Characterization of the shape stability for nonlinear elliptic problems

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Abstract

We characterize all geometric perturbations of an open set, for which the solution of a nonlinear elliptic PDE of p-Laplacian type with Dirichlet boundary condition is stable in the L^∞-norm. The necessary and sufficient conditions are jointly expressed by a geometric property associated to the γ_p-convergence.

If the dimension of the space N satisfies $N-1 < p \leq N$ and if the number of the connected components of the complements of the moving domains are uniformly bounded, a simple characterization of the uniform convergence can be derived in a purely geometric frame, in terms of the Hausdorff complementary convergence. Several examples are presented.

1 Introduction

Let $N \geq 2$, $p \in (1, N]$, $\lambda \geq 0, \varepsilon > 0$ and $D \subseteq \mathbb{R}^N$ be a bounded open set. For some $f \in L^{\frac{N}{p}+\varepsilon}(D)$ and for every open set $\Omega \subseteq D$ we consider the following equation set in the sense of distribution set on Ω

$$\begin{cases} -\Delta_p u + \lambda |u|^{p-2} = f & \text{in } \Omega \\ u \in W_0^{1,p}(\Omega) \end{cases} \tag{1}$$

This equation has a unique solution denoted $u_{\Omega,f}$ which, from the choice of f, also belongs to $L^\infty(\Omega)$. Extended by zero on $D \setminus \Omega$, this solution can be seen as an element of $W_0^{1,p}(D) \cap L^\infty(D)$. The question we are concerned within this paper is to characterize the *convergence* of a sequence $(\Omega_n)_n$ towards Ω, such that

$$u_{\Omega_n,f} \longrightarrow u_{\Omega,f} \quad \text{in} \ \ L^\infty(D), \tag{2}$$

namely to identify all perturbations of an open set Ω for which the solution of (1) is stable into the L^∞-norm.

The convergence of solutions into the energy space, i.e. $u_{\Omega_n,f} \longrightarrow u_{\Omega,f}$ in $L^p(D)$, is related to the γ_p-convergence of the geometric domains (see [4, 8] and Definition 2.2 in Section 2) which can be characterized in terms of the local behavior in capacity of Ω_n^c. We refer to the pioneering paper of Dal Maso [8] for the main study and description of the γ_p-convergence via Γ-convergence methods and to [4] for a discussion of the same topic using tools of potential theory. In concrete situations, understanding whether a given sequence of domains γ_p-converges or not may be a complicated question. Nevertheless, different results obtained in the past years give a quite large number of sufficient conditions for the γ_p-convergence (see for instance [5]).

The convergence of solutions in $L^\infty(D)$ being stronger than the convergence in $L^p(D)$, the γ_p-convergence appears to be a necessary condition for (2). As simple examples show, and because $W_0^{1,p}(D)$ is not embedded into $L^\infty(D)$, the γ_p-convergence is not sufficient for (2). Since $L^\infty(D)$ is not the natural energy space, any approach based on the Γ or Mosco convergences fails to work. The missing step from the γ_p to the L^∞-convergence of solutions concerns only a purely geometric behavior of the moving sets. This geometric property (which turns out to be also a necessary condition) provides the key result for getting locally uniform oscillations of the solutions near the moving boundaries.

Assuming that Ω_n and Ω are regular in the sense of Wiener, the functions $u_{\Omega_n,f}, u_{\Omega,f}$ are continuous on D. For $p = 2$ the question of studying the uniform convergence $u_{\Omega_n,f} \to u_{\Omega,f}$ was raised by Arendt and Daners in [2], where they give a set of sufficient conditions on the convergence of domains which ensure the uniform convergence of solutions. In the particular case in which all Ω_n are contained in Ω, those conditions are also sufficient. Recent developments, still in the case $p = 2$, can be found in [3]. Here the authors make an extensive study of the L^∞-convergence of solutions and give a set of necessary and sufficient conditions for the uniform convergence under the hypothesis that Ω is stable in the sense of Keldysh. Although Keldysh stability does not require smoothness, this hypothesis excludes a quite large class of open sets, as for example domains with cracks.

In this paper we give a characterization of the geometric convergence of domains for which the solutions convergence in $L^\infty(D)$. The only assumption we made concerns the limit set Ω, which is required to be p-Wiener regular at every point of its boundary. This is the minimal constraint under which uniform convergence can be expected for non-smooth perturbations. Indeed, if this condition is dropped, then the sequence of increasing domains $(\Omega \setminus \overline{B}(x_0, \frac{1}{n}))_n$ would not give the L^∞-convergence of the solutions (see [2] for the necessity of the Wiener criterion to have uniform shape stability for increasing sequences). The necessary and sufficient conditions given in this paper are jointly expressed by a local capacity behavior of Ω_n^c (which is related to the γ_p-convergence) and a purely geometric condition. If Ω_n are also regular in the sense of Wiener, the $L^\infty(D)$-convergence becomes uniform convergence.

From a practical point of view, two consequences can be noticed. If $N - 1 < p \le N$ and if the number of the connected components of Ω_n^c is uniformly bounded, then we can give a simple characterization for the L^∞-stability of the solutions if the domains converge in the (compact) Hausdorff complementary topology. This is mainly possible relying on the generalization of Šverák's result obtained in [6] for p-Laplacian type operators. As a second consequence, we characterize all sets which are L^∞-stable for the so called compact convergence, i.e. we discuss the Keldysh like stability into the L^∞-norm of the solutions. We recover into a non-linear frame the result of [3]. An open set is L^∞-stable if and only if it is stable in the sense of Keldysh and p-Wiener regular at every point of its boundary.

Notice that the cases $N = 1$, $1 < p < +\infty$ and $N \geq 2$, $N < p < +\infty$ are not of interest, since the Sobolev space $W_0^{1,p}(D)$ is embedded in a Hölder space $C^{0,\alpha}(\mathbb{R}^N)$. Consequently, uniform convergence of solutions holds as soon as the geometric domains converge in the Hausdorff complementary topology (which is compact). Together with the fact that every point has positive p-capacity, this gives a complete characterization of the uniform shape stability. This is the reason why, throughout the paper we consider only the case $N \geq 2$ and $1 < p \leq N$.

For simplicity, we present our results for the p-Laplace operator, but most of the results extend without any modifications of the proofs to more general elliptic equations of the form $-\text{div } \mathcal{A}(x, \nabla u_\Omega) + \mathcal{B}(x, u_\Omega) = 0$, with non-homogeneous Dirichlet boundary conditions. The operator $\mathcal{A}$ is similar to the p-Laplacian and $\mathcal{B}$ satisfies the usual Carathéodory and monotonicity assumptions (see Section 5 and [20, 21]). In order to have solutions in $W^{1,p}(\Omega) \cap L^\infty(\Omega)$, the most important assumption is a boundedness hypothesis on $\mathcal{B}$ by a function belonging to the Morrey space $\mathcal{M}^{N/(p-\varepsilon)}(D)$ (see for instance [21, Chapter 3]).

