# Do optimal shapes exist?

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**Abstract**

This paper surveys the talk given by the author at "Seminario Matematico e Fisico di Milano" in November 2006. It deals with the existence question for shape optimization problems associated to the Dirichlet Laplacian. Existence of solutions is seen from both geometrical and functional ($\gamma$-convergence) point of view and is discussed in relationship with the optimality conditions and numerical algorithms. Several examples are given concerning isoperimetric inequalities for eigenvalues and shape control problems.

## 1 Introduction

A general shape optimization problem can formally be written as

$$\min_{A \in \mathcal{U}_{ad}} J(A). \tag{1}$$

In general, given an open set $D \subseteq \mathbb{R}^N$ (called the design region) and a constant $m > 0$, the family of admissible shapes is

$$\mathcal{U}_{ad} = \{A \subseteq D | A \text{ open}, \ |A| \le m\},$$

where $|A|$ stands for the Lebesgue measure of the set $A$.

The cost to be minimized is the shape functional $J : \mathcal{U}_{ad} \to \mathbb{R} \cup \{+\infty\}$. In our examples, $J$ depends on the set $A$ via an elliptic partial differential equation or the spectrum of the Dirichlet Laplacian which has the geometric domain supported by $A$. Although we discuss only the Dirichlet Laplacian, some of the results presented in this paper are true for more general elliptic operators in divergence form with Dirichlet boundary conditions. We point out that these results do not apply for problems with Neumann conditions on the free boundary.

As a typical example fitting in our frame we may consider the isoperimetric inequality for the $k$-th eigenvalue of the Dirichlet Laplacian

$$\min\{\lambda_k(A) \mid A \subseteq \mathbb{R}^N, \ A \text{ open}, \ |A| = m\}.$$

For $k = 1$ and $k = 2$ this problem was solved by Faber and Krahn in 1921-1924, the answer being a ball of measure $m$ and two disjoint balls of measure $m/2$, respectively. Starting with $k = 3$, the answer to the isoperimetric problem is not known.

In general, a shape optimization problem may not have a solution! An intuitive example is the following real life problem. Given a quantity of ice and a glass of water, find the best shape of the ice, such that once put in the water it will melt the quickest possible. If a shape solution would exist (say a cube, or whatever the shape could be, consisting on one or more small pieces), one could take one piece of ice and crush it in two smaller pieces. In this way, the contact surface between the water and the ice would increase, and intuitively the ice would melt quicker. The natural answer to this real life problem is : *crush all the ice.* An opposite question, is to find the best shape for which the ice would melt the slowest possible. Intuitively, one intends to minimize the contact surface hence the optimal shape should be the ball.

This "real life" problem is not written into a mathematical framework and the discussion above is only intuitive. The precise answer of existence or not of a solution is heavily depending on the mathematical model! Here are the main questions: *what is the right setting? what is the right PDE? what is the right class of shapes?* It is not the purpose of the paper to deal with the correctedness of different models, which may have both positive and negative features from different points of view. We only underline the fact that the answer we give is an answer to the mathematical model and not to the real life problem.

Some existence results may be completely "artificial", being a consequence of purely technical arguments introduced in the model. In practice, one may impose some a priori constraints on shapes in order to achieve existence. These kinds of constraints are usually of geometric type and impeach shapes to oscillate or to change topology (e.g. uniform cone condition). In order to distinguish an "artificial" result from a "natural" one, the optimality conditions play an important role. Since the family of shapes has not a vector space structure, the classical optimization theory does not apply, and in particular, geometric constraints on the shapes cannot be handled via Lagrange multipliers. From this point of view, a good existence result is a result for which optimality conditions may be written properly (see more explanations below).

Of course, sometimes the existence of a "natural" solution may fail, and a relaxation process may occur. This is the situation in which a minimizing sequence of shapes leads to a "mixture" between material and void which needs a profound mathematical explanation. Finally, there are also situations in which problem (1) is definitely ill posed, and neither "natural" existence nor relaxation occurs. Nevertheless, for engineering purposes, "optimal" shapes may be needed, and in this case the "artificial" constraints play an important role.

A tentative recipe for dealing with shape optimization problems in the frame of the calculus of variations may consist on the following four steps:

*Step 1. Prove the existence of a solution.* We shall discuss this question in the rest of the paper, by pointing out the main pathological behaviors of the minimizing sequences.

*Step 2. Investigate the regularity of a solution.* Investigating the smoothness of the boundary for the optimal set is a very difficult question. We refer to the pioneering paper of Alt and Caffarelli [2] and to the more recent paper and its references [7]. The information about regularity is not only a nice theoretical question but is of high importance when trying to

write down the optimality conditions, since differentiability of shape functionals often require smoothness.
*Step 3. Write the optimality conditions.* After regularity is obtained, one can write the optimality conditions by performing different types of differentiation. Sometimes, overdetermined problems are obtained. We recall the two main tools for writing optimality conditions: *the shape and the topological derivative.* A good existence result should be obtained in classes of shapes where optimality conditions can be written.

**The shape derivative.** For a vector field $V \in C_c^\infty(D, \mathbb{R}^N)$ the so called directional shape derivative

$$dJ(A^*; V) = \lim_{t \to 0} \frac{J((Id + tV)A^*) - J(A^*)}{t},$$

is computed. Under certain hypotheses, $dJ(A^*; V)$ is a distribution on the boundary of $A^*$ acting on the normal component of $V$. Consequently, optimality conditions can be written $dJ(A^*; V) \geq 0$.

