

On the characterization of the compact embedding of Sobolev spaces

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Abstract

For every positive regular Borel measure, possibly infinite valued, vanishing on all sets of p -capacity zero, we characterize the compactness of the embedding $W^{1,p}(\mathbb{R}^N) \cap L^p(\mathbb{R}^N, \mu) \hookrightarrow L^q(\mathbb{R}^N)$ in terms of the qualitative behavior of some characteristic PDE. This question is related to the well posedness of a class of geometric inequalities involving the torsional rigidity and the spectrum of the Dirichlet Laplacian introduced by Polya and Szegő [14] in 1951. In particular, we prove that finite torsional rigidity of an arbitrary domain (possibly with infinite measure), implies the compactness of the resolvent of the Laplacian.

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1 Introduction

The study of the compact embeddings of Sobolev spaces $W_0^{1,p}(\Omega)$ into $L^p(\Omega)$ has a long history; starting from the simplest case when Ω is bounded, in which the compactness always occurs, several generalizations have been found (see for instance [2]). For $p = 2$, this is related (actually equivalent) to the compactness of the resolvent operator $\mathcal{R}_\Omega : L^2(\Omega) \rightarrow L^2(\Omega)$ which associates to every function $f \in L^2(\Omega)$ the solution of the elliptic PDE

$$\begin{cases} -\Delta u + u = f & \text{in } \Omega \\ u \in H_0^1(\Omega), \end{cases}$$

then providing the existence of a discrete spectrum made by a nondecreasing sequence of eigenvalues.

Analogously, the same kind of problems arise for the Schrödinger operator

$$\begin{cases} -\Delta u + u + V(x)u = f & \text{in } \mathbb{R}^N \\ u \in H^1(\mathbb{R}^N); \end{cases}$$

the compactness of its resolvent operator $\mathcal{R}_V : L^2(\mathbb{R}^N) \rightarrow L^2(\mathbb{R}^N)$ is again an important issue for its spectral analysis.

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We unify the two topics considering the so-called *capacitary measures* and the related Sobolev spaces $W_\mu^{1,p}$ (see Section 2 for the precise definitions); when $\mu = \infty_{\mathbb{R}^N \setminus \Omega}$ we recover the usual Sobolev space $W_0^{1,p}(\Omega)$, while $\mu = V(x) dx$ provides the natural space for the Schrödinger operator.

The main results of the paper deal with some characterizations for the compact embeddings

$$W_\mu^{1,p} := W^{1,p}(\mathbb{R}^N) \cap L_\mu^p \hookrightarrow L^p(\mathbb{R}^N) \quad \text{and} \quad W_\mu^{1,p} \hookrightarrow L^1(\mathbb{R}^N)$$

in terms of the qualitative behavior of the formal solution (see the precise definition in Section 2) of the equation

$$-\Delta_p w + w^{p-2} w + \mu w^{p-2} w = 1.$$

Precisely, we prove that the (inclusion and) compactness $W_\mu^{1,p} \hookrightarrow L^1(\mathbb{R}^N)$ is equivalent to $\int_{\mathbb{R}^N} w dx < +\infty$ and the compactness $W_\mu^{1,p} \hookrightarrow L^p(\mathbb{R}^N)$ is equivalent to the uniform vanishing at infinity of w .

Of course, as soon as the compact embedding $W^{1,p}(\mathbb{R}^N) \cap L_\mu^p \hookrightarrow L^q(\mathbb{R}^N)$ holds for $q = 1$ or $q = p$, the embedding $W^{1,p}(\mathbb{R}^N) \cap L_\mu^p \hookrightarrow L^r(\mathbb{R}^N)$ is also compact for every $q \leq r < p^*$, where $p^* = Np/(N-p)$, by a standard argument based on Hölder inequality and completeness of $L^r(\mathbb{R}^N)$ (see for instance [2, Lemma 6.7]).

Clearly, if the torsional rigidity of the measure μ is finite (take $p = 2$) then by the maximum principle $\int_{\mathbb{R}^N} w dx < +\infty$ so that the embedding $H_\mu^1 \hookrightarrow L^q(\mathbb{R}^N)$ holds for every $1 < q < 2^*$. The torsional rigidity of μ is defined by

$$P(\mu) = \int_{\mathbb{R}^N} u dx, \quad \text{where } -\Delta u + \mu u = 1 \text{ in } (H_\mu^1)', \text{ and } u \in H_\mu^1.$$

The question of analysing the torsional rigidity and the torsion function in relationship with the geometric domain and the spectrum of the Dirichlet Laplacian was already addressed in [15] and [3]. Precisely, in these papers the authors are interested to situations when the torsion function belongs to $L^\infty(\mathbb{R}^N)$ and the torsional rigidity is finite. As well, sufficient conditions expressed in terms of the distance function to the boundary of the domains give information about summability of u .

For the simplicity of the exposition we prove all results for $p = 2$, which is more rich than the nonlinear framework. In the last section, we briefly consider the general case $1 < p < +\infty$ for which we point out the main differences with respect to the Hilbertian case.

2 The Sobolev space H_μ^1

We will use in the following the notion of *capacity* of a subset E of \mathbb{R}^N , defined by

$$\text{cap}(E) = \inf \left\{ \int_{\mathbb{R}^N} |\nabla u|^2 + u^2 dx : u \in \mathcal{U}_E \right\},$$

where \mathcal{U}_E is the set of all functions u of the Sobolev space $H^1(\mathbb{R}^N)$ such that $u \geq 1$ almost everywhere in a neighbourhood of E . Below we summarize the main properties of the capacity and the related convergences. For further details we refer to [5] or to [10].

If a property $P(x)$ holds for all $x \in E$ except for the elements of a set $Z \subset E$ with $\text{cap}(Z) = 0$, then we say that $P(x)$ holds *quasi-everywhere* (shortly *q.e.*) on E . The expression *almost everywhere* (shortly *a.e.*) refers, as usual, to the Lebesgue measure.

A subset Ω of \mathbb{R}^N is said to be *quasi-open* if for every $\varepsilon > 0$ there exists an open subset Ω_ε of \mathbb{R}^N , such that $\text{cap}(\Omega_\varepsilon \Delta \Omega) < \varepsilon$, where Δ denotes the symmetric difference of sets. Equivalently, a

quasi-open set Ω can be seen as the set $\{u > 0\}$ for some function u belonging to the Sobolev space $H^1(\mathbb{R}^N)$. Note that a Sobolev function is only defined quasi-everywhere, so that a quasi-open set Ω does not change if modified by a set of capacity zero.

A function $f : \mathbb{R}^N \rightarrow \mathbb{R}$ is said to be *quasi-continuous* (respectively *quasi-lower semicontinuous*) if for every $\varepsilon > 0$ there exists a continuous (respectively lower semicontinuous) function $f_\varepsilon : \mathbb{R}^N \rightarrow \mathbb{R}$ such that $\text{cap}(\{f \neq f_\varepsilon\}) < \varepsilon$. It is well known (see, e.g., Ziemer [16]) that every function u of the Sobolev space $H^1(\mathbb{R}^N)$ has a quasi-continuous representative, which is uniquely defined up to a set of capacity zero. We shall always identify the function u with its quasi-continuous representative, so that a pointwise condition can be imposed on $u(x)$ for quasi-every $x \in \mathbb{R}^N$. Notice that with this convention we have

$$\text{cap}(E) = \min \left\{ \int_{\mathbb{R}^N} |\nabla u|^2 + |u|^2 dx : u \in H^1(\mathbb{R}^N), u \geq 1 \text{ q.e. on } E \right\}.$$

For every quasi-open set $\Omega \subset \mathbb{R}^N$ we denote by $H_0^1(\Omega)$ the space of all functions $u \in H^1(\mathbb{R}^N)$ such that $u = 0$ q.e. on $\mathbb{R}^N \setminus \Omega$, endowed with the Hilbert space structure inherited from $H^1(\mathbb{R}^N)$. In this way $H_0^1(\Omega)$ is a closed subspace of $H^1(\mathbb{R}^N)$. If Ω is open, then the definition above of $H_0^1(\Omega)$ is equivalent to the usual one (see [1]). If Ω is bounded the linear operator $-\Delta$ on $H_0^1(\Omega)$ has a compact resolvent, hence a discrete spectrum, denoted by $\lambda_1(\Omega) \leq \lambda_2(\Omega) \leq \lambda_3(\Omega) \leq \dots$; for general Ω this is not true and the question is related to the compact embedding of $H_0^1(\Omega)$ into $L^2(\Omega)$ which will be considered in the next section.

More generally, we can consider the Sobolev spaces H_μ^1 made with respect to the so-called *capacitary measures*; precisely, we consider nonnegative regular Borel measures μ on \mathbb{R}^N , possibly $+\infty$ valued, that vanish on all sets of capacity zero. The family of these measures is denoted by $\mathcal{M}_0(\mathbb{R}^N)$. We stress the fact that the measures μ above do not need to be finite, and may take the value $+\infty$ even on large parts of \mathbb{R}^N .

