ASYMPTOTIC BEHAVIOUR OF OPTIMAL SPECTRAL PLANAR DOMAINS WITH FIXED PERIMETER

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Abstract. We consider the problem of minimizing the k^{th} Dirichlet eigenvalue of planar domains with fixed perimeter and show that, as k goes to infinity, the optimal domain converges to the ball with the same perimeter. We also consider this problem within restricted classes of domains such as $n-$polygons and tiling domains, for which we show that the optimal asymptotic domain is that which maximises the area for fixed perimeter within the given family, i.e. the regular $n-$polygon and the regular hexagon, respectively.

1. Introduction

Consider the eigenvalue problem for the Dirichlet Laplacian given by

$$\begin{array}{rl} -\Delta u = \lambda u & x \in \Omega \\ u = 0 & x \in \partial\Omega \end{array}, \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^n$. The isoperimetric structure of low eigenvalues of problem (1) is a classical topic in shape optimisation which may be traced back to Rayleigh's book *The Theory of Sound* [R] and, more specifically, to his conjecture that the disk should minimise the first Dirichlet eigenvalue of the Laplacian among domains of fixed area. This conjecture was proven by Faber [F] and Krahn [K1, K2] in the early 1920's and since then a lot of effort has been made to extend it to higher eigenvalues. However, in nearly a century no progress has been made along this direction with the minimisation of λ_3 with a volume restriction still an open problem, even in two dimensions where the minimiser has long been conjectured to be the disk.

Furthermore, recent numerical work indicates that for eigenvalues of planar domains higher than the fourth, the optimal shape under an area restriction will not have a boundary which may be described in terms of *known* functions [AF1, O]. Also, the optimal domains obtained in [AF1] for the first fifteen eigenvalues suggest that any underlying structure that one might expect there to exist, such as optimal sets having at least $\mathbb{Z}_2$ symmetry, will most likely be up against some exceptions.

The above should not give the impression that no progress at all has been made. In fact, quite the opposite is true and since the appearence of Faber's and Krahn's papers much work has been done extending the theory in other directions including

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the counterparts of the classical results to other boundary conditions and higher order operators, for instance.

This paper has at its origin two different problems which were considered in the literature recently. The first consists in the minimisation of the k^{th} eigenvalue of problem (1) with a restriction on the perimeter of Ω – by perimeter of a measurable set Ω we mean the $(N-1)-$Hausdorff measure of the reduced boundary of Ω (see [AFP, Definitions 2.46 and 3.54]); for simplicity, we shall use the generic notation $|\partial\Omega|$ for $\mathcal{H}^{n-1}(\partial^*\Omega)$. More precisely, we want to solve the problem

$$\lambda_k^* = \min\left\{\lambda_k(\Omega) : \Omega \subset \mathbb{R}^n,\ |\partial\Omega| = \alpha\right\}. \tag{2}$$

As far as we are aware, this problem was first studied in [BBH], where existence and properties of the optimal domain for the second eigenvalue were determined. While for the first eigenvalue the solution to this problem will still be the ball, as a consequence of the Faber-Krahn and isoperimetric inequalities, in the case of λ_2 the optimal domain in two dimensions will in fact be connected and not coincide with the solution in the case of an area restriction. A more involved study of problem (2) can be found in [BI] where, in particular, existence of a solution is proved in $\mathbb{R}^2$ and a series of qualitative properties of minimizers are given.

The second problem consists in determining the asymptotic behaviour of optimisers of this type of spectral problems. Denoting by Ω_k^* an optimiser of problem (2), we are interested in knowing if the limit of this sequence of optimal domains does in fact exist in some appropriate sense and, if so, what the limitting domain is. Whenever such domain exists, we shall denote it by Ω_∞^*. In the case of fixed area, it was recently shown in [AF2] that the sequence of optimal rectangles converges to the square as k goes to infinity. This problem is equivalent to determining the optimal ellipse which contains k integer lattice points in the first quadrant (excluding the axes), and the result obtained implies that, asymptotically, this is achieved by the circle. Again, we believe that this was the first time where a problem of this type was addressed.