Of course,in the previous relation the vector field $V$ has to be such that $(Id+tV)A^* \in \mathcal{U}_{ad}$ for small $t$. For example, in the case when $\mathcal{U}_{ad}$ consists only of convex sets, the family of admissible fields is very restrictive, and optimality conditions are very poor on the flat regions of the boundary of the optimal set. On the contrary, if the $\mathcal{U}_{ad}$ consists of a class which is stable for vector fields deformations (such as a class defined by a topological constraint, e.g. the class of simply connected sets), the optimality conditions have a full meaning.

**The topological derivative.** For every $x_0 \in A^*$, one computes the following asymptotic development

$$J(A^* \setminus \overline{B}_{x_0,\varepsilon}) = J(A^*) + g(x_0)f(\varepsilon) + o(f(\varepsilon)),$$

where $f(\varepsilon) > 0$ is such that $\lim_{\varepsilon \to 0} f(\varepsilon) = 0$. The optimality condition then writes $g(x_0) \geq 0$. Again, the family of admissible domains $\mathcal{U}_{ad}$ has to be stable to this operation. If the class consists of the simply connected sets in $\mathbb{R}^2$, to perform a hole is not admissible! Consequently, the topological derivative cannot be used for deriving optimality conditions. A suitable class where the topological derivative can be used properly, is the class of all open subsets of the design region.

We refer to [18], [27] for a detailed discussion of the topological derivative and for several applications.
*Step 4. Perform numerical computations.* If existence of a solution holds in stable classes (to the performance of shape and/or topological derivatives) the use of gradient based methods may be a good issue. If existence does not hold, or it holds in a class in which neither shape derivative nor topological derivative can be performed, numerical computations via gradient methods may not be justified and other types of algorithms may be employed, which do not require the knowledge of the gradient (e.g. genetic algorithms). In the specific situations in which a relaxation process occurs, the solutions associated to a family of shapes converge toward the solution of a problem associated to a "mixture" between shape and void. The description of this relaxed problem depends roughly on the PDEs which are involved, and is not always possible. We refer the reader to Allaire [1] for PDEs with Neumann conditions on the free boundary which lead to homogenization of materials and to [9] for PDEs with Dirichlet boundary conditions which lead to relaxed measures. If this issues are available, optimization can be performed in the class of relaxed domains. Since an optimal shape

may not exist, any tentative to "project" a relaxed solution on the family of shapes is not mathematically optimal, but may be driven by technical/engineering requirements.

# 2 Problems with Dirichlet boundary conditions

Throughout the paper we consider shape optimization problems related to the Dirichlet Laplacian. The Laplace equation with Dirichlet boundary conditions is well posed (in a weak sense) on an arbitrary open set. The most "general" sets on which this equation is well posed are the quasi open sets and a direct extension of the equation can be done for measures (see below). From a certain point of view, the quasi-open sets represent for the Sobolev spaces $H^1_0$, what the measurable sets represent for the $L^p$ spaces. Since the class of quasi open sets is the largest class of sets on which the Laplace equation is well posed, it is natural to study optimal design problems in this class.

We start by introducing the quasi open sets.

## 2.1 Basic facts on capacity and quasi open sets

Let $N \geq 2$ and $E \subseteq D$ be two sets in $\mathbb{R}^N$, such that $D$ is open. The capacity of $E$ in $D$ is

$$C(E,D) = \inf\{\int_D |\nabla u|^2 + |u|^2 dx, \quad u \in \mathcal{U}_{E,D}\}$$

where $\mathcal{U}_{E,D}$ stands for the class of all functions $u \in H^1_0(D)$ such that $u \geq 1$ a.e. in an open set containing $E$. If no specification is made, $D$ is assumed to be $\mathbb{R}^N$. A pointwise property is said to hold quasi everywhere on a set $E$ (shortly q.e. on $E$) if the set of all points where the property does not hold has capacity zero.

A set $A \subseteq \mathbb{R}^N$ is called quasi open if for every $\epsilon > 0$ there exists an open set $U_\epsilon$ such that $A \cup U_\epsilon$ is open and $C(U_\epsilon, \mathbb{R}^N) < \epsilon$. A function $u : \mathbb{R}^N \mapsto \mathbb{R}$ is said to be quasi continuous if for all $\epsilon > 0$ there exists an open set $U_\epsilon$ with $C(U_\epsilon, \mathbb{R}^N) < \epsilon$ such that $u_{|\mathbb{R}^N \setminus U_\epsilon}$ is continuous on $\mathbb{R}^N \setminus U_\epsilon$ (see [20]). Every function $u \in H^1(\mathbb{R}^N)$ has a quasi continuous representative, $\tilde{u}$, such that $\tilde{u}(x) = u(x)$ a.e., and this representative is unique up to a set of zero capacity. Implicitly, throughout the paper all the pointwise properties of Sobolev functions are intended for quasi continuous representatives.