Example 2.1 If $N - 2 < \alpha \leq N$ the α -dimensional Hausdorff measure \mathcal{H}^α is a capacitary measure (and consequently every μ absolutely continuous with respect to \mathcal{H}^α as well). In fact all Borel sets with capacity zero have a Hausdorff dimension which is less than or equal to $N - 2$.

Example 2.2 Another example of capacitary measure is, for every $S \subset \mathbb{R}^N$, the measure ∞_S defined by

$$\infty_S(B) = \begin{cases} 0 & \text{if } \text{cap}(B \cap S) = 0, \\ +\infty & \text{otherwise.} \end{cases} \quad (1)$$

The norm

$$\|u\|_{1,\mu}^2 = \int_{\mathbb{R}^N} (|\nabla u|^2 + |u|^2) dx + \int_{\mathbb{R}^N} |u|^2 d\mu$$

makes

$$H_\mu^1 = \{u \in H^1(\mathbb{R}^N) : \|u\|_{1,\mu} < +\infty\}$$

a Hilbert space, and for every $f \in L^2(\mathbb{R}^N)$ (or more generally for $f \in (H_\mu^1)'$) we may consider the elliptic PDE

$$-\Delta u + u + \mu u = f, \quad u \in H_\mu^1 \quad (2)$$

whose precise sense has to be given in the weak form

$$\int_{\mathbb{R}^N} (\nabla u \nabla \phi + u \phi) dx + \int_{\mathbb{R}^N} u \phi d\mu = \int_{\mathbb{R}^N} f \phi dx \quad \forall \phi \in H_\mu^1.$$

Notice that, since the Sobolev functions u are defined quasi-everywhere and the capacity measures μ vanish on all sets with capacity zero, the products $u\mu$ are well defined. In particular, if $\mu = \infty_S$ we have

$$H_\mu^1 = \{u \in H^1(\mathbb{R}^N) : u = 0 \text{ q.e. on } S\}.$$

By standard Lax-Milgram methods, for every $f \in L^2(\mathbb{R}^N)$ equation (2) has a unique solution, that we denote by $\mathcal{R}_\mu(f)$; in this way we may define the resolvent operator $\mathcal{R}_\mu : L^2(\mathbb{R}^N) \rightarrow L^2(\mathbb{R}^N)$ whose compactness will be discussed in the next section.

For every $\mu \in \mathcal{M}_0(\mathbb{R}^N)$ and every quasi-open set Ω we define the Dirichlet restriction

$$\mu \upharpoonright \Omega = \mu + \infty_{\mathbb{R}^N \setminus \Omega}$$

which takes the value $+\infty$ outside Ω ; in other words, solving the PDE (2) with $\mu \upharpoonright \Omega$ means that we are considering the same PDE but with Dirichlet condition $u \in H_0^1(\Omega) \cap H_\mu^1$. The classical restriction of measures is denoted by $\mu \lfloor \Omega$.

The space $\mathcal{M}_0(\mathbb{R}^N)$ of all capacity measures can be endowed with an interesting convergence structure, called γ -convergence: we say that $\mu_n \rightarrow \mu$ in the γ convergence if for every ball B

$$\mathcal{R}_{\mu_n \upharpoonright B} \rightarrow \mathcal{R}_{\mu \upharpoonright B} \text{ in } \mathcal{L}(L^2(\mathbb{R}^N)).$$

It is well known (see for instance [5]), that the γ -convergence is equivalent to any of the assertions below

- $\mathcal{R}_{\mu_n \upharpoonright B}(f) \rightarrow \mathcal{R}_{\mu \upharpoonright B}(f)$ weakly in $L^2(\mathbb{R}^N)$, for all balls B and for all $f \in L^2(\mathbb{R}^N)$
- $\mathcal{R}_{\mu_n \upharpoonright B}(1) \rightarrow \mathcal{R}_{\mu \upharpoonright B}(1)$ weakly in $L^2(\mathbb{R}^N)$, for all $R > 0$

The γ -convergence is metrizable and the family of measures $\mathcal{M}_0(\mathbb{R}^N)$ is compact for the γ -convergence. Clearly, the spectrum of the operator in (2) is γ -continuous on the families $\{\mu \upharpoonright B_R : \mu \in \mathcal{M}_0(\mathbb{R}^N)\}$, but in general is not γ -continuous on $\mathcal{M}_0(\mathbb{R}^N)$. Moreover, the family of measures of the form ∞_S with S smooth and compact (that we often identify with the domain $\mathbb{R}^N \setminus S$) is γ -dense in $\mathcal{M}_0(\mathbb{R}^N)$ as well as the family of measures of the form $a(x) dx$ with $a(x)$ smooth.

In the following we shall use the function w_μ that *formally* solve the PDE

$$-\Delta u + u + \mu u = 1.$$

Since in general the constant 1 does not belong to $(H_\mu^1)'$ we define w_μ as

$$w_\mu = \lim_{R \rightarrow +\infty} \mathcal{R}_{\mu \upharpoonright B_R}(1).$$

By the maximum principle the limit above exists since the solutions $\mathcal{R}_{\mu \upharpoonright B_R}(1)$ are monotonically increasing with R ; moreover, it is easy to see that $0 \leq \mathcal{R}_{\mu \upharpoonright B_R}(1) \leq 1$, so that $0 \leq w_\mu \leq 1$.

In this way, there is a classical extension of the operator \mathcal{R}_μ on $L^\infty(\mathbb{R}^N)$, defined by

$$\mathcal{R}_\mu(f) = \sup_R \mathcal{R}_{\mu \upharpoonright B_R}(f^+) - \sup_R \mathcal{R}_{\mu \upharpoonright B_R}(f^-),$$

which is linear and continuous (see for instance [4]).

For every measure μ we denote by $\lambda_1(\mu)$ the spectral abscissa of the Laplacian associated to the measure μ by

$$\lambda_1(\mu) = \inf_{u \in H_\mu^1, u \neq 0} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 dx + \int_{\mathbb{R}^N} u^2 dx + \int_{\mathbb{R}^N} u^2 d\mu}{\int_{\mathbb{R}^N} u^2 dx}.$$

Clearly if H_μ^1 is compactly embedded in $L^2(\mathbb{R}^N)$, the spectral abscissa is the first eigenvalue of the operator in (2). By an abuse of notation, we still denote it $\lambda_1(\mu)$, even if the compact embedding does not hold.

Throughout the paper we consider a given function $\theta \in C_c^\infty(B_2)$ such that $0 \leq \theta \leq 1$ and $\theta = 1$ on B_1 . For every $R > 0$, we set $\theta_R(x) = \theta(\frac{x}{R})$. We shall often use the fact that for every function $u \in H_\mu^1$ we have $u\theta_R \rightarrow u$ strongly in H_μ^1 as $R \rightarrow +\infty$, and the estimate

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla(u\theta_R)|^2 dx &\leq 2 \int_{\mathbb{R}^N} |\nabla u|^2 dx + 2 \int_{\mathbb{R}^N} u^2 |\nabla \theta_R|^2 dx \\ &\leq 2 \int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{2|\nabla \theta|_\infty^2}{R^2} \int_{\mathbb{R}^N} u^2 dx. \end{aligned}$$

3 Characterization of the compactness in the linear frame

Here are the main results of the paper.

Theorem 3.1 *Let $\mu \in \mathcal{M}_0(\mathbb{R}^N)$. Then the embedding $H_\mu^1 \hookrightarrow L^2(\mathbb{R}^N)$ is compact if and only if*

$$w_\mu \cdot 1_{B_R^c} \rightarrow 0 \text{ in } L^\infty(\mathbb{R}^N) \text{ as } R \rightarrow +\infty.$$

Theorem 3.2 *Let $\mu \in \mathcal{M}_0(\mathbb{R}^N)$. The following assertions are equivalent.*

1. $w_\mu \in L^1(\mathbb{R}^N)$.
2. $H_\mu^1 \subset L^1(\mathbb{R}^N)$ with continuous injection.

Moreover, if one of the two assertions above holds, then the embedding $H_\mu^1 \hookrightarrow L^1(\mathbb{R}^N)$ is compact.

For the sake of completeness, in Proposition 3.1 below we collect several characterizations of the compact embedding in $L^2(\mathbb{R}^N)$ in the case of capacity measures. Some of them are only direct extensions to capacity measures of well known characterizations holding for domains or positive potentials absolutely continuous with respect to the Lebesgue measure, of the form $V(x)dx$. In the case of open sets, condition 4) is a consequence of [4, Theorem 3.1] while condition 8) is an extension to measures of the characterization of the compactness in [2, Theorem 6.19].