2 Preliminary results

In what follows, we always denote Ω an open set in $\mathbb{R}^N$ and by $\Omega^c = \mathbb{R}^N \setminus \Omega$ its complement.

The Sobolev capacity of a set $E \subseteq \mathbb{R}^N$ is

$$\text{cap}_p(E) = \inf\{\int_{\mathbb{R}^N} |\nabla\varphi|^p + |\varphi|^p dx \mid \varphi \in W^{1,p}(\mathbb{R}^N), E \subseteq \{\varphi \geq 1\}^o\}.$$

For $x \in \mathbb{R}^N, r > 0$ and a set E such that $E \subseteq B(x, r)$, the condenser capacity of E in the ball $B(x, r)$ is:

$$\text{cap}_p(E, B(x, r)) = \inf\{\int_{B(x,r)} \mid \nabla\varphi \mid^p \, dx \ \mid \varphi \in W_0^{1,p}(B(x, r)), E \subseteq \{\varphi \geq 1\}^o\}.$$

A function $u : \Omega \to \mathbb{R}$ is said to be p-quasi continuous if for all $\varepsilon > 0$ there exists an open set $G_\varepsilon \subseteq \Omega$ with $\text{cap}_p(G_\varepsilon) < \varepsilon$ such that the restriction $u|_{\Omega \setminus G_\varepsilon}$ is continuous on $\Omega \setminus G_\varepsilon$. A property is said to hold p-quasi everywhere (written p-q.e.) if it holds in the complement of a set of zero p-capacity.

We refer the reader to [15, 21] for an extensive presentation of properties of capacities in relation with Sobolev spaces. We only recall that every function $u \in W_0^{1,p}(\Omega)$ has a p-quasi continuous representative, which is unique up to a set of p-capacity zero. We also recall the following characterization of the $W_0^{1,p}$-spaces (see the paper of Hedberg [14] or [15]).

Lemma 2.1 *Let $\Omega \subseteq \mathbb{R}^N$ be an open set. If $u \in W^{1,p}(\mathbb{R}^N)$ then $u \in W_0^{1,p}(\Omega)$ if and only if $u = 0$ p-q.e. on Ω^c.*

Throughout the paper, $W_0^{1,p}(\Omega)$ is seen as a subspace in $W^{1,p}(\mathbb{R}^N)$, the embedding mapping being the extension by zero on Ω^c.

Let D be a bounded open set of $\mathbb{R}^N$.

Definition 2.2 *It is said that a sequence $(\Omega_n)_n$ of open subsets of D γ_p-converges to an open set Ω if*

$$\forall \lambda \geq 0,\ \forall f \in W^{-1,p'}(D) \quad u_{\Omega_n,f} \to u_{\Omega,f} \text{ stronlgy in } W_0^{1,p}(D).$$

Here $p' = p/(p-1)$.

We refer the reader to [5] for detailed presentation of the γ_p convergence. We recall (see also [4], [8]) the following characterization of the γ_p-convergence in terms of the local behavior in capacity of the moving domains.

Theorem 2.3 *A sequence (Ω_n) of open subsets of D γ_p-converges to an open set Ω if and only if $\forall x \in \mathbb{R}^N, \forall r > 0$ the following two conditions hold*

$$\operatorname{cap}_p(\Omega^c \cap \overline{B}_{x,r}, B_{x,2r}) \geq \limsup_{n\to\infty} \operatorname{cap}_p(\Omega_n^c \cap \overline{B}_{x,r}, B_{x,2r}). \tag{3}$$

$$\operatorname{cap}_p(\Omega^c \cap B_{x,r}, B_{x,2r}) \leq \liminf_{n\to\infty} \operatorname{cap}(\Omega_n^c \cap B_{x,r}, B_{x,2r}). \tag{4}$$

We recall from [4] that (3) and (4) are equivalent with the first and the second Mosco conditions, respectively:
1. $\forall \varphi \in W_0^{1,p}(\Omega)$ $\exists$ $\varphi_n \in W_0^{1,p}(\Omega_n)$ such that $\varphi_n \to \varphi$ strongly in $W_0^{1,p}(D)$;
2. $\forall \varphi_{n_k} \in W_0^{1,p}(\Omega_{n_k})$ such that $\varphi_n \to \varphi$ weakly in $W_0^{1,p}(D)$ we have $\varphi \in W_0^{1,p}(\Omega)$.

It is worth to notice that the γ_p-convergence is also equivalent to

$$u_{\Omega_n,1} \to u_{\Omega,1} \text{ weakly in } L^p(D)$$

for some $\lambda \geq 0$, namely with the continuity of the solution with respect to the shape only in the case $f \equiv 1$ and for a single value of λ. Moreover, the γ_p-convergence can be seen via the Γ-convergence of the energy functionals associated to (1) or via the Mosco convergence of the moving Sobolev spaces $W_0^{1,p}(\Omega_n)$. As a consequence of the characterization via the Mosco convergence, if Ω_n γ_p-converges to Ω, then for more general equations of the form $-\text{div}\ \mathcal{A}(x, \nabla u_\Omega) + \mathcal{B}(x, u_\Omega) = 0$ one has $u_{\Omega_n} \to u_\Omega$ strongly in $W_0^{1,p}(D)$. It is not clear whether the converse is true since the right hand side f which is implicitly contained in $\mathcal{B}$ may produce solutions which are not positive p-q.e. (see [11]).

Notice also that the γ_p-convergence is metrizable but not compact. From the weak compactness of the unit ball of $W_0^{1,p}(D)$ and from the compact embedding into $L^p(D)$, one can extract from every sequence $(u_{\Omega_n,1})_n$ a subsequence which converges strongly in $L^p(D)$ to some function u. In general, one can not find an open set Ω such that $u = u_{\Omega,1}$. Nevertheless, following [11], there exists a positive Borel measure absolutely continuous with respect to the p-capacity such that for every $f \in W^{-1,p'}(D)$

$$u_{\Omega_n,f} \overset{L^p(D)}{\longrightarrow} u_{\mu,f},$$

where $u_{\mu,f} \in W_0^{1,p}(D) \cap L^p(D,\mu)$ and

$$-\Delta_p u_{\mu,f} + (\lambda + \mu)|u_{\mu,f}|^{p-2} u_{\mu,f} = f$$

in the weak variational sense. So, $u = u_{\mu,1}$. This phenomenon is called relaxation (see [11]).

For an open set $U \subseteq \mathbb{R}^N$, $x \in \mathbb{R}^N$, $0 < r < R$ we use the following notation

$$w(U, x, r, R) = \int_r^R \Big(\frac{\mathrm{cap}(U^c \cap B_{x,t}, B_{x,2t})}{\mathrm{cap}(B_{x,t}, B_{x,2t})}\Big)^{p'-1} \frac{dt}{t}.$$

If $h : U \to \mathbb{R}$ is a continuous function, we denote

$$\mathrm{osc}(h, U) = \sup_{x \in U} h(x) - \inf_{x \in U} h(x).$$

We recall from [21, Theorem 4.22] (see also [15, Lemma 4.6.5]) the following estimate for $u_{\Omega,f}$, the solution of (1).