For a quasi open set $A$, the Sobolev space $H^1_0(A)$ is defined as follows:

$$H^1_0(A) = \{u \in H^1(\mathbb{R}^N) \ : \quad u = 0 \quad \text{q.e. on} \quad \mathbb{R}^N \setminus A\}. \tag{2}$$

If $A$ is open, the space $H^1_0(A)$ defined above coincides with the usual Sobolev space (see [21])

$$H^1_0(A) = cl_{H^1(\mathbb{R}^N)} C^\infty_0(A).$$

Let $A$ be a quasi open set of finite measure. Then the injection $H^1_0(A) \hookrightarrow L^2(A)$ is compact and the constant of the Poincaré inequality depends only on the measure of $A$ and the dimension of the space.

Let us denote $\mathcal{M}_0(\mathbb{R}^N)$ the set of all positive Borel measures $\mu$, possibly infinite, which vanish on sets of zero capacity. The regular set $A_\mu$ of a measure $\mu \in \mathcal{M}_0(\mathbb{R}^N)$ is the smallest quasi open set (in the sense of inclusions up to zero capacity sets) containing all the quasi open sets of finite $\mu$ measure, see [14]. We identify a set $A$ (open or quasi open) with the measure $\mu$ defined by $\mu(E) = 0$ if $C(E \cap A^c, \mathbb{R}^N) = 0$ and $\mu(E) = +\infty$ if $C(E \cap A^c, \mathbb{R}^N) > 0$.

More than defining the Laplace equation on quasi open sets, we introduce the Laplace equation on measures. For the measure we consider below, we assume that $|A_\mu| < +\infty$. The resolvent operator associated to the measure $\mu$ is $R_\mu : L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ defined by $R_\mu f = u$ where $u$ is the weak variational solution of

$$\left\{ \begin{array}{rcl} \int_{\mathbb{R}^N} \nabla u \nabla \phi dx + \int_{\mathbb{R}^N} u\phi d\mu & = & \int_A f\phi dx \ \ \forall \phi \in H^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N, \mu) \\ u & \in & H^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N, \mu). \end{array} \right. \tag{3}$$

As soon as the measure $\mu$ is the measure associated to a quasi open set $A$, the resolvent operator $R_\mu$ is precisely the usual resolvent operator $R_A$ associated to the Dirichlet problem for the Laplacian, which is defined by $R_A : L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ and $R_A(f) = u$ where $u$ is the weak variational solution of

$$\left\{ \begin{array}{rcl} \int_A \nabla u \nabla \phi dx & = & \int_A f\phi dx \ \ \forall \phi \in H^1_0(A) \\ u & \in & H^1_0(A). \end{array} \right. \tag{4}$$

We recall the following compactness result issued from the $\Gamma$-convergence theory (see [16] for the uniform bounded case and [6] for the unbounded case).

**Theorem 2.1** *Let $(\mu_n)_n$ be a sequence of measures $\mathcal{M}_0(\mathbb{R}^N)$. There exists a subsequence (denoted using the same index) and a measure $\mu \in \mathcal{M}_0(\mathbb{R}^N)$ such that for every $R > 0$ and $f \in L^2(\mathbb{R}^N)$*

$$R_{\mu_n \lfloor B(0,R)} f \to R_{\mu \lfloor B(0,R)} f \quad \textit{strongly in } \ L^2(\mathbb{R}^N).$$

*We say that $\mu_n$ $\gamma$-converges to $\mu$.*

Here, by $\mu \lfloor B(0,R)$ we denote the measure $\nu$ defined by $\nu(E) = \mu(E)$ if $\mu(E) < +\infty$ and $C(E \cap B(0,R)^c, \mathbb{R}^N) = 0$, and $\nu(E) = +\infty$ if not.

Note that if a sequence of uniformly bounded sets $A_n$ $\gamma$-converges to a measure $\mu$, then the regular set of the measure $\mu$ is also bounded and the operators $R_{A_n}$ converge in norm to $R_\mu$. This is a consequence of the compact embedding $H^1_0(D) \hookrightarrow L^2(D)$ for sets $D$ of finite measure. Also note that if the limit measure $\mu$ is the measure corresponding to an open (or quasi open) set $A$, then the convergence of the resolvent operators is equivalent to the so called Mosco convergence of the Sobolev spaces (see [9]) i.e.

1. $\forall \phi \in H^1_0(A)\ \exists \phi_n \in H^1_0(A_n)$ such that $\phi_n \to \phi$ strongly in $H^1(\mathbb{R}^N)$;
2. If $\phi_{n_k} \in H^1_0(A_{n_k})$ is such that $\phi_{n_k} \to \phi$ weakly in $H^1(\mathbb{R}^N)$, then $\phi \in H^1_0(A)$.

Finally, we also recall that the family of open sets is dense in the family of measures, for the $\gamma$-convergence ([16]).

## 2.2 Examples of shape optimization problems

In this section we give some classical examples of shape optimization problems associated to the Dirichlet Laplacian, and discuss the existence of a solution.

**Example 2.2** (Distribution of heat sources.) Let $D$ be a bounded open set of $\mathbb{R}^N$ and $f \in L^2(D)$. For every open set $A \subseteq D$, we define $u_{A,f} \in H_0^1(A)$ being the weak solution of

$$\begin{cases} -\Delta u_{A,f} & = f \text{ in } A \\ \quad u_{A,f} & = 0 \text{ on } \partial A \end{cases} \tag{5}$$

By extension with 0 on $D \setminus A$ we can consider $u_{A,f}$ as an element of $H_0^1(D)$. Let us denote by $u^* \in L^2(D)$ the target temperature. The problem of the distribution of heat sources can be formally written

$$\min_{A \in \mathcal{U}_{ad}} \int_D (u_{A,f} - u^*)^2 dx,$$

where

$$\mathcal{U}_{ad} = \{A \subseteq D \mid \ A \text{ open}\}.$$

As we will see below, this problem may have or not a solution, depending on $f$ and $u^*$. In order to justify the non existence of the optimal solution, we recall the following pioneering example of relaxation (see [15]).