Proposition 3.3 *Let $\mu \in \mathcal{M}_0(\mathbb{R}^N)$. The following assertions are equivalent.*

1. $H_\mu^1 \hookrightarrow L^2(\mathbb{R}^N)$ is compact;
2. \mathcal{R}_μ is compact;
3. $\mathcal{R}_{\mu \upharpoonright B_R} \rightarrow \mathcal{R}_\mu$ in $\mathcal{L}(L^2(\mathbb{R}^N))$ as $R \rightarrow +\infty$;
4. $\mathcal{R}_{\mu \upharpoonright B_R} \rightarrow \mathcal{R}_\mu$ in $\mathcal{L}(L^\infty(\mathbb{R}^N))$ as $R \rightarrow +\infty$;
5. $\lambda_1(\mu \upharpoonright B_R^c) \rightarrow +\infty$ as $R \rightarrow +\infty$;
6. for every $h > 0$, $\lambda_1(\mu \upharpoonright B_h(x)) \rightarrow +\infty$ as $\|x\| \rightarrow +\infty$;
7. there exists $h > 0$, $\lambda_1(\mu \upharpoonright B_h(x)) \rightarrow +\infty$ as $\|x\| \rightarrow +\infty$;

8. there exists $h > 0$, such that for every $\varepsilon > 0$ there exists $r \geq 0$ such that

$$\inf_{u \in H_{\mu|_H}^1(\mathbb{R}^N), u \neq 0} \frac{\int_H |\nabla u|^2 dx + \int_H u^2 d\mu}{\int_H u^2 dx} \geq \frac{1}{\varepsilon},$$

for every N -cube $H = [x, x + h)$, with $\|x\| \geq r$.

In the sequel we often use the notation $[x, y)$ for the set $\prod_{i=1}^N [x_i, y_i)$. We now give some examples, some of them classical, in order to highlight the various conditions in Theorems 3.1, 3.2 and Proposition 3.3.

Example 3.4 Let $\Omega \subseteq \mathbb{R}^N$ be an open set such that for some $h > 0$ $|\Omega \cap B_h(x)| \rightarrow 0$ as $\|x\| \rightarrow +\infty$. It is well known that in this case the embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact. Indeed, the Faber-Krahn inequality gives that

$$\lambda_1(\Omega \cap B_h(x)) \geq \lambda_1([\Omega \cap B_h(x)]^*),$$

where $[\Omega \cap B_h(x)]^*$ is the ball of the same volume as $\Omega \cap B_h(x)$. Since $|\Omega \cap B_h(x)| \rightarrow 0$, then $\lambda_1([\Omega \cap B_h(x)]^*) \rightarrow +\infty$, so that condition 7) in Proposition 3.3 is satisfied.

Example 3.5 Let $\Omega \subseteq \mathbb{R}^N$ be an open set such that $|\Omega| < +\infty$. Then $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact. This is an immediate consequence of Example 3.4

Example 3.6 There exist situations when the conditions of Example 3.5 is only sufficient and not necessary for compactness. Simply consider in \mathbb{R}^2

$$\Omega = (0, +\infty) \times (0, 1) \setminus \bigcup_{n \in \mathbb{N}} \{x_n\} \times (0, 1),$$

where

$$x_n = \log(1 + n).$$

Clearly, $|\Omega| = +\infty$ and also $|\Omega \cap B_1(x_n, 0)| \not\rightarrow 0$, but condition 7) of Proposition 3.3 is satisfied since

$$\lambda_1((x_n, x_{n+1}) \times (0, 1)) \rightarrow +\infty.$$

Example 3.7 We give here an example showing that higher order sumability of w_μ is not related to the compact embedding of H_μ^1 in $L^2(\mathbb{R}^N)$. Consider as in the previous example

$$\Omega = (0, +\infty) \times (0, 1) \setminus \bigcup_{n \in \mathbb{N}} \{x_n\} \times (0, 1),$$

where $(x_n)_n$ is an increasing sequence of positive numbers such that $x_{n+1} - x_n \rightarrow 0$. This readily gives the compact embedding of $H_0^1(\Omega)$ in $L^2(\mathbb{R}^N)$. Clearly, we can tune the x_n such that for some $\alpha > 0$ $\int w_\mu^\alpha dx = +\infty$. In particular, if $\alpha = 1$ there is no compact embedding in $L^1(\mathbb{R}^N)$.

Example 3.8 Obviously, the measure μ has a decisive influence on the compactness. It is well known that if $\mu = V(x) dx$, with $V : \mathbb{R}^N \rightarrow [0, +\infty)$, measurable and $V(x) \rightarrow +\infty$ as $\|x\| \rightarrow +\infty$, then $H_\mu^1 \hookrightarrow L^2(\mathbb{R}^N)$ is compact as a consequence of condition 7) in Proposition 3.3. In other words, the Schrödinger operator $-\Delta u + u + V(x)u$ has a compact resolvent.

Example 3.9 In order to get the compactness embedding $H_\mu^1 \hookrightarrow L^2(\mathbb{R}^N)$ for measures of the form $\mu = V(x) dx$, it is not necessary to require that $V(x) \rightarrow +\infty$ as $\|x\| \rightarrow +\infty$. Indeed, in \mathbb{R}^2 one can consider for instance $V(x_1, x_2) = |x_1|^\alpha |x_2|^\beta$, for some $\alpha, \beta > 0$. In this case, one can prove easily that condition 7) in Proposition 3.3 is still satisfied, by analyzing the γ -convergence of the measures $V dx \llcorner B_h(x_n)$.

Example 3.10 Let $\mu = V(x) dx$, with $V : \mathbb{R}^N \rightarrow (0, +\infty)$, measurable. If there exists $\beta \in (0, 1]$ such that $\frac{1}{V^\beta} \in L^1(\mathbb{R}^N)$, then H_μ^1 is compactly embedded in $L^1(\mathbb{R}^N)$ (hence in $L^2(\mathbb{R}^N)$). Indeed, we have

$$\int_{\mathbb{R}^N} w_{\mu \llcorner B_R} dx = \int_{\mathbb{R}^N} V^\alpha w_{\mu \llcorner B_R} \frac{1}{V^\alpha} dx \leq \left(\int_{\mathbb{R}^N} [V^\alpha w_{\mu \llcorner B_R}]^p dx \right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^N} \frac{1}{V^{\alpha q}} dx \right)^{\frac{1}{q}},$$

where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Choosing $\alpha = \frac{1}{p}$ and noticing that for $p \geq 2$ we have $\alpha q \in (0, 1]$, we fix p such that $\alpha q = \beta$, so $\int_{\mathbb{R}^N} \frac{1}{V^{\alpha q}} dx < +\infty$. For the first term, we use the fact that $w_{\mu \llcorner B_R} \leq 1$ and $p \geq 2$, so that

$$\int_{\mathbb{R}^N} w_{\mu \llcorner B_R} dx \leq \left(\int_{\mathbb{R}^N} V w_{\mu \llcorner B_R}^2 dx \right)^{\frac{1}{p}} |V^\beta|_{L^1(\mathbb{R}^N)}^{\frac{1}{q}}.$$

Using the equation satisfied by $w_{\mu \llcorner B_R}$, we obtain

$$\int_{\mathbb{R}^N} V w_{\mu \llcorner B_R}^2 \leq \int_{\mathbb{R}^N} w_{\mu \llcorner B_R} dx$$

thus we get

$$\left(\int_{\mathbb{R}^N} w_{\mu \llcorner B_R} dx \right)^{1-\frac{1}{p}} \leq |V^\beta|_{L^1(\mathbb{R}^N)}^{\frac{1}{q}}.$$

Passing to the limit as $R \rightarrow \infty$, we conclude that $w_\mu \in L^1(\mathbb{R}^N)$ so that Theorem 3.2 applies.

4 Further remarks and applications

The main motivation of the paper originates from a conjecture of Polya and Szegö [14] which states that among all simply connected membranes $\Omega \subseteq \mathbb{R}^2$, the minimum of the product $P(\Omega)\lambda_1^2(\Omega)$ is attained on balls. Here $P(\Omega)$ stands for the torsional rigidity and $\lambda_1(\Omega)$ for the first Dirichlet eigenvalue of the Laplacian on Ω . By definition

$$P(\Omega) = \int_{\Omega} u dx,$$

where $u \in H_0^1(\Omega)$ solves in $\mathcal{D}'(\Omega)$

$$-\Delta u = 1.$$

The conjecture was proved in 1978 by Köhler-Jobin in [11] and extended to inhomogeneous membranes in [12] by a rather sophisticated “dearrangement” procedure. Naturally, we can reframe the problem as

$$\min\{\lambda_1(\Omega) : \Omega \subseteq \mathbb{R}^N, P(\Omega) = c\} \quad (3)$$

and obtain a sort of “isoperimetric” problem, where the usual constraint on Ω set in terms of area or perimeter is replaced by a constraint on torsional rigidity. A natural generalization of this problem is the following: for $k \in \mathbb{N}$ solve

$$\min\{\lambda_k(\Omega) : \Omega \subseteq \mathbb{R}^N, P(\Omega) = c\}. \quad (4)$$

More general functionals of the form $F(\lambda_1(\Omega), \dots, \lambda_k(\Omega))$ can also be considered.