Lemma 2.4 *Suppose that Ω is a bounded open set and $f \in L^{\frac{N}{p}+\varepsilon}(D)$. If $x_0 \in \partial\Omega$, then $\forall\ 0 < r \leq R$ it is true that*

$$osc(u_{\Omega,f}, \Omega \cap B(x_0, r)) \leq C exp\Big(- \frac{1}{C} w(\Omega, x, r, R)\Big) \tag{5}$$

where C depends on N, p, ε and $|u_{\Omega,f}|_\infty$.

Definition 2.5 *A point $x_0 \in \partial\Omega$ is called p-Wiener regular for Ω if $\lim_{r \to 0} w(\Omega, x_0, r, R) = +\infty$.*

We recall from [17] the following result.

Lemma 2.6 *Let $x_0 \in \partial\Omega$. The following assertions are equivalent:*
i) For $f \equiv 1$, $\lim_{x \to x_0, x \in \Omega} u_{\Omega,1}(x) = 0$.
ii) x_0 is p-Wiener regular.

Definition 2.7 *Let Ω be open. A point $x \in \partial\Omega$ is a called p-capacity point if $\forall \varepsilon > 0$ $\mathrm{cap}_p(\Omega^c \cap B(x, \varepsilon), B(x, 2\varepsilon)) > 0$.*

The following result has an immediate proof.

Lemma 2.8 *Let Ω be a bounded open set and let*

$$\Omega^* = \Omega \cup \{x \in \partial\Omega : x \textit{ is not a p-capacity point}\}.$$

Then Ω^* is open and $\mathrm{cap}_p(\Omega^* \setminus \Omega) = 0$.

3 The shape stability result

Let us set $N \geq 2$ and $1 < p \leq N$. Let D be a smooth bounded open set and let Ω_n, Ω be open subsets of D. We assume that Ω is p-Wiener regular at every point of its boundary.

Theorem 3.1 *For every $\varepsilon > 0, \lambda \geq 0$ and $f \in L^{\frac{N}{p}+\varepsilon}(D)$ we have $u_{\Omega_n,f} \to u_{\Omega,f}$ in $L^\infty(D)$ if and only if the following two relations hold.*

1. $\forall K \subset\subset \Omega$, $\exists N_K$, $\forall n \geq N_K$ *we have* $K \subset\subset \Omega_n^*$.

2. $\forall x \in \mathbb{R}^N$, $\forall r > 0$

$$\mathrm{cap}_p(B(x,r) \cap \Omega^c, B(x,2r)) \leq \liminf_{n\to\infty} \mathrm{cap}_p(B(x,r) \cap \Omega_n^c, B(x,2r)).$$

Proof **Necessity.** Assume that for every $\varepsilon > 0, \lambda \geq 0$ and $f \in L^{\frac{N}{p}+\varepsilon}(D)$ we have $u_{\Omega_n,f} \to u_{\Omega,f}$ in $L^\infty(D)$. In particular, we consider $f \equiv 1$, $\lambda = 0$ and get $u_{\Omega_n,1} \to u_{\Omega,1}$ in $L^\infty(D)$. Since D is bounded, this convergence holds also in $L^p(D)$. Consequently Ω_n γ_p-converges to Ω, hence from [4] relation 2. holds.

Assume for contradiction that 1. does not hold. Then, there exists a compact set $K \subset \Omega$, there exists $n_k \to \infty$ such that $K \not\subset \Omega_{n_k}^*$. Let $x_k \in K \setminus \Omega_{n_k}^*$ and assume (maybe extracting a subsequence) that $x_k \to x \in K$. The point x being interior to Ω, we can find $r, \delta > 0$ such that $\overline{B}(x,r) \subseteq \Omega$ and

$$u_{\Omega,1}(x) \geq \delta > 0 \text{ a. e. on } B(x,r). \tag{6}$$

By hypotheses, the L^∞-convergence gives that for n large enough $u_{\Omega_n,1} \geq \delta/2$ a.e. on $B(x,r)$. This inequality is also true p-q.e. for a quasi continuous representative. But $x_k \in (\Omega_{n_k}^*)^c$ and $x_k \to x$. Thus, for k large enough we have $x_k \in B(x, r/2) \cap (\Omega_{n_k}^*)^c$, and since x_k is a p-capacity point for $\Omega_{n_k}^*$, we get that

$$\mathrm{cap}_p(B(x_k, r/2) \cap (\Omega_{n_k}^*)^c) > 0.$$

This contradicts relation (6) since on the set $B(x_k, r/2) \cap (\Omega_{n_k}^*)^c$, which is of positive capacity, $u_{\Omega_{n_k},1}$ vanishes p-q.e.

Sufficiency. Relations 1. and 2. give that Ω_n γ_p-converges to Ω. This is a direct consequence of the local behavior in capacity of Ω_n^c (see Theorem 2.3 and [4]). Indeed, relation 1. gives the upper semicontinuity of the local capacity on closed balls, namely (3), and 2. gives the lower semicontinuity on open balls, namely (4).

Consequently, for every $f \in L^{\frac{N}{p}+\varepsilon}(D)$ the convergence of solutions holds in $W_0^{1,p}(D)$. It remains to prove that the convergence holds also in $L^\infty(D)$. Two cases are to be treated. On compact subsets of Ω the uniform convergence holds as a consequence of the equi continuity of $(u_{\Omega_n,f})_n$. The difficult part is to control the oscillations of $u_{\Omega_n,f}$ near the boundaries $\partial\Omega_n$ and to prove that they behave somehow uniformly with respect to n.

Let $\varepsilon > 0$ be fixed. We have to prove the existence of $N_\varepsilon \in \mathbb{N}$ such that $\forall n \geq N_\varepsilon$, and a.e. $x \in D$

$$|u_{\Omega_n,f}(x) - u_{\Omega,f}(x)| \leq \varepsilon.$$

Let us fix $R > 0$ and take $x \in \partial\Omega$. By hypothesis, x is a regular point, hence

$$\lim_{r\to 0} w(\Omega, x, r, R) = +\infty.$$

Thus, there exists $r = r_x > 0$ such that

$$C \exp\Big(-\frac{\overline{c}}{2C} w(\Omega, x, \frac{r_x}{2}, \frac{R}{2})\Big) \leq \frac{\varepsilon}{4}, \tag{7}$$

$$C\exp\Big(-\frac{1}{C}w(\Omega,x,r_x,R)\Big)\leq\frac{\varepsilon}{4}. \tag{8}$$

Here, C is the constant given in relation (5) and $\overline{c}$ is the constant given by the following lemma (for the proof, see [4]).