**Theorem 2.3** *(Cioărănescu-Murat) Let $c > 0$ and $u^* \in H_0^1(D)$ the solution of*

$$\begin{cases} -\Delta u^* + cu^* & = f \textit{ in } D \\ \qquad u^* & = 0 \textit{ on } \partial D \end{cases} \tag{6}$$

*There exists a family of open sets $A_n \subseteq D$ such that*

$$u_{A_n,f} \overset{H_0^1(D)}{\rightharpoonup} u^*.$$

The construction of $A_n$ is done by extracting a family of balls of equal radius centered on regular grids (the radius and the grid depend on $n$). We also notice that this result is a particular constructive case of approximation in the $\gamma$-convergence of the measure $cdx\lfloor D$ by open sets.

The main consequence of this result is that the shape optimization problem

$$\min_{A \subseteq D} \int_D (u_{A,f} - u^*)^2 dx \tag{7}$$

does not have a solution. Indeed, the infimum in (7) is equal to zero, but, in general there is no $A$ such that $u_{A,f} = u^*$. If $f = 1$, by the maximum principle $u^* > 0$ on $D$, hence the optimal shape should coincide with $D$. Nevertheless $u^* \neq u_{D,f}$.

An intuitive proof of the non existence of smooth shapes can avoid the use of Theorem 2.3. Assume that $A^*$ is a solution of (7) for a constant function $f \equiv 1$ and $u^* = c1_D$. If $D \setminus \overline{A}^* \neq \emptyset$, we can choose $x \in D \setminus \overline{A}^*$ and $\varepsilon$ small enough such that $J(A^* \cup B_{x,\varepsilon}) < J(A^*)$, where $J$ is the cost functional introduced in (7)! This is an easy exercise, but this proof works well only if $A^*$ is not dense in $D$.

**Example 2.4** (Maximization of torsional rigidity) Let $f \equiv 1$ in (5). The shape optimization problem consists in

$$\max_{A \subseteq D, |A|=m} \int_A u_{A,1} dx. \tag{8}$$

If $D$ is large enough to contain a ball $B_m$ of measure $m$, problem (8) has $B_m$ as solution! The proof comes straight forwardly from the direct comparison

$$\int_A u_{A,1} dx \leq \int_{B_m} u_{B_m,1} dx.$$

Indeed, by the Schwarz symmetrization procedure, we get (see for instance [24])

$$-\frac{1}{2}\int_A u_{A,1} dx = \frac{1}{2}\int_A |\nabla u_{A,1}|^2 dx - \int_A u_{A,1} dx \geq \frac{1}{2}\int_{B_m} |\nabla u^*_{A,1}|^2 dx - \int_{B_m} u^*_{A,1} dx$$

$$\geq \frac{1}{2}\int_{B_m} |\nabla u_{B_m,1}|^2 dx - \int_{B_m} u_{B_m,1} dx = -\frac{1}{2}\int_{B_m} u_{B_m,1} dx.$$

Consequently, problem(8) has a solution without any constraint on the admissible class of shapes.

**Example 2.5** (Isoperimetric inequalities for eigenvalues.) We refer the reader to [22] for a survey of this topic, and to [9] for a detailed variational approach of the question.

For every quasi open set $A \subseteq \mathbb{R}^N$ of finite measure, the Dirichlet Laplacian has a spectrum consisting only of eigenvalues which can be ordered (multiplicity is counted)

$$0 < \lambda_1(A) \leq \lambda_2(A) \leq ...$$

and for each $\lambda_k$ there exists $u \in H^1_0(A)$ such that the equation

$$\begin{cases} -\Delta u &= \lambda_k(A) u \text{ in } A \\ \quad u &= 0 \text{ on } \partial A \end{cases} \tag{9}$$

holds in the usual weak sense

$$u \in H^1_0(A) \;\; s.t. \;\; \forall \varphi \in H^1_0(A) \quad \int_A \nabla u \nabla \varphi dx = \lambda_k(A) \int_A u \varphi dx.$$

Given $j : \mathbb{R}^k \to \mathbb{R}$, an open set $D \subseteq \mathbb{R}^N$ and a constant $m > 0$, a general isoperimetric inequality for the eigenvalues can be written

$$\min_{A \in \mathcal{U}_{ad}} j(\lambda_1(A), .., \lambda_k(A))$$

where

$$\mathcal{U}_{ad} = \{A \subseteq D \mid A \text{ quasi open}, \; |A| = m\}.$$

For $J_1(A) = \lambda_1(A)$ and $J_2(A) = \lambda_2(A)$ the answer to the isoperimetric problem was given by Faber and Krahn and is based on direct comparison by Schwarz rearrangement (see [24]). Already, for $J_3(A) = \lambda_3(A)$ the answer to the isoperimetric problem is not known. A conjecture states that the ball is the optimum in two and three dimensions. We notice that for $J_5(A) = \lambda_5(A)$ in two dimensions of the space, the solution (if it exists!) is neither a ball nor a union of balls (see [26]). Yet, there is not a known example of non-existence of solution for such a problem. In the next section we present several existence techniques for solutions to isoperimetric problems of eigenvalues.