Notice, that in usual isoperimetric inequalities, the constraint on Ω is of the form $|\Omega| = c$ or $\mathcal{H}^{N-1}(\partial\Omega) = c$. Both of them imply that the resolvent operator of the Dirichlet Laplacian is compact so that the spectrum is well defined. A priori, it is not obvious that finite torsional rigidity alone would imply the same property. Of course, one can add an artificial constraint by setting that Ω is bounded. Nevertheless, from a variational point, this amounts to restrict the class of admissible domains to a non-closed one.

It is not difficult to observe that for $k = 2$ the solution of (4) consists on two disjoint and equal balls. For $k = 3$ and $k = 4$ numerical computations based on a genetic algorithm (see [8, 6]) lead to the intuition that the solution consists on 3, respectively 4, equal and disjoint balls. Although it is clear that for classical isoperimetric inequality for eigenvalues, the minimizer for λ_3 and λ_4 is not the union of 3 or 4 equal balls (in 2D), the same arguments are not valid for problem (4). Starting from the numerical computations above, we are led to the following problems.

Problem 1. Prove or disprove that for $k = 3, 4$ in \mathbb{R}^2 the solution consists of k equal and disjoint balls.

Problem 2. Is a similar assertion true for every k and every dimension of the space?

The main consequence of Theorem 3.2 is that problem (4) is well posed in the family of all open sets (possibly unbounded or of infinite measure of \mathbb{R}^N) with finite torsional rigidity. This is a consequence of the fact that if $P(\Omega) < +\infty$, then $w_\Omega \in L^1(\Omega)$. By Theorem 3.2 the resolvent operator is compact, so the spectrum of the Dirichlet Laplacian consists only on eigenvalues, thus (4) is well posed.

Moreover, the problem has a natural extension on the family of measures $\mathcal{M}_0(\mathbb{R}^N)$, and we know that the family of domains is “dense” in the sense of γ -convergence in $\mathcal{M}_0(\mathbb{R}^N)$ (see section 2). If classical isoperimetric inequalities are hardly well written on measures, since the perimeter or the area of a measure μ has no mechanical meaning, the torsional rigidity of a measure is well defined.

5 Proofs of the main results

Proof [of Theorem 3.1] Necessity. Assume by contradiction that there exists $\delta > 0$ and a sequence of points x_n with $\|x_n\| \rightarrow +\infty$ such that for every $r > 0$

$$|B_r(x_n) \cap \{x : w_\mu(x) \geq \delta\}| > 0.$$

In view of the definition of w_μ , for every $n \in \mathbb{N}$ there exists R_n such that for every $r > 0$

$$|B_r(x_n) \cap \{x : w_{\mu \upharpoonright B_{R_n}}(x) \geq \delta/2\}| > 0.$$

We introduce the functions

$$\varphi_n(x) = w_{\mu \upharpoonright B_{R_n}}(x)\theta(x + x_n),$$

and we prove that φ_n is bounded in H_μ^1 , converges to 0 weakly in H_μ^1 but does not converge strongly in $L^2(\mathbb{R}^N)$.

In order to bound the H_μ^1 -norm, we take φ_n as test function in the equation satisfied by $w_{\mu \lceil B_{R_n}}$. So

$$\int_{\mathbb{R}^N} \nabla w_{\mu \lceil B_{R_n}} \nabla \varphi_n dx + \int_{\mathbb{R}^N} w_{\mu \lceil B_{R_n}} \varphi_n dx + \int_{\mathbb{R}^N} w_{\mu \lceil B_{R_n}} \varphi_n d\mu = \int_{\mathbb{R}^N} \varphi_n dx.$$

Simple computations lead to

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla w_{\mu \lceil B_{R_n}}|^2 \theta(\cdot + x_n) dx + \int_{\mathbb{R}^N} \nabla w_{\mu \lceil B_{R_n}} \nabla \theta(\cdot + x_n) w_{\mu \lceil B_{R_n}} dx + \int_{\mathbb{R}^N} w_{\mu \lceil B_{R_n}}^2 \theta(\cdot + x_n) dx \\ + \int_{\mathbb{R}^N} w_{\mu \lceil B_{R_n}}^2 \theta(\cdot + x_n) d\mu = \int_{\mathbb{R}^N} w_{\mu \lceil B_{R_n}} \theta(\cdot + x_n) dx. \end{aligned}$$

Since $0 \leq w_{\mu \lceil B_{R_n}} \leq 1$, $0 \leq \theta \leq 1$, and θ has its support in B_2 , we have in the right hand side

$$\int_{\mathbb{R}^N} w_{\mu \lceil B_{R_n}} \theta(\cdot + x_n) dx \leq |B_2|.$$

Performing an integration by parts on the second term in left hand side, we get

$$\int_{\mathbb{R}^N} \nabla w_{\mu \lceil B_{R_n}} \nabla \theta(\cdot + x_n) w_{\mu \lceil B_{R_n}} dx = -\frac{1}{2} \int_{\mathbb{R}^N} w_{\mu \lceil B_{R_n}}^2 \Delta \theta(\cdot + x_n) dx,$$

so that

$$\left| \int_{\mathbb{R}^N} \nabla w_{\mu \lceil B_{R_n}} \nabla \theta(\cdot + x_n) w_{\mu \lceil B_{R_n}} dx \right| \leq \frac{1}{2} |B_2| \|\Delta \theta\|_\infty.$$

Finally, we get that

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla w_{\mu \lceil B_{R_n}}|^2 \theta(\cdot + x_n) dx + \int_{\mathbb{R}^N} w_{\mu \lceil B_{R_n}}^2 \theta(\cdot + x_n) dx \\ + \int_{\mathbb{R}^N} w_{\mu \lceil B_{R_n}}^2 \theta(\cdot + x_n) d\mu \leq |B_2| \left(1 + \frac{1}{2} \|\Delta \theta\|_\infty \right). \end{aligned}$$

Using again that $0 \leq \theta \leq 1$ we obtain

$$\|\varphi_n\|_{H_\mu^1}^2 \leq 2|B_2| \left(1 + \frac{1}{2} \|\Delta \theta\|_\infty + \|\nabla \theta\|_\infty^2 \right),$$

so $(\varphi_n)_n$ is bounded in H_μ^1 .

We notice that $\varphi_n \rightharpoonup 0$ weakly in H_μ^1 since the support of φ_n lies in a ball of radius 2 centered at the point x_n which goes to infinity. In order to prove that φ_n does not converge strongly in L^2 to 0, it is enough to show that its L^1 -norm does not converge to 0, since

$$\int_{\mathbb{R}^N} \varphi_n^2 dx = \int_{B_2(x_n)} \varphi_n^2 dx \geq \frac{1}{|B_2|} \left(\int_{B_2(x_n)} \varphi_n dx \right)^2 = \frac{1}{|B_2|} \left(\int_{\mathbb{R}^N} \varphi_n dx \right)^2.$$

We have

$$\int_{B_2(x_n)} \varphi_n dx \geq \int_{B_1(x_n)} w_{\mu \lceil B_{R_n}} dx.$$

Since

$$-\Delta w_{\mu \lceil B_{R_n}} \leq 1 \text{ in } \mathcal{D}'(\mathbb{R}^N),$$

we have

$$\Delta \left(w_{\mu \upharpoonright B_{R_n}}(x) + \frac{|x - x_n|^2}{2N} \right) \geq 0 \text{ in } \mathcal{D}'(\mathbb{R}^N).$$

Consequently, since there are Lebesgue points of the set $\{x : w_{\mu \upharpoonright B_{R_n}}(x) \geq \frac{\delta}{2}\}$ in any neighborhood of x_n , for every $r > 0$ we get

$$|B_r(x_n)| \frac{\delta}{2} \leq \int_{B_r(x_n)} \left(w_{\mu \upharpoonright B_{R_n}}(x) + \frac{|x - x_n|^2}{2N} \right) dx = \int_{B_r(x_n)} w_{\mu \upharpoonright B_{R_n}}(x) dx + c_N r^{N+2}.$$

Therefore, there exists r small enough depending only on δ , such that

$$\int_{B_r(x_n)} w_{\mu \upharpoonright B_{R_n}}(x) dx \geq |B_r(x_n)| \frac{\delta}{4}, \quad (5)$$

hence φ_n does not converge strongly in L^2 to 0.

Sufficiency. We start with the following

Lemma 5.1 *Let $\mu \in \mathcal{M}_0(\mathbb{R}^N)$ be such that for some $0 < \varepsilon < 1$ we have that $w_\mu \leq \varepsilon$. Then*

$$\lambda_1(\mu) \geq \frac{1}{\varepsilon}.$$

Proof [of Lemma 5.1] From the density of $\{u\theta_R : u \in H_\mu^1, R > 0\}$ in H_μ^1 , it is enough to prove the assertion for a measure $\mu \upharpoonright B_R$. Moreover, using the density for the γ -convergence of bounded open sets in the family measures with bounded regular set, it is enough to prove the assertion of the theorem only for bounded open sets.