Lemma 3.2 *There exists a positive constant $\overline{c}$, depending only on N and p such that $\forall R > r > 0$, $\forall x_1, x_2 \in \mathbb{R}^N$ with $|x_1-x_2|\leq r/2$ we have*

$$w(\Omega,x_1,r,R)\geq\overline{c}w(\Omega,x_2,\frac{r}{2},\frac{R}{2}). \tag{9}$$

We cover $\partial\Omega$ with the balls $B(x,r_x/4)$ obtained using (7)-(8), and since $\partial\Omega$ is compact there exists a finite covering

$$\partial\Omega\subseteq\bigcup_{i\in I}B\Big(x_i,\frac{r_{x_i}}{4}\Big).$$

Let

$$K_1=\Omega\setminus\bigcup_{i\in I}B(x_i,\frac{r_{x_i}}{4})=\overline{\Omega}\setminus\bigcup_{i\in I}B(x_i,\frac{r_{x_i}}{4}).$$

Then K_1 is compact and $K_1\subset\subset\Omega$. By hypothesis 1), we have for large enough that

$$K_1\subset\subset\Omega_n^*.$$

Let us denote

$$K=\Omega\setminus\bigcup_{i\in I}B(x_i,\frac{r_{x_i}}{2}).$$

Then $K\subseteq\overset{\circ}{K_1}$ and

$$u_{\Omega_n,f}\longrightarrow u_{\Omega,f}\text{ uniformly on }\ K.$$

This is a consequence of the uniform boundedness of all functions, their convergence in $W_0^{1,p}(D)$ and of their equi-continuity (Lemma 3.3 below). Indeed, the following equi-continuity result is a direct consequence of [21, Theorem 4.11].

Lemma 3.3 *There exists $\alpha>0$ and a constant C such that for every open set $\Omega\subseteq D$, $f\in L^{\frac{N}{p}+\varepsilon}(D)$ and for every ball $B(x,R)\subset\subset\Omega$ and $0<r<R$ we have*

$$osc_{B(x,r)}u_{\Omega,f}\leq Cr^{\alpha}.$$

The constant C depends on N,p,R,ε and $\|f\|_{L^{\frac{N}{p}+\varepsilon}(D)}$.

It remains to prove the L^∞-convergence on $\bigcup_{i\in I}B(x_i,\frac{r_{x_i}}{2})$ and on $D\setminus(\Omega\cup_{i\in I}B(x_i,r_{x_i}/2))$, respectively.

On $\bigcup_{i\in I}B(x_i,\frac{r_{x_i}}{2})$ we shall control the oscillations of $u_{\Omega_n,f}$ with the help of the Wiener modulus of Ω.

Let us fix an index $i\in I$ and let $x\in B(x_i,\frac{r_{x_i}}{2})$. The modulus inequality gives

$$|u_{\Omega_n,f}(x)-u_{\Omega,f}(x)|\leq|u_{\Omega_n,f}(x)|+|u_{\Omega,f}(x)|.$$

We will estimate separately both $|u_{\Omega_n,f}(x)|$ and $|u_{\Omega,f}(x)|$. Since x_i is a regular point, we have from (8) and from Lemma 2.4 that for every $x \in B(x_i, r_{x_i}) \cap \Omega$

$$|u_{\Omega,f}(x)| \quad \leq C \exp\Big(-\frac{1}{C} w(\Omega, x, r_x, R)\Big) \leq \frac{\varepsilon}{4} \tag{10}$$

The constant C depends on $|u_{\Omega,f}|_\infty$ but can be chosen independently with respect to Ω since from the maximum principle all solutions $u_{\Omega,f}$ are uniformly bounded on D by $|u_{D,|f|}|_\infty$.

If $x \in \Omega^c \cap B(x_i, r_{x_i})$, then p-q.e. $u_{\Omega,f}(x) = 0$. Let us now estimate $|u_{\Omega_n,f}(x)|$ on $B(x_i, r_{x_i})$. From the γ_p-convergence we get for n large enough

$$\mathrm{cap}_p(B(x_i, r_{x_i}) \cap \Omega_n^c, B(x_i, 2r_{x_i})) > 0.$$

There are two possibilities. Either $B(x_i, r_{x_i}/2) \cap \Omega_n = \emptyset$ and in this case $u_{\Omega_n,f}(x) = 0$ p- q.e. on $B(x_i, r_{x_i}/2)$, or $B(x_i, r_{x_i}/2) \cap \Omega_n \neq \emptyset$. From [4] we recall the following technical result.

Lemma 3.4 *Let Ω be an open set such that $\Omega \cap B(x, \delta) \neq \emptyset$ and $\mathrm{cap}_p(\Omega^c \cap B(x, \delta), B(x, 2\delta)) > 0$. Then $\mathrm{cap}_p(\partial\Omega \cap B(x, \delta), B(x, 2\delta)) > 0$.*

In the latter case, Lemma 3.4 applies and gives that

$$\mathrm{cap}_p(\partial\Omega_n \cap B(x_i, r_{x_i}/2), B(x, r_{x_i})) > 0.$$

We observe that hypotheses 1. and 2. imply the γ_p-convergence via Theorem 2.3. From the Fatou lemma and the equality

$$\mathrm{cap}_p(\Omega^c \cap B(x, \delta), B(x, 2\delta)) = \mathrm{cap}_p(\Omega^c \cap \overline{B}(x, \delta), B(x, 2\delta))$$

which holds a.e. $\delta > 0$ we get directly that (see [4, 5] for finer results) for every $x \in \mathbb{R}^N$ and $\forall 0 < r < R$

$$\liminf_{n\to\infty} w(\Omega_n, x, r, R) = w(\Omega, x, r, R). \tag{11}$$

By (11) we get that

$$w(\Omega, x_i, \frac{r_{x_i}}{2}, \frac{R}{2}) = \lim_{n\to\infty} w(\Omega_n, x_i, \frac{r_{x_i}}{2}, \frac{R}{2}). \tag{12}$$

Consequently, there exists $N \in \mathbb{N}$ such that for every $n \geq N$ we have

$$w(\Omega_n, x_i, \frac{r_{x_i}}{2}, \frac{R}{2}) \geq \frac{1}{2} w(\Omega, x_i, \frac{r_{x_i}}{2}, \frac{R}{2}). \tag{13}$$

Thus, using Lemma 3.2, for every $y \in \overline{B}(x_i, \frac{r_{x_i}}{2})$ we have that

$$w(\Omega_n, y, r_{x_i}, R) \geq \frac{\overline{c}}{2} w(\Omega, x_i, \frac{r_{x_i}}{2}, \frac{R}{2}). \tag{14}$$

Since $\mathrm{cap}_p(\partial\Omega_n \cap B(x_i, r_{x_i}/2), B(x, r_{x_i})) > 0$, we can find a point $y_n \in \partial\Omega_n \cap B(x_i, r_{x_i}/2)$ which is p-regular for Ω_n. We apply estimate (8) for Ω_n and get for every $x \in B(y_n, r_{x_i}) \cap \Omega_n$

$$\begin{aligned} |u_{\Omega_n,f}(x)| \quad &\leq C \exp(-\frac{1}{C} w(\Omega_n, y_n, r_{x_i}, R)) \\ &\leq C \exp(-\frac{\overline{c}}{2C} w(\Omega, x_i, \frac{r_{x_i}}{2}, \frac{R}{2})) \end{aligned} \tag{15}$$

Using (7) we get

$$|u_{\Omega_n,f}(x)| \leq \frac{\varepsilon}{4}.$$

On the other hand, on $B(y_n, r_{x_i}) \cap \Omega_n^c$ we have $u_{\Omega_n,f}(x) = 0$ p-quasi everywhere.