# 3 How to prove existence of optimal shapes

In several classical isoperimetric inequalities for which the solution is the ball (or union of balls), the method to prove this assertion is based on direct comparisons and rearrangements of test functions. Nevertheless, as soon as the expected answer is not the ball (or simply not known), the direct comparison method cannot be applied, and a variational approach based on Steps 1 to 4 is more suitable.

## 3.1 Local existence: bounded design region

A fundamental existence result was proved by Buttazzo and Dal Maso in 1993 [14] and relies on the relaxation of the the Dirichlet problems associated to the Laplace operator. Let $D$ be a bounded open set and

$$\mathcal{U}_{ad} = \{A \subseteq D | A \text{ quasi open, } |A| = m\}.$$

**Theorem 3.1** *(Buttazzo and Dal Maso) Let $J : \mathcal{U}_{ad} \to \overline{\mathbb{R}}$ be a function which is $\gamma$-lower semicontinuous and monotone decreasing with respect to the set inclusion (up to sets of zero capacity). Then the optimization problem*

$$\min\{J(A) : A \in \mathcal{U}_{ad}\}$$

*has at least one solution.*

**Proof** Hint: the proof is a consequence of the following result (see [11]). Let $A_n \xrightarrow{\gamma} \mu$. Then, there exists a subsequence (still denoted using the same index) and a sequence of quasi open sets $\tilde{A}_n$ such that

$$D \supseteq \tilde{A}_n \supseteq A_n \text{ and } \tilde{A}_n \xrightarrow{\gamma} A_\mu,$$

where $A_\mu$ is the regular set of the measure $\mu$.

As a consequence, if $(A_n)$ is a minimizing sequence such that $A_n \xrightarrow{\gamma} \mu$, we consider the sets $\tilde{A}_n$ given by the previous property. The lower semi-continuity of $J$ for the $\gamma$-convergence and its monotonicity with respect to inclusions give

$$J(A_\mu) \le \liminf_{n\to\infty} J(\tilde{A}_n) \le \liminf_{n\to\infty} J(A_n).$$

In order to close the proof we notice in a first step that

$$|A_\mu| \le \liminf_{n\to\infty} |A_n| = m.$$

This is a consequence of the strong $L^2$-convergence

$$R_{A_n}1 \longrightarrow R_\mu 1.$$

In a second step, if $|A_\mu| < m$, we replace $A_\mu$ by $[A_\mu \cup B(0,R)] \cap D$ for a suitable $R$ such that $|[A_\mu \cup B(0,R)] \cap D| = m$. □

A fundamental hypothesis in this theorem is the boundedness of the design region $D$. In particular, this implies that the $\gamma$-convergence of a sequence of measures implies the norm convergence of the resolvent operators, hence of their spectrum! Consequently, the eigenvalues of the Dirichlet Laplacian are $\gamma$-continuous provided the design region is bounded.

**Example 3.2** We consider

$$J(A) = j(\lambda_1(A), .., \lambda_k(A)),$$

where $j : \mathbb{R}^k \to \mathbb{R}$ is lower semicontinuous and increasing in each variable. Then the theorem 3.1 of Buttazzo and Dal Maso applies.

In particular, typical examples are

$$J_k(A) = \lambda_k(A).$$

There are two important questions for these particular examples: the first one is to eliminate the boundedness of the design region and to consider quasi open sets in $\mathbb{R}^N$. We discuss this topic in the sequel. A second question is to prove the regularity of the solution, or at least to prove that the solution is an open set. We refer to [7] for recent results concerning $\lambda_1$. We notice that regularity results are available (up to our knowledge) only for the first eigenvalue, and not for the other ones. This is mainly due to the fact that the first eigenvalue acts as an energy, consequently the technique of Alt and Caffarelli can be adapted.

**Remark 3.3** If $J$ is not monotone decreasing, the existence of a solution to the shape optimization problem can still be achieved in some particular situations. We notice that there is not a known example of shape optimization problem of eigenvalues which has only relaxed solutions.

We recall the following result from [10]. The convention $\lambda_k(\emptyset) = +\infty$ is applied.

**Theorem 3.4** *Let*

$$j : \mathbb{R}^2 \to \mathbb{R} \cup \{+\infty\}$$

*be lower semicontinuous. Then the optimization problem*

$$\min\{j(\lambda_1(A), \lambda_2(A)) \mid A \textit{ quasi open}, \ A \subseteq D, \ |A| \le m\}$$

*has at least one solution.*

**Proof** Hint: the proof of this result relies on the closedeness of the set

$$E = \{(\lambda_1(A)\lambda_2(A)) : \ A \text{ quasi open}, \ A \subseteq D, \ |A| \le m\} \subseteq \mathbb{R}^2,$$

which is achieved using the relaxation and the convexity of $E$ in the vertical and horizontal directions. The estimate of Ashbaugh and Benguria (see [4])

$$\frac{\lambda_2(A)}{\lambda_1(A)} \leq \frac{\lambda_2(ball)}{\lambda_1(ball)}$$

is a key result result necessary in the proof.