Let Ω be a bounded open set and let u_1 denote a nonzero first eigenfunction for the operator $-\Delta u + u$. Then $u_1 \in L^\infty(\Omega) \cap H_0^1(\Omega)$, and we have

$$-\Delta u_1 + u_1 \leq \lambda_1(\Omega) \|u_1\|_\infty,$$

so by monotonicity we get

$$0 \leq u_1 \leq w_\Omega \lambda_1(\Omega) \|u_1\|_\infty,$$

thus passing to the supremum on the left hand side and using the hypothesis $w_\Omega \leq \varepsilon$ we obtain

$$1 \leq \varepsilon \lambda_1(\Omega),$$

which gives the conclusion. \square

Coming back to to the proof of the sufficiency part, we can use the lemma above and the hypothesis $w_\mu \cdot 1_{B_R^c} \rightarrow 0$ in $L^\infty(\mathbb{R}^N)$ as $R \rightarrow +\infty$. So, for every $\varepsilon > 0$ there exists R_ε such that for $R \geq R_\varepsilon$

$$w_\mu \cdot 1_{B_R^c} \leq \varepsilon \quad \text{a.e.}$$

By monotonicity, we get that

$$w_{\mu \upharpoonright B_R^c} \leq w_\mu \cdot 1_{B_R^c} \leq \varepsilon \quad \text{a.e.}$$

hence Lemma 5.1 gives that $\lambda_1(\mu \upharpoonright B_R^c) \geq \frac{1}{\varepsilon}$. Making $\varepsilon \rightarrow 0$ we get that $\lambda_1(\mu \upharpoonright B_R^c) \rightarrow +\infty$, as $R \rightarrow +\infty$.

Let $\{u_n\}_n \subseteq H_\mu^1$ be a bounded sequence and assume $u_n \rightarrow 0$ weakly in H_μ^1 , with $\|u_n\|_{H_\mu^1} \leq M$. Then for every $R > 0$ we have that $u_n \theta_R \rightarrow 0$ strongly in $L^2(\mathbb{R}^N)$. But $u_n(1 - \theta_R) \in H_{(\mu \lceil B_R^c)}^1$ so that it can be taken as test function for $\lambda_1(\mu \lceil B_R^c)$. Hence

$$\begin{aligned} \lambda_1(\mu \lceil B_R^c) &\leq \frac{\int_{\mathbb{R}^N} |\nabla(u_n(1 - \theta_R))|^2 dx + \int_{\mathbb{R}^N} u_n^2(1 - \theta_R)^2 dx + \int_{\mathbb{R}^N} u_n^2(1 - \theta_R)^2 d\mu}{\int_{\mathbb{R}^N} u_n^2(1 - \theta_R)^2 dx} \\ &\leq \frac{2M(1 + R^{-2}|\nabla\theta|_\infty^2)}{\int_{\mathbb{R}^N} u_n^2(1 - \theta_R)^2 dx}, \end{aligned}$$

so that

$$\int_{\mathbb{R}^N} u_n^2(1 - \theta_R)^2 dx \leq \frac{2M(1 + R^{-2}|\nabla\theta|_\infty^2)}{\lambda_1(\mu \lceil B_R^c)}.$$

By a standard argument we get that $u_n \rightarrow 0$ strongly in $L^2(\mathbb{R}^N)$. \square

Proof [of Theorem 3.2] **1) \Rightarrow 2)** Assume $w_\mu \in L^1(\mathbb{R}^N)$. By the definition of w_μ we obtain that $w_\mu \in H_\mu^1$, and that $w_{\mu \lceil B_R} \rightharpoonup w_\mu$ weakly in H_μ^1 and that w_μ satisfies the equation

$$-\Delta w_\mu + w_\mu + \mu w_\mu = 1 \tag{6}$$

in the weak sense, for test functions $v \in H_\mu^1$, with compact support.

Indeed, we have that $R \rightarrow w_{\mu \lceil B_R}(x)$ is not decreasing and $w_{\mu \lceil B_R}(x) \rightarrow w_\mu(x)$ a.e. But

$$\|w_{\mu \lceil B_R}\|_{H_\mu^1}^2 = \|w_{\mu \lceil B_R}\|_{L^1(\mathbb{R}^N)}.$$

The mapping $R \rightarrow \|w_{\mu \lceil B_R}\|_{L^1(\mathbb{R}^N)}$ is not decreasing, and for $R \rightarrow +\infty$ we have $\|w_{\mu \lceil B_R}\|_{L^1(\mathbb{R}^N)} \rightarrow \|w_\mu\|_{L^1(\mathbb{R}^N)}$ by the monotone convergence theorem. This proves that $(w_{\mu \lceil B_R})_R$ is bounded in H_μ^1 and weakly converges in H_μ^1 to w_μ . Consequently, taking a test function $\varphi \in H_\mu^1$, with compact support, in the equation satisfied by $w_{\mu \lceil B_R}$ for R large enough, we obtain by passage to the limit (6).

Take now an arbitrary function $v \in H_\mu^1$, $v \geq 0$. We prove first that $v \in L^1(\mathbb{R}^N)$. We may take $\theta_R v$ as test function in (6) and get

$$\int_{\mathbb{R}^N} \nabla w_\mu \nabla(\theta_R v) dx + \int_{\mathbb{R}^N} w_\mu \theta_R v dx + \int_{\mathbb{R}^N} w_\mu \theta_R v d\mu = \int_{\mathbb{R}^N} \theta_R v dx.$$

Passing to the limit as $R \rightarrow +\infty$, we obtain

$$\int_{\mathbb{R}^N} \nabla w_\mu \nabla v dx + \int_{\mathbb{R}^N} w_\mu v dx + \int_{\mathbb{R}^N} w_\mu v d\mu = \int_{\mathbb{R}^N} v dx.$$

Since the left hand side is finite, then $v \in L^1(\mathbb{R}^N)$, and

$$\|v\|_{L^1(\mathbb{R}^N)} \leq \|w_\mu\|_{H_\mu^1} \|v\|_{H_\mu^1}.$$

Therefore the embedding $H_\mu^1 \subset L^1(\mathbb{R}^N)$ is continuous. This also means that $1 \in (H_\mu^1)'$ and that equation (6) holds in $(H_\mu^1)'$.

In order to prove the compactness of the embedding, we assume that $v_n \in H_\mu^1$ has norm bounded by M and converges weakly to 0. Again, we may assume $v_n \geq 0$. We consider $(1 - \theta_R)v_n$ as test function for (6). Thus

$$\begin{aligned} & \int_{\mathbb{R}^N} \nabla w_\mu \nabla ((1 - \theta_R)v_n) dx + \int_{\mathbb{R}^N} w_\mu (1 - \theta_R)v_n dx \\ & + \int_{\mathbb{R}^N} w_\mu (1 - \theta_R)v_n d\mu = \int_{\mathbb{R}^N} (1 - \theta_R)v_n dx. \end{aligned} \quad (7)$$

Since $w_\mu \in H_\mu^1$, for every $\varepsilon > 0$ there exists $R > 0$ such that

$$\int_{\mathbb{R}^N \setminus B_R} |\nabla w_\mu|^2 dx + \int_{\mathbb{R}^N \setminus B_R} w_\mu^2 dx + \int_{\mathbb{R}^N \setminus B_R} w_\mu^2 d\mu \leq \varepsilon.$$

So, by (7) and Hölder inequality

$$\begin{aligned} \int_{\mathbb{R}^N} (1 - \theta_R)v_n dx & \leq \varepsilon^{1/2} \left(\int_{\mathbb{R}^N} |\nabla((1 - \theta_R)v_n)|^2 dx + \int_{\mathbb{R}^N} (1 - \theta_R)v_n^2 dx + \int_{\mathbb{R}^N} (1 - \theta_R)v_n^2 d\mu \right)^{1/2} \\ & \leq \varepsilon^{1/2} \left(2 \int_{\mathbb{R}^N} |\nabla v_n|^2 dx + \frac{2|\nabla \theta|_\infty^2}{R^2} \int_{\mathbb{R}^N} v_n^2 dx + \int_{\mathbb{R}^N} v_n^2 dx + \int_{\mathbb{R}^N} v_n^2 d\mu \right)^{1/2}. \end{aligned}$$

Hence

$$\int_{\mathbb{R}^N} (1 - \theta_R)v_n dx \leq (2\varepsilon)^{1/2} M \left(1 + \frac{|\nabla \theta|_\infty^2}{R^2} \right)^{1/2}.$$

Since for every fixed R we have

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} \theta_R v_n dx = 0,$$

by a standard argument we get $|v_n|_{L^1(\mathbb{R}^N)} \rightarrow 0$ as $n \rightarrow \infty$.