A crucial step in the proof of convergence of optimal rectangles to the square in [AF2], was to show that the perimeter of the optimal sequence remained uniformly bounded in k. This makes it natural to consider what happens in the case where the perimeter is uniformly bounded a priori, such as in problem (2) above. In fact, in this case we are able to give a complete answer to this problem within the class of general planar domains, where we show that the asymptotic optimal domain is the disk.

Theorem 1. *The sequence Ω_k^* of optimal planar domains with fixed perimeter converges as k goes to infinity and $\Omega_\infty^* = D$, where D denotes the disk with perimeter α.*

Remark 1.1. The geometric convergence above is understood in the sense that the characteristic functions converge in L^1, i.e.

$$1_{\Omega_k^*} \stackrel{L^1(\mathbb{R}^2)}{\longrightarrow} 1_{\Omega_k^*}.$$

Meanwhile, since in Theorem 1 the sets Ω_k^* are convex, the convergence also holds in a geometric sense, given by the Hausdorff distance d_H

$$d_H(\Omega_k^*, \Omega_\infty^*) \to 0,$$

and

$$d_H(\mathbb{R}^2 \setminus \Omega_k^*, \mathbb{R}^2 \setminus \Omega_\infty^*) \to 0.$$

Remark 1.2. In higher dimensions there are difficulties of different types which, for the moment, we are not able to overcome. Firstly, it has not yet been proven that problem (2) has a solution for $k \geq 4$. Although the existence question seems quite reasonable and several hints lead to this conclusion, one cannot yet ensure the existence of such an optimal sequence. Moreover, convexification in dimension greater than 2 does not necessarily decrease the perimeter of a set. This also means that one cannot prove equi-boundedness of the suspected minimizers of problem (2). By comparison, in the case of a measure constraint, the best bound obtained for the diameter so far depends on k, although it is not known if it is uniformly bounded independently of k (see [Bu, MP]). Nevertheless, in the absence of a uniform bound on diameters, there will be a new difficulty for the construction of the limit of solutions of (2). In particular, there may possibly be a sort of vanishing of the sequence without any concentration part. Also, in the absence of convexity, the argument we use below to uniformly compare the k-th eigenvalue on Ω_k^* with the k-th eigenvalue on the presumable limit is a difficult task for varying k.

It is possible to consider the same problem within other families of domains, of which a natural case is that of polygons with n sides with the same perimeter, for which again the set which is optimal for the geometric isoperimetric inequality in that class is the limiting domain.

Theorem 2. *Let $\mathcal{P}_n$ denote the family of $n-$sided planar polygons and consider the minimisation problem*

$$\lambda_k^* = \min\{\lambda_k(P) : P \in \mathcal{P}_n,\ |\partial P| = \alpha\} \tag{3}$$

Then the sequence P_k^* of optimal $n-$polygons with fixed perimeter converges as k goes to infinity and $P_\infty^* = P_n^{reg}$, where P_n^{reg} denotes the regular $n-$polygon with perimeter α.

Remark 1.3. The geometric convergence above holds in the same way as in Remark 1.1, since the sets P_k^* are convex.

We remark that in this case, and with the exception of triangles and quadrilaterals, the analogue of the Faber-Krahn inequality is still open. More specifically, the fact that the equilateral triangle (resp. the square) minimises the first Dirichlet eigenvalue among triangles (resp. quadrilaterals) of given area was proven in [PS], where it was also conjectured that the corresponding optimal domain among the set of $n-$polygons is the regular $n-$polygon with the same area. The optimiser with a perimeter restriction is not known either.

Finally, we shall consider a family of domains of a different type, namely, planar tiling domains, still with a perimeter restriction, for which a similar type of result may be shown.

Let $\mathcal{T}_\alpha$ denote the family of planar tiling domains (open sets, not necessarily connected) satisfying the following properties:

P_1: $\mathbb{R}^2 \setminus T$ is connected;

P_2: $\min\{\mathcal{H}^1(K) : K \text{connected}, \partial T \subset K\} \leq \alpha$.

