The rugosity effect

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Abstract

This paper surveys the series of lectures given by the author at the Nečas Center for Mathematical Modelling in 2006 and 2007. The main purpouse is the study of the boundary behaviour of solutions of some partial differential equations in domains with rough boundaries. Several classical examples are recalled: the strange term "coming from somewhere else" of Cioranescu-Murat, Babuška's paradox, the Courant-Hilbert example and the rugosity effect in fluid dynamics. Some classical and recent results on the shape stability of partial differential equations with Dirichlet boundary conditions are presented. In particular we describe different ways to deal with the rugosity effect in fluid dynamics or contact mechanics.

1 Some classical examples

1.1 Introduction

The behaviour of the solutions of partial differential equations or the spectrum of some differential operators as a consequence of geometric domains perturbations is a classical question which has both theoretical and numerical issues. It is natural to expect that if Ω_ε is a "nice" perturbation of a smooth open set Ω, then the solution of some partial differential equation defined on Ω_ε converges to the solution of the same equation on Ω. While this is indeed a reasonable guess corresponding to the reality, there are many "simple" situations where dramatic changes can be produced by "small" geometric perturbations.

We recall some classical examples of such geometric perturbations and give the main tools for handling the particular case of Dirichlet boundary conditions and of the rugosity effect. We underline the fact that the Dirichlet boundary conditions are much easier to deal with than Neumann or Robin boundary conditions (see [7, 25, 20]). The rugosity effect can be seen as sort of effect of *partial* Dirichlet boundary conditions for vector valued solutions, which interact with the geometric perturbation.

In the sequel, we show how *small* geometric perturbations can produce huge effects on the solution of the partial differential equations, or on the spectrum of some differential operators. The word *small* is not clear and may have significantly different interpretations. Overall, the perturbations are certainly small in terms of Lebesgue measure but they have also some other features which at a first sight may lead to the false intuition that the perturbations would leave the behaviour of the partial differential equation unchanged.

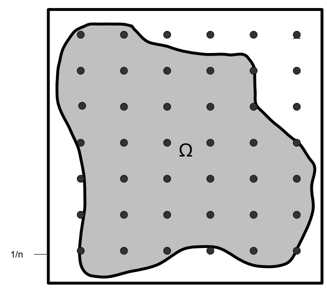
*The visit of D.B. was supported by the Nečas Center for Mathematical Modelling

1.2 The example of Cioranescu and Murat: a strange term coming from somewhere else

We consider an open set Ω contained in the unit square S in $\mathbb{R}^2$ and $f \in L^2(S)$. For every $n \in \mathbb{N}$ we introduce

$$C_n = \bigcup_{i,j=0}^{n} \overline{B}_{(i/n,j/n),r_n}, \qquad \Omega_n = \Omega \setminus C_n,$$

where $r_n = e^{-cn^2}$, $c > 0$ being a fixed positive constant.



$$r_n = e^{-cn^2}$$

If we denote by u_n the weak solution of

$$\begin{cases} -\Delta u_n = f \text{ in } \Omega_n \\ u_n \in H_0^1(\Omega_n). \end{cases} \tag{1}$$

one can prove that that $u_n \rightharpoonup u$ weakly in $H_0^1(S)$, where u solves

$$\begin{cases} -\Delta u + cu = f \text{ in } \Omega \\ u \in H_0^1(\Omega). \end{cases} \tag{2}$$

We refer the reader to [14] for a detailed proof of the passage to the limit as $n \to \infty$. The proof is elementary and comes from a direct computation as follows: one introduces the functions $z_n \in H^1(S)$:

$$z_n = \begin{cases} 0 & \text{on } C_n \\ \dfrac{\ln \sqrt{(x-i/n)^2 + (y-j/n)^2} + cn^2}{cn^2 - \ln(2n)} & \text{on } \overline{B}_{(i/n,j/n),1/2n} \setminus C_n \\ 1 & \text{on } S \setminus \bigcup_{i,j=0}^{n} \overline{B}_{(i/n,j/n),1/2n}. \end{cases}$$

Then, for every $\varphi \in C_0^\infty(\Omega)$, one can take $z_n\varphi$ as test function in equation (1). The passage to the limit for $n \to \infty$ can be performed completely to arrive to the weak form of (2).