2) \Rightarrow 1) Let C be the norm of the continuous injection $H_\mu^1 \subset L^1(\mathbb{R}^N)$. Taking $w_{\mu \upharpoonright B_R}$ as test function for $w_{\mu \upharpoonright B_R}$ we get

$$\|w_{\mu \upharpoonright B_R}\|_{H_\mu^1}^2 = \|w_{\mu \upharpoonright B_R}\|_{L^1(\mathbb{R}^N)} \leq C \|w_{\mu \upharpoonright B_R}\|_{H_\mu^1}.$$

Consequently, $w_{\mu \upharpoonright B_R}$ is uniformly bounded in $L^1(\mathbb{R}^N)$ and in H_μ^1 , for every $R > 0$. Using the definition of w_μ and the monotone convergence theorem, we get that $w_\mu \in L^1(\mathbb{R}^N)$. \square

Proposition 3.3 gives a list of useful tools for proving the compact embedding in $L^2(\mathbb{R}^N)$.

Proof [of Proposition 3.3]

1) \Rightarrow 5) Let $u_R \in H_{\mu \upharpoonright B_R^c}^1$ be such that $\|u_R\|_{L^2(\mathbb{R}^N)} = 1$ and

$$\lambda_1(\mu \upharpoonright B_R^c) \leq \frac{\int_{\mathbb{R}^N} |\nabla u_R|^2 dx + \int_{\mathbb{R}^N} u_R^2 dx + \int_{\mathbb{R}^N} u_R^2 d\mu}{\int_{\mathbb{R}^N} u_R^2 dx} \leq 1 + \lambda_1(\mu \upharpoonright B_R^c).$$

Assume by contradiction that $\lambda_1(\mu \upharpoonright B_R^c) \not\rightarrow +\infty$. Then $(u_R)_R$ is bounded in H_μ^1 and converges weakly to 0 in H_μ^1 . This is a consequence of the fact that the support of u_R is located outside the ball B_R . Condition 1) implies that u_R has to converge strongly to 0 in $L^2(\mathbb{R}^N)$ which is in contradiction with the fact that $\|u_R\|_{L^2(\mathbb{R}^N)} = 1$.

5) \Rightarrow 1) The proof of this statement is implicitly contained in the proof of Theorem 3.1.

1) \Rightarrow 3) Assume 1) holds. In order to prove that $\mathcal{R}_{\mu \upharpoonright B_R} \rightarrow \mathcal{R}_\mu$ in $\mathcal{L}(L^2(\mathbb{R}^N))$ as $R \rightarrow +\infty$ it is enough to consider a sequence $(f_n)_n$ in $L^2(\mathbb{R}^N)$ such that $f_n \rightharpoonup f$ weakly in $L^2(\mathbb{R}^N)$ and prove that

$$\mathcal{R}_{\mu \upharpoonright B_{R_n}}(f_n) \rightarrow \mathcal{R}_\mu(f) \text{ strongly in } L^2(\mathbb{R}^N).$$

For simplicity we set $\mu_n = \mu \upharpoonright B_{R_n}$. Because of the compact injection assumption 1) and from the equiboundedness

$$\|\mathcal{R}_{\mu_n}(f_n)\|_{H_\mu^1} \leq \|f_n\|_{L^2}$$

it is enough to prove simply that $\mathcal{R}_{\mu_n}(f_n)$ converges to $\mathcal{R}_\mu(f)$ weakly in $L^2(\mathbb{R}^N)$. Since \mathcal{R}_{μ_n} and \mathcal{R}_μ are self-adjoint operators this means

$$\langle f_n, \mathcal{R}_{\mu_n}(\psi) \rangle_{L^2} \rightarrow \langle f, \mathcal{R}_\mu(\psi) \rangle_{L^2} \quad \forall \psi \in L^2(\mathbb{R}^N).$$

This will be a consequence of the fact that

$$\mathcal{R}_{\mu_n}(\psi) \rightharpoonup \mathcal{R}_\mu(\psi) \quad \text{weakly in } L^2(\mathbb{R}^N), \quad \forall \psi \in L^2(\mathbb{R}^N) \quad (8)$$

again because of the compact injection hypothesis 1). In order to prove (8) it is enough to assume $\psi \geq 0$, so that the maximum principle gives that $\mathcal{R}_{\mu_n}(\psi)$ is a nondecreasing sequence of functions. If we denote $u_n = \mathcal{R}_{\mu_n}(\psi)$ then u_n solves

$$\begin{cases} -\Delta u_n + u_n + u_n \mu_n = \psi \text{ in } (H_{\mu_n}^1)' \\ u_n \in H_{\mu_n}^1. \end{cases} \quad (9)$$

There exists a constant C such that for every n we have $\|u_n\|_{H_\mu^1} \leq C$, so without loss of generality we may assume that $u_n \rightharpoonup u$ weakly in H_μ^1 , so by 1) strongly in $L^2(\mathbb{R}^N)$. Summarizing, in order to prove 3) it is enough to show that $u = \mathcal{R}_\mu(\psi)$, which is equivalent to

- $u \in H_\mu^1$: this is obvious;
- u solves the equation

$$-\Delta u + u + u\mu = \psi \text{ in } (H_\mu^1)'. \quad (10)$$

We consider a test function $\varphi \in H_\mu^1$, with compact support. In order to prove that

$$\int_{\mathbb{R}^N} \nabla u \nabla \varphi \, dx + \int_{\mathbb{R}^N} u \varphi \, dx + \int_{\mathbb{R}^N} u \varphi \, d\mu = \int_{\mathbb{R}^N} \psi \varphi \, dx,$$

we take φ as test function for u_n , with n large enough. Passing to the limit as $n \rightarrow \infty$ we readily get (10).

3) \Rightarrow 2) This is an obvious consequence of the fact that $\mathcal{R}_{\mu \upharpoonright B_R}$ are compact operators.

2) \Rightarrow 5) Assume by contradiction that $\lambda_1(\mu \upharpoonright B_R^c) \leq M$ for every $R > 0$. We denote $f_n = \lambda_1(\mu \upharpoonright B_n^c) u_n$, where $u_n \in H_{\mu \upharpoonright B_n^c}^1$ is a first positive eigenfunction associated to the measure $\mu \upharpoonright (B_n^c \cap B_{n'})$, with $\|u_n\|_{L^2} = 1$, where n' is large enough such that

$$\lambda_1(\mu \upharpoonright (B_n^c \cap B_{n'})) \leq 1 + \lambda_1(\mu \upharpoonright B_n^c).$$

By monotonicity, we have that

$$u_n \leq \mathcal{R}_\mu(\lambda_1(\mu \llcorner (B_n^c \cap B_{n'}))u_n),$$

so that $\mathcal{R}_\mu(\lambda_1(\mu \llcorner (B_n^c \cap B_{n'}))u_n)$ cannot converge strongly to 0 in $L^2(\mathbb{R}^N)$. On the other hand, if we denote $f_n = \lambda_1(\mu \llcorner (B_n^c \cap B_{n'}))u_n$, we notice that f_n converges to 0 weakly in $L^2(\mathbb{R}^N)$, which contradicts hypothesis 2).

4) \Rightarrow 1) Clearly, if 4) holds, then $\mathcal{R}_{\mu \llcorner B_R}(1) \rightarrow \mathcal{R}_\mu(1)$ in $L^\infty(\mathbb{R}^N)$, hence

$$w_\mu - w_{\mu \llcorner B_R} \rightarrow 0 \text{ in } L^\infty(\mathbb{R}^N),$$

so by monotonicity we have that $1_{B_R^c} \cdot w_\mu \rightarrow 0$ in L^∞ , hence we can use Theorem 3.1.

1) \Rightarrow 4) In the case of a domain Ω , i.e. $\mu = \infty_{\mathbb{R}^N \setminus \Omega}$, this assertion is a consequence of Theorem 3.1 and [4, Theorem 3.31]. For measures, the proof is the same and is a direct consequence of Theorem 3.1 and of the inequality

$$0 \leq \mathcal{R}_\mu(f) - \mathcal{R}_{\mu \llcorner B_R^c}(f) \leq \|f\|_\infty (\mathcal{R}_\mu(1) - \mathcal{R}_{\mu \llcorner B_R^c}(1)) \quad \forall f \in L^\infty(\mathbb{R}^N), f \geq 0.$$

5) \Rightarrow 6) This is obvious, by monotonicity of the eigenvalues with respect to measures.

6) \Rightarrow 7) This is obvious.

7) \Rightarrow 5) Let h be given by 7) and consider $h' < h$ such that a N -cube $C_{h'}$ of edge of length h' centered at the origin is contained in the ball B_h . We cover the space \mathbb{R}^N by closed cubes of edges of length h' parallel to the axes and with centers in the points of the lattice $(\frac{h'}{2}\mathbb{Z})^N$. We denote these cubes by C_i for $i \in I$. We consider a function $\varphi \in C_c^\infty(C_{h'})$, such that $0 \leq \varphi \leq 1$, and $\varphi = 1$ on the cube $C_{h'/2}$. We denote by φ_i the function φ supported by the cube C_i .