Consequently

$$|u_{\Omega_n,f}(x)| \leq \frac{\varepsilon}{2} \ \ p\text{-q.e. } x \in B(y_n, r_{x_i}). \tag{16}$$

Since $B(x_i, r_{x_i}/2) \subseteq B(y_n, r_{x_i})$, we get that

$$|u_{\Omega_n,f}(x)| + |u_{\Omega,f}(x)| \leq \varepsilon \ \ p\text{-q.e. } x \in B(x_i, \frac{r_{x_i}}{2}).$$

It remains to prove that $\forall x \in D \setminus (\Omega \cup_{i\in I} B(x_i, r_{x_i}/2))$ we have (for n large enough)

$$|u_{\Omega_n,f}(x) - u_{\Omega,f}(x)| \leq \varepsilon.$$

Since $\overline{D} \setminus (\Omega \cup_{i\in I} B(x_i, r_{x_i}/2))$ is compact, we will follow a similar argument as for the neighborhood of $\partial\Omega$. For every $x \in \overline{D} \setminus (\Omega \cup_{i\in I} B(x_i, r_{x_i}/2))$ we fix r_x as in (7)-(8). Such r_x exists since for r small enough

$$\mathrm{cap}_p(\Omega^c \cap B(x, r), B(x, 2r)) = \mathrm{cap}_p(B(x, r), B(x, 2r)).$$

We cover $\overline{D} \setminus (\Omega \cup_{i\in I} B(x_i, r_{x_i}/2))$ by a finite family of balls

$$\overline{D} \setminus (\Omega \cup_{i\in I} B(x_i, r_{x_i}/2)) \subseteq \cup_{j\in J} B(x_j, r_{x_j}/4).$$

We fix an index j and get for every $B(x_j, r_{x_j}/4)$

$$|u_{\Omega_n,f}(x) - u_{\Omega,f}(x)| = |u_{\Omega_n,f}(x)|.$$

In order to estimate $|u_{\Omega_n,f}(x)|$ two possibilities may occur. Either $B(x_j, r_{x_j}/4) \cap \Omega_n = \emptyset$ or not. In the first case, obviously $u_{\Omega_n,f}(x) = 0$. In the second case, by Lemma 3.4 we get

$$\mathrm{cap}_p(\partial\Omega_n \cap B(x_j, r_j/2), B(x_j, r_j)) > 0,$$

and a similar argument as for the neighborhood of $\partial\Omega$ in (12)-(16) holds true. Finally,

$$|u_{\Omega_n,f}(x)| \leq \varepsilon \ p\text{-q.e. } x \in B(x_j, r_{x_j}/4).$$

□

Remark 3.5 Assume that for some $\lambda \geq 0$ and $f \equiv 1$ we have $u_{\Omega_n,1} \rightarrow u_{\Omega,1}$ in $L^\infty(D)$. The proof of Theorem 3.1 yields that for every $\lambda \geq 0, \varepsilon > 0$ and for every $f \in L^{\frac{N}{p}+\varepsilon}(D)$ we have $u_{\Omega_n,f} \rightarrow u_{\Omega,f}$ in $L^\infty(D)$.

This kind of behavior is typical for the γ_p-convergence. We refer the reader to [3] for further developments of this topic into the linear case.

Let us set the following notation.

Notation 3.6 *Let Ω_n, Ω be open subsets of D. If assertions 1. and 2. of Theorem 3.1 hold, we denote*

$$\Omega_n \xrightarrow{\infty_p} \Omega.$$

We formulate the following corollary which is a clear consequence of Theorem 3.1.

Corollary 3.7 *Let Ω_n, Ω be open subsets of D. Then $\Omega_n \xrightarrow{\infty_p} \Omega$ if and only if the following relations hold.*

1. *$\forall K \subset\subset \Omega$, $\exists N_K$, $\forall n \geq N_K$ we have $K \subset\subset \Omega_n^*$.*
2. *Ω_n γ_p-converges to Ω.*

Proof For the necessity we use Theorem 3.1 and get the first assertion. In order to get the γ_p-convergence, we notice that the first assertion gives (3) which associated to (4) and Theorem 2.3 gives the γ_p-convergence.

For the sufficiency, we apply Theorems 3.1 and 2.3.

□

For searching the "minimal" intuitive conditions which provide shape stability into the L^∞-norm, one may use the following.

Corollary 3.8 *Let Ω_n, Ω be open subsets of D. Then $\Omega_n \xrightarrow{\infty_p} \Omega$ if and only if the following relations hold.*

1. *$\forall K \subset\subset \Omega$, $\exists N_K$, $\forall n \geq N_K$ we have $K \subset\subset \Omega_n^*$.*
2. *$\forall \varphi_{n_k} \in W_0^{1,p}(\Omega_{n_k})$ such that φ_{n_k} converges weakly in $W_0^{1,p}(D)$ to φ, we have $\varphi \in W_0^{1,p}(\Omega)$.*

Proof The proof is a consequence of Theorem 3.1 and of the equivalence between the second Mosco condition (condition 2. above) and the lower semicontinuity of the local capacity (condition 2. in Theorem 3.1). We refer the reader to [4] for the proof of this equivalence. □

For applications in concrete situations, the conditions expressed in this corollary are the most intuitive. Indeed, the first condition is purely geometric and can be easily verified in practical situations. Using Hedberg's result (Lemma 3.3) on the description of $W_0^{1,p}$-spaces via quasi-continuous representatives (see [14]), the second condition can be easily checked as soon as Ω has *some* smoothness.

Remark 3.9 We notice that the L^∞-convergence of solutions can not hold if relaxation for the γ_p-convergence occurs. Indeed, let $f \equiv 1$ and fix $\lambda \geq 0$. Then $u_\mu > 0$ p-q.e. on the regular set A_μ of the measure μ. Assuming that relaxation occurs means that $\mu(A_\mu) > 0$.