Remark 1.4. The constraints P_1 and P_2 above ensure the existence of an optimal tiling domain in the class $\mathcal{T}_\alpha$ for problem (4) below. Clearly this includes the connected and simply connected tilling domains with a perimeter less than or equal to α.

Theorem 3. *Let consider the minimisation problem*

$$\lambda_k^* = \min\{\lambda_k(T) : T \in \mathcal{T}_\alpha\} \tag{4}$$

Then the sequence T_k^* of optimal tiling domains with fixed perimeter converges as k goes to infinity and $T_\infty^* = P_6^{reg}$, that is, the regular hexagon with the same perimeter.

Remark 1.5. In Theorem 3, the convergence is slightly weaker, since it holds in the sense of characteristic functions and only the complements converge in the Hausdorff metric

$$d_H(\mathbb{R}^2 \setminus T_k^*, \mathbb{R}^2 \setminus T_\infty^*) \to 0.$$

A remark similar to that following Theorem 2 above also holds here, in that the minimiser for the first eigenvalue of tiling domains either with fixed area or perimeter is not known.

2. General domains

The existence of Ω_k^* relies on the following convexification argument (see [BI, Theorem 1]): if Ω is open, then there exists a convex set with lower perimeter and lower eigenvalues. Indeed, if Ω is a union of at most k pairwise disjoint open sets such that $\partial\Omega$ is connected, by convexification, the perimeter decreases. From the monotonicity of eigenvalues with respect to inclusions, the new domain has no larger eigenvalues. In the family of convex sets of prescribed perimeter, the existence of Ω_k^* is based on the compactness properties of the Hausdorff metric and the good behaviour of the eigenvalues on moving convex sets.

For general domains in $\mathbb{R}^2$ we have [Be, LY]

$$\frac{2k\pi}{|\Omega|} \leq \lambda_k(\Omega)$$

which, when applied to optimal domains, yields

$$\frac{2k\pi}{|\Omega_k^*|} \leq \lambda_k(\Omega_k^*) \leq \lambda_k(\Omega) = \frac{4k\pi}{|\Omega|} + \mathrm{o}(k),$$

where Ω is any (fixed) domain. Dividing by k and taking $\limsup$ throughout we obtain

$$\liminf_{k\to+\infty} |\Omega_k^*| \geq \frac{1}{2}|\Omega|$$

for any planar domain Ω with perimeter α. In particular, this implies that the sequence of optimisers cannot degenerate to a one dimensional set (of zero area) and, if we take Ω to be the disk D with perimeter α, we obtain

$$|\Omega_\infty^*| \geq \frac{1}{2}|D| = \frac{\alpha^2}{8\pi}.$$

Relying on the compactness theorem of convex sets in the Hausdorff metric (see for instance [HP, Theorem 2.4.10]), there exists a closed convex set F and a subsequence (of translations of) $\Omega_{k_l}^*$ converging in the Hausdorff metric to F. From the arguments above, the interior of the set F is non-empty. If we denote it by Ω_∞^*, we notice that the perimeter and the measure of Ω_∞^* are limits of the corresponding quantities associated to $\Omega_{k_l}^*$.

We thus have that, for all positive δ there exists $m = m(\delta)$ such that, for all k greater than or equal to $m(\delta)$, $\Omega^*_{k_l} \subset \Omega^*_\infty + B_\delta$, where B_δ is a disk of radius δ. In other words, for sufficiently large l, depending on δ, all optimal domains will be in a $\delta-$neighbourhood of the limitting optimal domain Ω^*_∞ (modulus a rigid planar motion). We thus have

$$\lambda_{k_l}(\Omega^*_\infty + B_\delta) \leq \lambda_{k_l}(\Omega^*_{k_l}) \leq \lambda_{k_l}(D), \tag{5}$$

where the first inequality follows from the inclusion above. Dividing by k_l and taking limits as l goes to infinity yields

$$\frac{4\pi}{|\Omega^*_\infty + D_\delta|} \leq \frac{4\pi}{|D|}.$$

Since we may take δ to be arbitrarily small, it follows that $|D| \leq |\Omega^*_\infty|$ and, by the geometric isoperimetric inequality, Ω^*_∞ must be the disk D.