From hypothesis 7) for every $\varepsilon > 0$ there exists $R > 0$ such that for all $\|x\| \geq R$

$$\lambda_1(\mu \llcorner B_h(x)) \geq \frac{1}{\varepsilon}.$$

Let us consider a function $u \in H_{\mu \llcorner B_R^c}^1, u \geq 0$ such that

$$\lambda_1(\mu \llcorner B_R^c) \leq \frac{\int_{\mathbb{R}^N} |\nabla u|^2 dx + \int_{\mathbb{R}^N} u^2 dx + \int_{\mathbb{R}^N} u^2 d\mu}{\int_{\mathbb{R}^N} u^2 dx} \leq \lambda_1(\mu \llcorner B_R^c) + 1. \quad (11)$$

Then $\varphi_i u$ is a test function for $\lambda_1(\mu \llcorner C_i)$, and by monotonicity for $\lambda_1(\mu \llcorner B_i)$, where B_i is the ball centered at the same point as C_i , with radius h . Consequently, we may write

$$\lambda_1(\mu \llcorner B_i) \int_{\mathbb{R}^N} |\varphi_i u|^2 dx \leq \int_{\mathbb{R}^N} |\nabla(\varphi_i u)|^2 dx \int_{\mathbb{R}^N} |\varphi_i u|^2 dx + \int_{\mathbb{R}^N} |\varphi_i u|^2 d\mu.$$

Summing over $i \in I$ and decomposing $\nabla(\varphi_i u)$, we get

$$\begin{aligned} \sum_i \lambda_1(\mu \llcorner B_i) \int_{\mathbb{R}^N} |\varphi_i u|^2 dx &\leq 2 \sum_i \int_{\mathbb{R}^N} |\varphi_i|^2 |\nabla u|^2 dx + 2 \sum_i \int_{\mathbb{R}^N} u^2 |\nabla \varphi_i|^2 dx \\ &\quad + \sum_i \int_{\mathbb{R}^N} |\varphi_i u|^2 dx + \sum_i \int_{\mathbb{R}^N} |\varphi_i u|^2 d\mu. \end{aligned}$$

We notice that every point of the space is covered by at most 2^N cubes C_i so we can write

$$\begin{aligned} \frac{1}{\varepsilon} \sum_i \int_{\mathbb{R}^N} |\varphi_i u|^2 dx &\leq 2^{N+1} \left(\int_{\mathbb{R}^N} |\nabla u|^2 dx + \|\nabla \varphi\|_\infty^2 \int_{\mathbb{R}^N} u^2 dx \right) \\ &\quad + 2^N \int_{\mathbb{R}^N} u^2 dx + 2^N \int_{\mathbb{R}^N} u^2 d\mu. \end{aligned}$$

For the left hand side, the average inequality gives

$$\sum_i \int_{\mathbb{R}^N} |\varphi_i u|^2 dx \geq \frac{1}{2^N} \int_{\mathbb{R}^N} u^2 (\sum_i \varphi_i)^2 dx \geq \frac{1}{2^N} \int_{\mathbb{R}^N} u^2 dx.$$

Consequently,

$$\begin{aligned} \frac{1}{\varepsilon} \frac{1}{2^N} \int_{\mathbb{R}^N} u^2 dx &\leq 2^{N+1} \left(\int_{\mathbb{R}^N} |\nabla u|^2 dx + \|\nabla \varphi\|_\infty^2 \int_{\mathbb{R}^N} u^2 dx \right) \\ &\quad + 2^N \int_{\mathbb{R}^N} u^2 dx + 2^N \int_{\mathbb{R}^N} u^2 d\mu, \end{aligned}$$

so that, using (11)

$$\frac{1}{\varepsilon} \frac{1}{2^N} \leq 2^{N+1} (\lambda_1(\mu \llcorner B_R^c) + \|\nabla \varphi\|_\infty^2) + 2^N.$$

From this inequality, obviously 5) is a consequence of 7).

8) \Rightarrow 5) Let us consider a function $u \in H_{\mu \llcorner B_R^c}^1$, $u \geq 0$, such that

$$\lambda_1(\mu \llcorner B_R^c) \leq \frac{\int_{\mathbb{R}^N} |\nabla u|^2 dx \int_{\mathbb{R}^N} u^2 dx + \int_{\mathbb{R}^N} u^2 d\mu}{\int_{\mathbb{R}^N} u^2 dx} \leq \lambda_1(\mu \llcorner B_R^c) + 1.$$

We consider the cubes $H_i = [x_i, x_i + h)$, where x_i belongs to the lattice $(h\mathbb{Z})^N$, and $\|x_i\| \geq r$. Then we can write

$$\int_{H_i} |\nabla u|^2 dx + \int_{H_i} u^2 d\mu \geq \frac{1}{\varepsilon} \int_{H_i} u^2 dx,$$

and summing over $i \in I$ we obtain

$$\lambda_1(\mu \llcorner B_R^c) + 1 \geq \frac{1}{\varepsilon} + 1,$$

and 5) holds.

7) \Rightarrow 8) Assume by contradiction that there exists $\varepsilon > 0$ such that on a sequence of cubes $H_n = [x_n, x_n + h)$, we have functions $u_n \in H_{\mu \llcorner H_n}^1$ with $\int_{H_n} u_n^2 dx = 1$ and

$$\int_{H_n} |\nabla u_n|^2 dx + \int_{H_n} u_n^2 d\mu \leq \frac{1}{\varepsilon}.$$

Let $v_n = \min\{w_n, u_n\}$, where $w_n = w_{H_n}$. We get

$$\int_{H_n} |\nabla v_n|^2 dx + \int_{H_n} v_n^2 d\mu \leq \int_{H_n} |\nabla u_n|^2 dx + \int_{H_n} |\nabla w_n|^2 dx + \int_{H_n} u_n^2 d\mu \leq \frac{1}{\varepsilon} + |H_n|.$$

We shall prove that 7) fails, since $v_n \in H^1(\mu \llcorner H_n)$ satisfies

$$\lambda_1(\mu \llcorner H_n) \int_{H_n} v_n^2 dx \leq \int_{H_n} |\nabla v_n|^2 dx + \int_{H_n} v_n^2 dx + \int_{H_n} v_n^2 d\mu \leq \frac{1}{\varepsilon} + |H_n| + \int_{H_n} v_n^2 d\mu.$$

If $\limsup_{n \rightarrow \infty} \int_{H_n} v_n^2 dx > 0$, then 7) fails, in contradiction with the hypothesis.

Assume that $\lim_{n \rightarrow \infty} \int_{H_n} v_n^2 dx = 0$. Making translations at the origin and passing to the limit, we get, by an abuse of notation,

$$\int_H (\min\{w_H, u\})^2 dx = 0,$$

where u is a weak limit of the translations of u_n , hence has its L^2 -norm equal to 1. Obviously, this is impossible. \square

6 The general case

Let $1 < p < +\infty$. We denote by $\mathcal{M}_0^p(\mathbb{R}^N)$ the family of all nonnegative regular Borel measures, possibly $+\infty$ valued, that vanish on all sets of p -capacity zero (see the precise definition in [9]). For $\mu \in \mathcal{M}_0^p(\mathbb{R}^N)$, we denote by $W_\mu^{1,p}$ the Banach space $W^{1,p}(\mathbb{R}^N) \cap L^p(\mathbb{R}^N, \mu)$, endowed with the norm

$$\|u\|_{1,\mu}^p = \int_{\mathbb{R}^N} (|\nabla u|^p + |u|^p) dx + \int_{\mathbb{R}^N} |u|^p d\mu.$$

If $\mu \in \mathcal{M}_0^p(\mathbb{R}^N)$, for every $f \in L^{p'}(\mathbb{R}^N)$ (where $\frac{1}{p} + \frac{1}{p'} = 1$, or more generally for $f \in (W_\mu^{1,p})'$) we may consider the elliptic PDE

$$-\Delta_p u + |u|^{p-2}u + \mu|u|^{p-2}u = f, \quad u \in W_\mu^{1,p} \quad (12)$$

whose precise sense has to be given in the weak form

$$\int_{\mathbb{R}^N} (|\nabla u|^{p-2} \nabla u \nabla \varphi + |u|^{p-2} u \varphi) dx + \int_{\mathbb{R}^N} |u|^{p-2} u \varphi d\mu = \int_{\mathbb{R}^N} f \varphi dx \quad \forall \varphi \in W_\mu^{1,p}.$$

We still denote $\mathcal{R}(f) = u$, the nonlinear operator $\mathcal{R} : L_\mu^p \rightarrow L_\mu^p$ which associates to every f the unique solution u of the equation (12). Again, we note $w_\mu = \mathcal{R}(1)$, defined by approximation on the ball B_R , as in the linear case. We refer to [7] for the study of monotonicity properties of \mathcal{R} with respect to the measures μ .

Then we have the following results.