Consider $\delta > 0$ and the p-quasi open set $U_\delta = \{u_\mu > \delta\}$ which is also of positive Lebesgue measure (for δ enough small). For n large enough we would have

$$|u_{\Omega_n,1} - u_\mu| \leq \frac{\delta}{2} \text{ a.e. on } A_\mu,$$

hence p-q.e. since A_μ is p-quasi open. Then $\mathrm{cap}_p(U_\delta \cap \Omega_n^c) = 0$ and consequently from the γ_p convergence $\mu(U_{\delta'}) = 0$ for $\delta' > \delta$. Finally, taking $\delta \to 0$ we would get $\mu(A_\mu) = 0$, which is a contradiction with our relaxation assumption (see [9]).

Remark 3.10 If Ω would not be p-Wiener regular at every point of its boundary, following Lemma 2.6, at such a point and for $f = 1$, the solution of (1) on Ω would be discontinuous. Therefore, if $\Omega_n \xrightarrow{\infty_p} \Omega$, then $u_{\Omega_n,1}$ should be discontinuous either. This means that every p-irregular point of Ω should be, for n large enough an irregularity point for Ω_n. As a consequence, the sequence $\Omega \setminus \overline{B}(x_0, 1/n)$ does not ∞_p-converge to Ω!

Nevertheless, the shape stability in $L^\infty(D)$ could steel hold if Ω is not p-Wiener regular at every point of its boundary, but the perturbation is highly restrictive and the expression of the stability conditions is certainly more complicated.

4 Examples of ∞_p-convergence of domains

The main interest in applications is to understand for a specific perturbation whether or not the solution of (1) is stable in the L^∞-norm. Although the second condition in Theorem 3.1 seems difficult to understand in practice, following Corollaries 3.7 and 3.8 this condition can be replaced with the γ_p-convergence which is well studied in the literature or with the second Mosco condition which sometimes can be proved easily. Besides the case of *uniformly* smooth domains (e.g. domains satisfying a uniform cone condition), the γ_p-convergence can be obtained into the following frame: geometric convergence of the domains in the Hausdorff complementary topology associated to some geometrical, topological or capacity assumptions on the moving domains. We also notice another particular case of γ_p-convergence which is more restrictive, namely the compact convergence of domains associated to a limit domain which is stable in the sense of Keldysh (see [4]).

The Hausdorff complementary topology is given by the metric:

$$d_{H^c}(\Omega_1, \Omega_2) = \sup_{x \in \mathbb{R}^n} |d(x, \Omega_1^c) - d(x, \Omega_2^c)|.$$

Note that if $\Omega_n \xrightarrow{H^c} \Omega$, then condition 1. of Theorem 3.1 is automatically satisfied. Then $\Omega_n \xrightarrow{\infty_p} \Omega$, provided that Ω is p-Wiener regular at every point of its boundary and that $\Omega_n \xrightarrow{\gamma_p} \Omega$.

The most general situation in which the γ_p-convergence is known to be equivalent to the Hausdorff complementary convergence involves a sort of locally uniform Wiener criterion (see [6] and paper [13] for a first result into this direction).

This is for example the case in $\mathbb{R}^N$ as soon as there exists $c, r > 0$ such that for every $n \in \mathbb{N}$ the sets Ω_n satisfy the following uniform capacity density condition [7, 5]:

$$\forall x \in \partial\Omega_n \ \forall t \in (0, r) \quad \frac{\mathrm{cap}_p(\Omega_n^c \cap B_{x,t}, B_{x,2t})}{\mathrm{cap}_p(B_{x,t}, B_{x,2t})} \geq c. \tag{17}$$

A geometric situation when the uniform capacity density condition is satisfied, is the so called *flat cone condition*, i.e. there is a closed cone T of dimension $N-1$ such that for every point x_0 on the boundary of every Ω_n there exists a cone congruent with T with vertex in x_0, lying in the complement of Ω_n.

In N dimensions of the space and for $p \in]N-1, N]$, the generalization of Šverák's result proved in [6] gives that the γ_p-convergence is equivalent to the H^c-convergence in the class of domains for which the complementary sets have at most a fixed number, say l, of connected components. We denote by $\sharp\Omega^c$ the number of the connected components of $\mathbb{R}^N \setminus \Omega$.

Proposition 4.1 *Let $p \in]N-1, N]$ and $l \in \mathbb{N}$ be fixed, and let $\Omega_n \subseteq D$ be such that $\sharp\Omega_n^c \leq l$. Assume $\Omega_n^* = \Omega_n$ and $\Omega_n \xrightarrow{H^c} \Omega$. Then $\Omega_n \xrightarrow{\infty_p} \Omega$ if and only if $\Omega = \Omega^*$.*

The assumption $\Omega_n^* = \Omega_n$ is not restrictive at all since this means that $\partial\Omega_n$ should not have isolated points (so just remove the isolated points of $\partial\Omega_n$; there are at most l, hence of zero p-capacity). Moreover, the assumption $\Omega_n \xrightarrow{H^c} \Omega$ is not restrictive either since the H^c metric topology is compact. In fact, an equivalent formulation of this proposition is the following: *let $p \in]N-1, N]$ and $l \in \mathbb{N}$ be fixed, and let $\Omega_n \subseteq D$ be such that $\sharp\Omega_n^c \leq l$. Assume $\Omega_n \xrightarrow{H^c} \Omega$. Then $\Omega_n \xrightarrow{\infty_p} \Omega$ if and only if $\Omega_n^* \xrightarrow{H^c} \Omega^*$.*

Proof (of Proposition 4.1) Let $\Omega_n \xrightarrow{H^c} \Omega$. If $\Omega^* = \Omega$, then Ω is p-Wiener regular at every point of its boundary, since it does not have isolated points, and every point of the boundary which is not isolated belongs to a continua of positive diameter. Consequently, Ω is p-Wiener regular at every point of the boundary and therefore Theorem 3.1 gives $\Omega_n \xrightarrow{\infty_p} \Omega$. Indeed, condition 1. is a consequence of the H^c-convergence and condition 2. is proved in [6].

For the converse, assume that $\Omega_n \xrightarrow{\infty_p} \Omega$. Then obviously $\Omega_n \xrightarrow{\infty_p} \Omega^*$ since $\mathrm{cap}_p(\Omega^* \setminus \Omega) = 0$, and Ω^* is p-Wiener regular at every point of its boundary. Suppose for contradiction that $\Omega \neq \Omega^*$, i.e. Ω^c has an isolated point x_0. This means that there exists a sequence of continua of positive diameter K_n which are connected components of Ω_n^c such that K_n converges in the Hausdorff sense to $\{x_0\}$. Consequently condition 1. of Theorem 3.1 is violated for the sequence $(\Omega_n)_n$ associated to the limit set Ω^* by simply taking $K = \overline{B}(x_0, \varepsilon)$, with ε small enough. $\square$

Remark 4.2 Given $\delta > 0$, in N dimensions of the space and for $p \in]N-1, N]$ one can consider the following class of domains:

$$\{\Omega \subseteq D : \Omega^c = \cup_\alpha K_\alpha,\ K_\alpha \text{ connected diam } K_\alpha \geq \delta\},$$

which satisfy a uniform capacity density condition. Then

$$\Omega_n \xrightarrow{H^c} \Omega \implies \Omega_n \xrightarrow{\infty_p} \Omega.$$

All examples below (Figures 1, 2 and 3) give ∞_p-convergence in $2D$ for $1 < p \leq 2$.