Convexity in the horizontal direction means that for every $(x, y) \in E$ the closed segment joining $(x, y)$ and $(y, y)$ is contained in $E$, and convexity in the vertical direction means that for every $(x, y) \in E$ the closed segment joining $(x, y)$ and $(x, x\frac{\lambda_2(ball)}{\lambda_1(ball)})$ is contained in $E$. $\square$

## 3.2 Global existence: $\mathcal{D} = \mathbb{R}^N$

When the design region coincides with $\mathbb{R}^N$, the compactness of the $\gamma$-convergence (Theorem 2.1) does not imply the norm convergence of the resolvent operators, consequently the eigenvalues are not continuous for the $\gamma$-convergence. For example, a sequence of equal balls of centers in $x_n$, such that $\|x_n\| \to +\infty$, $\gamma$-converges to the empty set but the spectrum of the Dirichlet Laplacian remains unchanged! In fact, this behavior is produced by the lack of collective compactness of the union of all $H_0^1$ spaces on the moving domains in $L^2(\mathbb{R}^N)$. Since the spectrum of the Dirichlet Laplacian is invariant upon translations, one can use the uniform concentration compactness result for the resolvent operators [8] to deal with the isoperimetric inequalities for eigenvalues. This method is less interesting when dealing with non translation independent problems.

Let us consider the problem

$$\min_{A \subseteq \mathbb{R}^N, |A| = m} j(\lambda_1(A), ..., \lambda_k(A))$$

and a minimizing sequence of quasi open sets $(A_n)$. Before stating the main result, let us intuitively remark three typical behaviors which can occur for the sequence $(A_n)$. This is to be related to the concentration compactness principle of P.L. Lions [25]. Roughly speaking we are in one of the following situations.

**Case 1.** The mass of $A_n$ remains concentrate, but the center of mass goes to infinity. In the particular case of $J_1(A) = \lambda_1(A)$, for which we know that the ball is the solution, this situation can be given for example by the sequence

$$A_n = B(x_n, r)$$

with $\|x_n\| \to +\infty$. From this sequence, as such, we can not extract a *suitably* $\gamma$-convergent subsequence toward a measure, but if we translate the balls and center them around the origin, the spectrum on $A_n$ remains unchanged and we obtain a constant sequence (hence the resolvent operators converge in norm). Above, by *suitably* we precisely mean that the $\gamma$-convergence provides the norm convergence of the resolvent operators and not only the simple convergence.

**Case 2.** There are two regions which concentrate the mass, and the distance between these regions is going to infinity. If $J_2(A) = \lambda_2(A)$, for which we know that two equal and disjoint balls of mass $m/2$ is a solution, this situation can be give, for example, by

$$A_n = B(x_n, r) \cup B(-x_n, r)$$

with $\|x_n\| \to +\infty$. The spectrum is unchanged when $n$ varies but the balls are distancing. Again, there is no subsequence which suitably $\gamma$-converges! Translation is not enough to deal with this case and one should somehow deal independently with each region of concentration of mass.

**Case 3.** For any translation, there is no concentration of mass. This situation is theoretically possible, but it implies that each eigenvalue converges to $+\infty$ and for shape functionals like $A \to \lambda_k(A)$ this is to be excluded (see [8]). A typical example would be

$$A_n = \cup_{i=1}^n B(x_{i,n}, r_{i,n})$$

with $r_{i,n} \to 0$ and $\|x_{i,n} - x_{j,n}\| \to +\infty$ when $n \to \infty$.

In the sequel we recall the main results dealing with the concentration compactness of the resolvent operators (see [8]). The first one proves that if $R_{A_n}(1)$ converges strongly in $L^2(\mathbb{R}^N)$ to some function, then any weakly convergent sequence $u_n \in H^1(\mathbb{R}^N)$, such that, $u_n \in H_0^1(A_n)$ is strongly convergent in $L^2(\mathbb{R}^N)$. As a consequence, the sequence $A_n$ $\gamma$-converges to a measure and each eigenvalue of the Dirichlet Laplacian on $A_n$ converges to the corresponding eigenvalue associated to the measure.

**Theorem 3.5** *Let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of open (or quasi open) sets of uniformly bounded measure. If $R_{A_n}1 \xrightarrow{L^2(\mathbb{R}^N)} w$, then for any sequence $\{u_n\}_{n\in\mathbb{N}}$ such that $u_n \in H_0^1(A_n)$ and $u_n \overset{H^1(\mathbb{R}^N)}{\rightharpoonup} u$ we have $u_n \xrightarrow{L^2(\mathbb{R}^N)} u$, i.e. injection*

$$\bigcup_n H_0^1(A_n) \subseteq L^2(\mathbb{R}^N)$$

*is compact.*

The second theorem describes the uniform behavior of the sequence of Sobolev spaces through the resolvent operators.