Theorem 6.1 *Let $\mu \in \mathcal{M}_0^p(\mathbb{R}^N)$. The following assertions are equivalent.*

1. $W_\mu^{1,p} \hookrightarrow L^p(\mathbb{R}^N)$ is compact;
2. $w_\mu \cdot 1_{B_R^c} \rightarrow 0$ in $L^\infty(\mathbb{R}^N)$ as $R \rightarrow +\infty$;
3. $\mathcal{R}_\mu(f_n) \rightarrow \mathcal{R}_\mu(f)$ strongly in $L^p(\mathbb{R}^N)$, as soon as $f_n \rightharpoonup f$ weakly in $L^{p'}(\mathbb{R}^N)$;
4. $\mathcal{R}_{\mu \llcorner B_{R_n}}(f_n) \rightarrow \mathcal{R}_\mu(f)$ strongly in $L^p(\mathbb{R}^N)$, as soon as $f_n \rightharpoonup f$ weakly in $L^{p'}(\mathbb{R}^N)$ and $R_n \rightarrow +\infty$;
5. $\lambda_1(\mu \llcorner B_R^c) \rightarrow +\infty$ as $R \rightarrow +\infty$;

6. for every $h > 0$, $\lambda_1(\mu \llcorner B_h(x)) \rightarrow +\infty$ as $\|x\| \rightarrow +\infty$;
7. there exists $h > 0$ such that $\lambda_1(\mu \llcorner B_h(x)) \rightarrow +\infty$ as $\|x\| \rightarrow +\infty$;
8. there exists $h > 0$ such that for every $\varepsilon > 0$ there exists $r \geq 0$ such that

$$\inf_{u \in W_{\mu \llcorner H}^{1,p}(\mathbb{R}^N), u \neq 0} \frac{\int_H |\nabla u|^p dx + \int_H u^p d\mu}{\int_H u^p dx} \geq \frac{1}{\varepsilon},$$

for every N -cube $H = [x, x + h)$, with $\|x\| \geq r$.

Proof The only points different from those of Theorem 3.1 and Proposition 3.3 and which need some attention are listed below.

1) \Rightarrow 2) The proof is similar to the necessity part of Theorem 3.1. The only different point is concerned with the uniform bound of the $W_{\mu}^{1,p}$ -norm of φ_n . We replace θ by another function $\beta \in C_c^\infty(B_2)$, with $0 \leq \beta \leq 1$, which will be precised later.

Notice first that

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla(w_{\mu \llcorner B_{R_n}} \beta(\cdot + x_n))|^p dx &\leq 2^{p-1} \int_{\mathbb{R}^N} |\nabla w_{\mu \llcorner B_{R_n}}|^p \beta(\cdot + x_n)^p dx \\ &\quad + 2^{p-1} \int_{\mathbb{R}^N} w_{\mu \llcorner B_{R_n}}^p |\nabla \beta(\cdot + x_n)|^p dx. \end{aligned}$$

Taking $w_{\mu \llcorner B_{R_n}} \beta(\cdot + x_n)$ as test function for the equation satisfied by $w_{\mu \llcorner B_{R_n}}$, we get

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla w_{\mu \llcorner B_{R_n}}|^p \beta(\cdot + x_n) dx + \int_{\mathbb{R}^N} w_{\mu \llcorner B_{R_n}} |\nabla w_{\mu \llcorner B_{R_n}}|^{p-2} \nabla w_{\mu \llcorner B_{R_n}} \nabla \beta(\cdot + x_n) dx \\ + \int_{\mathbb{R}^N} w_{\mu \llcorner B_{R_n}}^p \beta(\cdot + x_n) dx + \int_{\mathbb{R}^N} w_{\mu \llcorner B_{R_n}}^p \beta(\cdot + x_n) d\mu \leq |B_2|. \end{aligned} \quad (13)$$

Setting

$$I = \int_{\mathbb{R}^N} w_{\mu \llcorner B_{R_n}} |\nabla w_{\mu \llcorner B_{R_n}}|^{p-2} \nabla w_{\mu \llcorner B_{R_n}} \nabla \beta(\cdot + x_n) dx,$$

by Hölder inequality we get

$$|I| \leq \int_{\mathbb{R}^N} |\nabla w_{\mu \llcorner B_{R_n}}|^{p-1} |\nabla \beta(\cdot + x_n)| dx \leq \left(\int_{\mathbb{R}^N} |\nabla w_{\mu \llcorner B_{R_n}}|^p |\nabla \beta(\cdot + x_n)|^{\frac{p}{p-1}} dx \right)^{\frac{p-1}{p}} |B_2|^{\frac{1}{p}}.$$

We choose $\beta(x) = (1 - |x|)^p \cdot 1_{B_1}$ and notice that

$$|\nabla \beta|^{\frac{p}{p-1}} \leq p^{\frac{p}{p-1}} \beta.$$

Thus, there exists a constant C such that

$$|I| \leq C \left(\int_{\mathbb{R}^N} |\nabla w_{\mu \llcorner B_{R_n}}|^p \beta(\cdot + x_n) dx \right)^{\frac{p-1}{p}}. \quad (14)$$

Since $0 \leq \beta \leq 1$ we get the uniform bound of φ_n by plugging (14) in (13).

For a similar inequality to (5), we may use [13, Theorem 3.9].

1) \implies 4) One has only to prove that the weak $W_{\mu}^{1,p}$ -limit of $\mathcal{R}_{\mu \llcorner B_{R_n}}(f_n)$ is $\mathcal{R}_{\mu}(f)$. Assume that for some subsequence (still denoted with the same index) we have that $\mathcal{R}_{\mu \llcorner B_{R_n}}(f_n)$ converges weakly

in $W_\mu^{1,p}$ to some function v . We shall prove that $v = \mathcal{R}_\mu(f)$ relying on the Γ -convergence principle. Indeed, on the one hand we have

$$\int_{\mathbb{R}^N} |\nabla v|^p dx + \int_{\mathbb{R}^N} |v|^p dx + \int_{\mathbb{R}^N} |v|^p d\mu \leq \liminf \|\mathcal{R}_{\mu \upharpoonright B_{R_n}}(f_n)\|_{W_\mu^{1,p}}^p.$$

If the inequality above is strict, using hypothesis 1), there exists R large enough such that

$$\begin{aligned} & \frac{1}{p} \left(\int_{\mathbb{R}^N} |\nabla v|^p dx + \int_{\mathbb{R}^N} |v|^p dx + \int_{\mathbb{R}^N} |v|^p d\mu \right) - \int_{\mathbb{R}^N} f v dx \\ & < \frac{1}{p} \left(\int_{\mathbb{R}^N} |\nabla(\theta_R v)|^p dx + \int_{\mathbb{R}^N} |\theta_R v|^p dx + \int_{\mathbb{R}^N} |\theta_R v|^p d\mu \right) - \int_{\mathbb{R}^N} f \theta_R v dx \\ & < \liminf \frac{1}{p} \|\mathcal{R}_{\mu \upharpoonright B_{R_n}}(f_n)\|_{W_\mu^{1,p}}^p - \int_{\mathbb{R}^N} f_n \mathcal{R}_{\mu \upharpoonright B_{R_n}} dx. \end{aligned}$$

Consequently, for some n large enough, we also have

$$\begin{aligned} & \frac{1}{p} \left(\int_{\mathbb{R}^N} |\nabla(\theta_R v)|^p dx + \int_{\mathbb{R}^N} |\theta_R v|^p dx + \int_{\mathbb{R}^N} |\theta_R v|^p d\mu \right) - \int_{\mathbb{R}^N} f_n \theta_R v dx \\ & < \frac{1}{p} \|\mathcal{R}_{\mu \upharpoonright B_{R_n}}(f_n)\|_{W_\mu^{1,p}}^p - \int_{\mathbb{R}^N} f_n \mathcal{R}_{\mu \upharpoonright B_{R_n}} dx, \end{aligned}$$

which is in contradiction with the variational definition of $\mathcal{R}_{\mu \upharpoonright B_{R_n}}$.

Finally, we get that $\mathcal{R}_{\mu \upharpoonright B_{R_n}}(f_n)$ converges strongly in $W_\mu^{1,p}$ to v , thus one can pass to the limit the weak form of the equations associated to $\mathcal{R}_{\mu \upharpoonright B_{R_n}}$, for a fixed test function with compact support. \square

Theorem 6.2 *Let $\mu \in \mathcal{M}_0^p(\mathbb{R}^N)$. The following assertions are equivalent.*

1. $w_\mu \in L^1(\mathbb{R}^N)$;
2. $W_\mu^{1,p} \subset L^1(\mathbb{R}^N)$ and the injection is continuous.

Moreover, if one of the two assertions above holds, then the embedding $W_\mu^{1,p} \hookrightarrow L^1(\mathbb{R}^N)$ is compact.

Proof The proof follows the same steps as for Theorem 3.2. The only point which is slightly different is to prove that w_μ satisfies the equation

$$-\Delta_p w_\mu + w_\mu^{p-2} w_\mu + \mu w_\mu^{p-2} w_\mu = 1 \tag{15}$$

in the weak sense, for test functions $v \in W_\mu^{1,p}$ with compact support. The argument above for the proof of 1) \implies 4) can be repeated. \square

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