)}}{\Gamma_2 \equiv (\Gamma_1, a,b{:}\texttt{NoType}) \vdash \text{even}(x+y)}}{\Gamma_1 \equiv (\Gamma_0,\ x,y{:}\text{Integer} \wedge \text{even}(x) \wedge \text{even}(y)) \vdash \text{even}(x+y)}$$

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Coverage and Extensibility

- The language of axiom, definition, theorem and proof is universal.
- So these languages in MathAbs need not to be extensible.

On the other hand:

- The language for statement is domain dependent.
- So the language of `formula` need to be extensible.

Conclusion

- A detailed account of MathAbs, its formal definition and semantics.
- A translation of natural deduction rules in MathAbs to establish its completeness.
- MathAbs tested so far for: elementary number theory, algebra and analysis.
- In principle, MathAbs abstraction seems adequate and promising for mathematical texts.
- But we can only be 100% sure after proof checking of MathAbs.

Questions

Questions?

MathNat homepage:
`www.lama.univ-savoie.fr/~humayoun/phd/mathnat.html`