Remark 4.3 Let $p = 2$ in (1). A more precise estimate can be derived into the class defined by relation (17). For a given $\varepsilon > 0$, here exists $\alpha \in (0, 1]$ and $C > 0$ such that for every $f \in L^{\frac{N}{2}+\varepsilon}(D)$ and every $\Omega_1, \Omega_2 \in D$ we have

$$|u_{\Omega_1,f} - u_{\Omega_2,f}|_{L^\infty(D)} \leq C(d_{H^c}(\Omega_1, \Omega_2))^\alpha |f|_{L^{\frac{N}{2}+\varepsilon}(D)}. \tag{18}$$

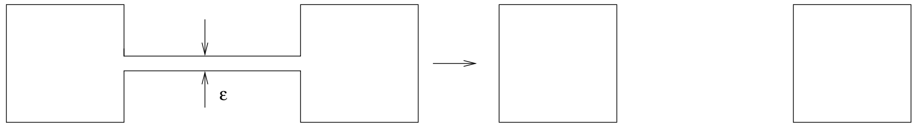

Figure 1: The thickness ε of the tube converges to zero.

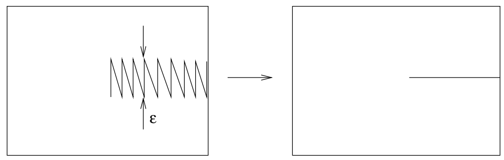

Figure 2: Oscillating crack with vanishing "amplitude" ε.

Indeed, in order to compute $|u_{\Omega_1,f} - u_{\Omega_2,f}|_{L^\infty(D)}$ one has only to look for

$$M = \max\{\sup_{x\in\Omega_1\setminus\Omega_2} |u_{\Omega_1,f}(x)|, \sup_{x\in\Omega_2\setminus\Omega_1} |u_{\Omega_2,f}(x)|\},$$

since

$$|u_{\Omega_1,f|\Omega_1\cap\Omega_2} - u_{\Omega_2,f|\Omega_1\cap\Omega_2}|_{L^\infty(\Omega_1\cap\Omega_2)} \le 2M.$$

Since Ω_1, Ω_2 satisfy (17), the solutions $u_{\Omega_1,f}$ and $u_{\Omega_1,f}$ satisfy

$$|u_{\Omega_i,f}|_{0,\alpha} \le C|f|_{L^{\frac{N}{2}+\varepsilon}(D)},$$

with C and α independent on Ω and f. Consequently, relation (18) follows. This result of estimating the continuity modulus of the mapping *shape* $\to$ *solution* is to be related to [22]. Savarè and Schimperna obtained in [22] estimates of the H^1 and L^2 norms with respect to the Hausdorff distance for equi-Lipschitz domains.

5 Further remarks

5.1 Localization of the ∞_p-convergence

Proposition 5.1 *Let Ω be p-Wiener regular at every point of its boundary and Ω_n open subsets of D. The following assertions are equivalent.*

i) $\Omega_n \xrightarrow{\infty_p} \Omega$,

ii) $\exists (U_i)_{i\in I}$ a family of open sets, with union covering D, such that every U_i is p-Wiener regular and

$$\forall i \in I \;\; \Omega_n \cap U_i \xrightarrow{\infty_p} \Omega \cap U_i.$$

Proof Implication $i) \to ii)$ is a consequence of the localization property of the γ_p-convergence (see [11, Corollary 6.13]).

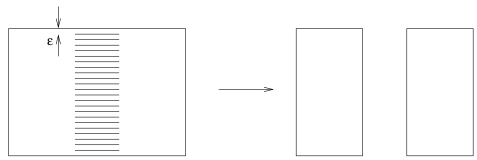

Figure 3: Infinite number of cracks of minorated diameter (two neighboring cracks are spaced by ε).

To prove $ii) \to i)$ notice first that if Ω and U_i are p-Wiener regular at every point of their boundary, then $\Omega \cap U_i$ is Wiener regular at every point of the boundary. From the localization property of the γ_p-convergence (see [11]) it is enough to prove property 1. of Corollary 3.7. Let us consider the compact $K \subset\subset \Omega$. Then there exists a finite covering of K by $U_1 \cup ... \cup U_q$. We denote $K_j = K \setminus (U_1 \cup ... \cup U_{j-1} \cup U_{j+1} \cup ... \cup U_q)$. Then for $j = 1, .., q$ the sets K_j are compact, and their union is K. Using hypothesis ii), for $n \geq N_j$ with N_j large enough we have

$$K_j \subset (\Omega_n \cap U_j)^*.$$

Since $(\Omega_n \cap U_j)^* \subseteq \Omega_n^*$ taking the union in j we get condition 1. of Corollary 3.7. $\square$

5.2 Convergence of eigenfunctions

We begin with the following preliminary result concerning moving right hand sides.

Lemma 5.2 *Let $f_n \in L^{\frac{N}{p}+\varepsilon}(D)$ and let $f_n \rightharpoonup f$ weakly in $L^{\frac{N}{p}+\varepsilon}(D)$. If Ω is Wiener regular at every point of its boundary and if $\Omega_n \xrightarrow{\infty_p} \Omega$ then*

$$u_{\Omega_n, f_n} \xrightarrow{L^\infty(D)} u_{\Omega, f}.$$

Proof From the γ_p-convergence we get that $u_{\Omega_n, f_n} \longrightarrow u_{\Omega, f}$ strongly in $W_0^{1,p}(D)$. In order to prove that the convergence holds in $L^\infty(D)$, one reproduces the sufficiency part of Theorem 3.1.

To get the uniform convergence on a compact set of Ω one uses the equicontinuity given by Lemma 3.3 and the uniform boundedness in $L^\infty(D)$ of $(u_{\Omega_n, f_n})_n$.

For the oscillations of u_{Ω_n, f_n} on $\partial\Omega_n$, the same argument as in Theorem 3.1 stands true, the main point being that the constant C in (5) is the same for every u_{Ω_n, f_n}. Indeed, following [21] the estimate of the L^∞-norm of u_{Ω_n, f_n} (which is crucial for the constant C) depends on the norm of f_n in $L^{\frac{N}{p}+\varepsilon}(D)$ (which is uniformly bounded with respect to n). $\square$

In the sequel, we denote by $R_\Omega : L^{\frac{N}{2}+\varepsilon}(D) \to L^\infty(D)$ the resolvent operator associated to problem (1), for $p = 2$.