The argument above shows only that the subseqence $(\Omega^*_{k_l})_l$ converges to the disk. As the limit is independent on the choice of the subsequence, we conclude that the full sequence $(\Omega^*_k)_k$ converges to the disk.

3. $n-$POLYGONS

The proof follows the main lines of the case of general domains. Let us first discuss the existence of P^*_k in (3). We begin by introducing the quantity

$$c^*_k = \min\left\{\lambda_k(P) : \exists\, l = 3, .., n, P \in \mathcal{P}_l,\ |\partial P| \leq \alpha\right\}. \tag{6}$$

Clearly, for every $P \in \mathcal{P}_n$ its convexification $\operatorname{conv} P$ belongs to the set above, so that from the monotonicity of eigenvalues with respect to inclusions,

$$\lambda^*_k \geq c^*_k.$$

The existence of a minimizer P^* in (6) follows by the compactness of the family of polygons with the number of edges less than or equal to k in the Hausdorff metric and the good behavior of the eigenvalues for convergence of convex sets.

The fact that P^* belongs to $\mathcal{P}_n$ comes from its optimality. If the number of sides was less than n, then cutting a vertex and introducing a new small edge of size ϵ would decrease the perimeter with an equivalent of ϵ and increase the eigenvalue with $o(\epsilon^2)$. Finally, the perimeter constraint is satisfied, otherwise one can dilate the optimal domain.

Consequently, $\lambda^*_k = c^*_k$ and the minimizer P^* is convex, belongs to $\mathcal{P}_n$ and satisfies the perimeter constraint. For the rest of the proof, one now follows the same steps as for the general case above.

4. Tiling domains

The first natural question is the existence of T^*_k for which in (4) we have $\lambda_k(T^*_k) = \lambda^*_k$. First of all, the competing tiling domains in $\mathcal{T}_\alpha$ have a diameter less than or equal to α. Thus, taking a minimizing sequence $(T_n)_n$ in (4), by the compactness of the Hausdorff metric, there exists an open set T such that (possibly extracting a subsequence and making translations) we have that

$$\mathbb{R}^2 \setminus T_n \xrightarrow{H} \mathbb{R}^2 \setminus T.$$

Relying on the properties of the Hausdorff convergence, we get in the limit a tiling domain $T \in \mathcal{T}_\alpha$ for which 1_{T_n} converges in L^1 to 1_T, and in particular

$$|T| = \lim_{n\to+\infty} |T_n|.$$

The L^1 convergence above is a consequence of the uniform bound on the perimeter in the sense stated by hypothesis P_2 above.

Moreover, since $\mathbb{R}^2 \setminus T_n$ are connected, by Sverak's [S] result one gets that

$$\lambda_k^* = \lambda_k(T) = \lim_{n\to+\infty} \lambda_k(T_n),$$

in particular this means that $|T| > 0$, thus we can set $T := T_k^*$.

In the case of tiling domains, Pólya's inequalty yields that for any domain T in $\mathcal{T}$ the k^{th} eigenvalue satisfies

$$\frac{4k\pi}{|T|} \leq \lambda_k(T).$$

Thus if in this case we proceed as in the previous sections we obtain directly

$$\liminf_{k\to+\infty} |T_k^*| \geq |T|.$$

From [H] we know that among all tiling domains with fixed perimeter the one with the largest area is the regular hexagon. By taking in the above inequality T to be the regular hexagon with perimeter α the result follows.

Note that the argument above is definitely different from the case of general domains or polygons. While in those situations, the convexity of the optimal domains was crucial for comparing the k-th eigenvalue on the k-th optimal domain and the k-th eigenvalue on the limit domain, for tiling domains, this comparison is difficult since no *a priori* information is known on the geometries. Simply connectedness is not enough to get uniform comparisons similar to (5) and the main point here is Pólya's inequalty together with Weyl's asymptotic formula.

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