**Theorem 3.6** *Let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of open (or quasi open) sets of uniformly bounded measure. There exists a subsequence still denoted with the same index such that one of the following situations occurs:*

**Compactness:** *There exists a sequence of vectors $\{y_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^N$ and a positive Borel measure $\mu$, vanishing on sets of zero capacity, such that $y_n + A_n$ $\gamma$-converges to the measure $\mu$ and $R_{y_n+A_n}$ converges in the uniform operator topology of $L^2(\mathbb{R}^N)$ to $R_\mu$.*

**Dichotomy:** *There exists a sequence of subsets $\tilde{A}_n \subseteq A_n$, such that*

$$\|R_{A_n} - R_{\tilde{A}_n}\|_2 \to 0, \quad \text{and} \quad \tilde{A}_n = A_n^1 \cup A_n^2$$

*with $d(A_n^1, A_n^2) \to \infty$ and $\liminf_{n\to\infty} |A_n^i| > 0$ for $i = 1, 2$.*

As main application of these two theorems, we recall the following existence result from [12].

**Theorem 3.7** *For $k \geq 2$, let us suppose that a bounded minimizer exists for $\lambda_1, ..., \lambda_{k-1}, \lambda_k$ in in the class of all quasi open sets of $\mathbb{R}^N$ of measure $m$. Then, at least one minimizer (bounded or unbounded) exists for $\lambda_{k+1}$.*

**Proof** Hint: take a minimizing sequence. As stated by Theorem 3.6, either compactness or dichotomy holds. If compactness occurs, the subsequence which $\gamma$-converges toward a measure will provide the optimal set as being the regular set of the measure (as in the bounded design region case).

If dichotomy occurs, than we replace the minimizing sequence by the disconnected sequence. Every connected component is solution of an isoperimetric inequality for a strictly lower index eigenvalue!

For $k = 2$, this leads to minimizers of $\lambda_1$ or $\lambda_2$ which are consisting on balls, hence either the optimal set is a (quasi) connected set issued from the compactness, or the optimal set is the union of three equal balls. Thus, a minimizer exists for $\lambda_3$!

For $k > 2$ and provided the minimizers for $j = 3, .., k$ are bounded, we construct the solution of the isoperimetric inequality for $\lambda_{k+1}$ as for $k = 2$. In the dichotomy case, the solution is a union of minimizers corresponding to isoperimetric inequalities for strictly lower index eigenvalues. Boundedness is a sufficient hypothesis in order to be able to construct this union! □

**Theorem 3.8** *Let $j : \mathbb{R}^2 \to \mathbb{R}$ be an increasing lower semicontinuous functional in both variables. Then problem*

$$\min\{j(\lambda_1(A), \lambda_2(A)) : A \text{ quasi open }, A \subseteq \mathbb{R}^N,\ |A| \leq c\} \tag{10}$$

*has at least one solution.*

**Proof** Hint: let us consider a minimizing sequence $\{A_n\}_{n\in\mathbb{N}}$, such that $\lambda_1(A_n) \to x$, $\lambda_2(A_n) \to y$ for $n \to \infty$, and $x, y \in \mathbb{R} \cup \{\infty\}$. Since $j$ is increasing in both variables, we get $x, y < \infty$. Using Theorem 3.6, we construct a quasi open set $A$, with $|A| \leq c$ such that $\lambda_1(A) \leq x$ and $\lambda_2(A) \leq y$ (see [8]). The set $A$ is a minimizer for $j$. □

# 4 Further remarks

The variational approach for shape optimization problems requires a topology on the family of shapes. There is no a standard topology which can be used in all the situations, since there are two requirements which can not be universally satisfied. We require from the topology to have good compactness properties and the shape functional to be continuous (or lower semicontinuous) in this topology. This last requirement relays on the relationship between the convergence of a sequence of shapes and the behavior of the shape functional and, implicitly of the resolvent operator and the partial differential equation. There are two points of view: either fix a geometric topology and see under which conditions the functional is lower semicontinuous, or define as topology the weakest topology providing continuity properties for the shape functional, and search compactness results. This last issue is related to the

$\gamma$-convergence. For the convenience of the reader, we recall the fundamental results of the first issue: fix a geometric convergence and seek the continuity of the resolvent operators.

There are several geometric convergences, but we will refer in the sequel only to the Hausdorff and the compact convergences. The Hausdorff complementary topology is defined by the distance

$$d_{H^c}(A_1, A_2) = \sup_{x \in \mathbb{R}^N} |d(x, A_1^c) - d(x, A_2^c)|,$$

where $d(x, F)$ is the Euclidean distance from the point $x$ to the set $F$. We notice that the family of open subsets of a bounded open set is compact in this topology.

The compact convergence is a sequential topology in the family of open sets and by definition: $A_n$ *compactly converges to* $A$ *if for any compact* $K \subset A \cup \overline{A}^c$ *there exists* $n_K \in \mathbb{N}$ *such that for all* $n \geq n_K$, $K \subseteq A_n \cup \overline{A}_n^c$.

Let $D$ be a bounded open set. A non-exhaustive list of compact classes of domains in which the $\gamma$-convergence is equivalent to the $H^c$-convergence is the following (see [9] for a detailed exposition of this question):

- The class of open convex sets contained in $D$.

- The class of open subsets $A \subseteq D$ satisfying a uniform exterior cone property, i.e. there exists a closed cone such that for every point $x_0$ on the boundary of $A$ there is a congruent cone having the vertex at $x_0$ and lying in $A^c$.

- The class of domains satisfying a uniform flat cone condition , i.e. as above, but with the weaker requirement that the cone may be flat, that is of dimension $N-1$.