Proposition 5.3 *Let $p = 2$. Suppose that Ω is Wiener regular at every point of its boundary and that $\Omega_n \xrightarrow{\infty_2} \Omega$. Then $R_{\Omega_n} \to R_\Omega$ in $\mathcal{L}(L^{\frac{N}{2}+\varepsilon}(D), L^\infty(D))$.*

Proof This is a consequence of Lemma 5.2 and of the fact that the unit ball is weakly compact in $L^{\frac{N}{2}+\varepsilon}(D)$. $\square$

Let us denote for every open set Ω by $\lambda_k(\Omega)$ the k-th eigenvalue (the multiplicities are counted) of the Dirichlet Laplacian on Ω and by $u_{\Omega,k}$ a corresponding eigenfunction which is L^2-normalized. It is well known that the γ_2-convergence gives the convergence of the spectrum as a consequence of the convergence $R_{\Omega_n} \to R_\Omega$ in $\mathcal{L}(L^2(D), L^2(D))$ (see for instance [5]). Moreover, every sequence of eigenfunctions corresponding to the k-th eigenvalue on Ω_n which weakly converges in $H^1_0(D)$ has as limit a k-th eigenfunction on Ω.

Corollary 5.4 *Suppose that $\Omega_n \xrightarrow{\infty_2} \Omega$ and $\Omega \neq \emptyset$. For every $k \in \mathbb{N}$ we have (up to a subsequence) that $u_{\Omega_n,k} \xrightarrow{L^\infty(D)} u_{\Omega,k}$, where $u_{\Omega,k}$ is an L^2-normalized eigenfunction associated to the k-th eigenvalue of the Dirichlet-Laplacian on Ω_n.*

Proof First, from the γ_2-convergence we have that

$$u_{\Omega_n,k} \longrightarrow u_{\Omega,k} \text{ strongly in } H^1_0(D).$$

From [12, Example 2.1.8] we have the following estimates for the L^∞-norm of the eigenfunctions

$$|u_{\Omega_n,k}|_{L^\infty(A)} \leq 3(8\pi\lambda_k(\Omega_n))^{\frac{N}{4}}.$$

Consequently, if $|\Omega| \neq 0$ then $\limsup_{n\to\infty} \lambda_k(\Omega_n) < +\infty$ hence one can find a uniform bound for the L^∞-norm of all $u_{\Omega_n,k}$. Thus, $u_{\Omega_n,k} \longrightarrow u_{\Omega,k}$ strongly in $L^{\frac{N}{p}+1}$ and the proof is concluded by using Lemma 5.2. $\square$

Remark 5.5 Into the nonlinear case (for the p-Laplacian with $p \neq 2$), the *right* characterization of all eigenvalues is not completely understood. We refer to [19] for a detailed description of the topic. Nevertheless, using the Rayleigh characterization for the first eigenvalue, one can easily establish the L^∞-convergence of a sequence of normalized first eigenfunctions provided that the geometric domains ∞_p-converge. Already for the second eigenfunctions this is not anymore clear.

5.3 Extensions to more general elliptic problems

Let us consider two nonlinear operators $u \mapsto -\text{div}\ \mathcal{A}(x, \nabla u)$, $u \mapsto \mathcal{B}(x, u)$ defined on $W^{1,p}_0(D)$ with values on $W^{-1,p'}(D)$. Under suitable assumption on $\mathcal{A}$ and $\mathcal{B}$, e.g. $-\text{div}\ \mathcal{A}$ be similar to the p-Laplacian and $\mathcal{B}$ be Carathéodory, nondecreasing in the second variable and satisfying

$$|\mathcal{B}(x, \xi)| \leq \alpha|\xi|^{p-1} + r,$$

where $\alpha > 0$ and $r \in L^{\frac{N}{p}+\varepsilon}(D)$ (or more general r belongs to the Morrey space $\mathcal{M}^{N/(p-\varepsilon)}(D)$, see [21, Chapter 3] and [20]), one can extend some of the results of Theorem 3.1.

Let $g \in C(\overline{D}) \cap W^{1,p}_0(D)$. For every $\Omega \subseteq D$ we consider the problem

$$\begin{cases} -\text{div}\ \mathcal{A}(x, \nabla u_\Omega) + \mathcal{B}(x, u_\Omega) = 0 & \text{in } \Omega \\ u_\Omega - g \in W^{1,p}_0(\Omega) \end{cases} \tag{19}$$

Existence and uniqueness of the solution follows using the standard approach via the Hartman-Stampacchia theorem. Let Ω be p-Wiener regular at every point of its boundary. One could prove one implication of Theorem 3.1, namely that $\Omega_n \xrightarrow{\infty_p} \Omega$ implies $u_{\Omega_n} \longrightarrow u_\Omega$ in $L^\infty(D)$ (the extension on Ω_n^c is g). The γ_p-convergence gives straight forwardly that $u_{\Omega_n} \longrightarrow u_\Omega$ in $W_0^{1,p}(D)$. For the uniform convergence, the proof follows the same lines as Theorem 3.1.

The converse is not so obvious since the answer clearly depends on $\mathcal{B}$. An involved study of the dependence of the solution u_Ω on $\mathcal{B}$ is necessary, in order to search the regions where the solution vanishes. Without any specific hypothesis on $\mathcal{B}$, the solution may vanish on sets of positive measure and on this region the geometry of the moving domains can not be anymore controlled.

5.4 Keldysh like stability

In [16], Keldysh introduced into the linear frame the following stability concept (the extension is natural to the nonlinear one): Ω is called p-stable if every sequence $(\Omega_n)_n$ which compactly converges to Ω do γ_p-converge to Ω. It is said that $(\Omega_n)_n$ compactly converges to Ω if

$$\forall K \subset\subset \Omega \cup int(\Omega^c)\ \ \exists N = N_K,\ \forall n \geq N \implies K \subseteq \Omega_n \cup int(\Omega_n^c).$$

Various characterizations of the stability were given in the literature; we refer the reader to [14] and, for an approach via γ_p-convergence, to [4].

Into the linear frame, the stability question into the L^∞-norm was raised in [3]. A domain is called L^∞-stable if every sequence of open sets which compactly converges to Ω do ∞_p-converge. A direct consequence of Theorem 3.1 is the following.

Proposition 5.6 *Let Ω be p-Wiener regular at each point of its boundary. Then Ω is p-stable if and only if is L^∞-stable.*

Proof If Ω_n compactly converges to Ω then condition 1. of Corollary 3.7 is satisfied. Consequently, ∞_p-stability is equivalent to γ_p-stability. □

Notice that domains with cracks may be p-Wiener regular at every point of the boundary, but they are not stable in the sense of Keldysh. This means that sequences of open set converging into the compact convergence are not necessarily ∞_p-converging. Nevertheless, ∞_p-convergence for such a situation can be achieved for other type of geometric convergences (e.g. those verifying the assumptions of Proposition 4.1, or more general Theorem 3.1).

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