- The class of open subsets $A \subseteq D$ satisfying a uniform capacitary density condition , i.e. such that there exist $c, r > 0$ (independent of $A$) such that for every $x \in \partial A$, we have

$$\forall t \in (0, r) \qquad \frac{\mathrm{cap}(A^c \cap B_{x,t}, B_{x,2t})}{\mathrm{cap}(B_{x,t}, B_{x,2t})} \geq c,$$

  where $B_{x,s}$ denotes the ball of radius $s$ centered at $x$ (see [13]).

- The class of open subsets $A \subseteq D$ satisfying a uniform Wiener condition (see [13]), i.e. for every point $x \in \partial A$

$$\int_r^R \frac{\mathrm{cap}(A^c \cap B_{x,t}, B_{x,2t})}{\mathrm{cap}(B_{x,t}, B_{x,2t})} \frac{dt}{t} \geq g(r, R, x) \text{ for every } 0 < r < R < 1$$

  where $g : (0,1) \times (0,1) \times D \to \mathbb{R}_+$ is independent of $A$, such that for every $R \in (0,1)$ $\lim_{r \to 0} g(r, R, x) = +\infty$ locally uniformly on $x$.

Another interesting class, which is only of topological type and is related to the capacity density condition, was given by Šverák [29] and consists in the following.

- For $N = 2$, the class of all open subsets $A$ of $D$ for which the number of connected components of $\overline{D} \setminus A$ is uniformly bounded.

The main result relating the compact convergence to the $\gamma$-convergence is due to Keldysh [19]. It involves the stability of the set $A$. An open set $A$ is called stable if for any function $u \in H^1(\mathbb{R}^N)$ vanishing a.e. on $\overline{A}^c$ we have $u_{|A} \in H^1(A)$. Roughly speaking, open sets with cracks are not stable (see also [19]). We give first a characterization theorem for stability (see [8] for a recent proof based on $\gamma$-convergence).

**Theorem 4.1** *A bounded open set $A$ is stable if and only if for any $x \in \mathbb{R}^N, r > 0$ we have*

$$C(B_{x,r} \setminus A, B_{x,2r}) = C(B_{x,r} \setminus \overline{A}, B_{x,2r}). \tag{11}$$

The relationship between the $\gamma$-convergence and the compact convergence is contained in the following result [19].

**Theorem 4.2** *If $A_n$ compactly converges to $A$ and if $A$ is stable, then $A_n$ $\gamma$-converges to $A$.*

**Proof** Hint: the proof is a direct consequence of the equivalence between the Mosco convergence of the Sobolev spaces and the $\gamma$-convergence in the family of quasi open sets, associated to the characterization of the $H^1_0$ spaces (see relation (2)). □

Here are some extensions to more general equations.

**Remark 4.3 Operators in divergence form.** Instead of the Laplace operator, one could consider an elliptic operator of the form $-\text{div}(A(x)\nabla u)$, where $A \in L^\infty(D, \mathbb{R}^{N\times N})$ is symmetric and such that $\alpha I_d \leq A \leq \beta I_d$. The local existence works in the same way, but the global existence may fail, because the eigenvalues of the operator are not translation invariant. In some particular situations, as for example $A$ periodic, global existence could be achieved.

**Remark 4.4 Nonlinear operators.** For nonlinear operators of the form $-\text{div}(A(x, \nabla u))$ the $\gamma$-convergence theory works as in the linear case. All shape continuity results presended above hold in similar classes of domains: convex, uniform cone, flat cone, $p$-capacity density condition, $p$-uniform Wiener criterion and the existence question for shape functionals depending on the state can be treated as in the linear case. The eigenvalues of the $p$-Laplacian are subject to intensive research and, despite the first and the second one, there is not relevant isoperimetric inequality.

**Remark 4.5 Systems of equations.** The case of elliptic systems, as the elasticity equations, where the functional space is a product of $H^1_0$ spaces, can be treated as the scalar case. New difficulties appear when dealing with the convergence in the sense of Mosco of free divergence spaces (like the Stokes equation), because for non smooth open sets $A$ we may have

$$\{u \in [H^1_0(A)]^N : \text{div}u = 0\} \neq \text{cl}_{[H^1_0(A)]^N}\{u \in [C^\infty_0(A)]^N : \text{div}u = 0\}.$$

**Remark 4.6 Higher order operators.** Several isoperimetric inequalities for the eigenvalues of the bi-Laplacian are referenced in [3] and some of them are open. Following Willms and Weinberger, a purely variational approach (see for instance [5]) could solve the conjecture on the buckling load of the clamped plate in two dimensions of the space.

**Remark 4.7 Neumann boundary conditions.** Isoperimetric inequalities for the eigenvalues of the Laplacian with Neumann boundary conditions are more difficult in the variational frame, and largely open. If $A_n, A$ are Lipschitz (but not uniformly Lipschitz) such that $A_n \xrightarrow{H^c} A$, then the convergence of the spectrum does not hold in general. This is mainly due to the fact that the union of the $H^1$-spaces does not inject compactly in $L^2$ (extensions by zero are considered in order to put all functions into a fixed $L^2$ space). Nevertheless, some isoperimetric inequalities could be obtained by direct comparison. For example the ball maximizes the first non-zero eigenvalue among all bounded Lipschitz open sets of given measure (Weinberger and Szego, see [22] and [3] for a more exhaustive discussion).

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