The explanation of the fact that a union of small perforations of measure less than $\pi n^2 e^{-2cn^2}$ rapidly converging to zero can produce a huge effect on the equation can be completely understood in terms of Γ-convergence (see [19]). The effect is observed by the presence of the "strange term" cu in the limit equation. For a complete description of this phenomenon in relationship with optimal design problems we refer to the recent book [7].

1.3 Babuška's paradox

We consider the sequence $(P_n)_n$ of regular polygons with n edges, inscribed in the unit circle in $\mathbb{R}^2$. As $n \to \infty$, it is reasonable to expect that the solutions of (some) partial differential equations set on P_n would converge to the solution on the disc. This is indeed the case for some partial differential equations of second order, like the Laplace equation with homogenous Dirichlet or Neumann boundary conditions (with a fixed admissible right hand side, see [7]).

Nevertheless, as Babuška noticed (see [2] and also [29]) this is not anymore the case for a fourth order equation of bi-laplacian type as equilibrium problems in the bending of simply supported Kirchhoff-Love plates (see for a detailed explanation [2] and also[29],[21]).

Precisely, we consider the constant force $f = 1$ and $0 \leq \sigma < \frac{1}{2}$. For every bounded Lipschitz open set $\Omega \subseteq \mathbb{R}^2$, the solution of the following minimization problem:

$$\min\{u \in H^2(\Omega) \cap H_0^1(\Omega) : \int_\Omega \frac{1}{2}|\Delta u|^2 + (1-\sigma)(u_{xy}^2 - u_{xx}u_{yy}) - u dx\}$$

is denoted u_Ω. Then u_Ω is a formal weak solution of the following partial differential equation

$$\begin{cases} \Delta^2 u = 1 \text{ in } \Omega \\ u = \Delta u - (1-\sigma)k\frac{u}{n} = 0 \text{ on } \partial\Omega \end{cases} \tag{3}$$

k being the curvature of the boundary.

It turns out that if Ω has a polygonal shape, as P_n does, then the term

$$\int_\Omega (1-\sigma)(u_{xy}^2 - u_{xx}u_{yy})dx$$

vanishes identically in the energy functional above (see [24, Lemma 2.2.2]). So that, the solution u_{P_n} is also solution of the minimization problem

$$\min\{u \in H^2(P_n) \cap H_0^1(P_n) : \int_{P_n} \frac{1}{2}|\Delta u|^2 - u dx\},$$

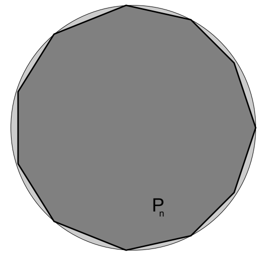
and formal weak solution of

$$\begin{cases} \Delta^2 u = 1 \text{ in } P_n \\ u = \Delta u = 0 \text{ on } \partial P_n \end{cases} \tag{4}$$

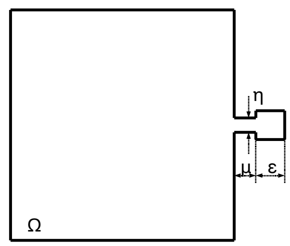
When $n \to \infty$, one can notice that u_{P_n} converges in L^∞ to the solution of (4) on the disc, which is different from the solution of (3) on the disc. This means, that the approximation of the disc by the sequence of regular polygons $(P_n)_n$ for equation (3) does not hold! The implications of this non-stability result for equation (3) in numerical analysis are obvious.

1.4 The Courant-Hilbert example for the Neumann-Laplacian spectrum

One considers the Neumann-Laplacian eigenvalues associated to the following Lipschitz domain, which depends on the small parameters $\varepsilon, \eta, \mu > 0$. Precisely, the values of η, μ will



The sequence $(P_n)_n$ of regular polygons "converges" to the disc



The parameters ε, η, μ vanish with different speeds

be chosen dependently on ε. By abuse of notation, let us denote Ω_ε the perturbed domain and by Ω the limit square.

Since Ω_ε is Lipschitz, the spectrum of the Neumann Laplacian consists only of eigenvalues satisifying formally

$$\begin{cases} -\Delta u = \lambda_k(\Omega_\varepsilon)u \text{ in } \Omega_\varepsilon \\ \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega_\varepsilon \end{cases} \tag{5}$$

for some function $u \in H^1(\Omega)$, $u \not\equiv 0$. The eigenvalues can be ordered, counting their multiplicities

$$0 = \lambda_1(\Omega_\varepsilon) < \lambda_2(\Omega_\varepsilon) \leq ...$$

Using the continuous dependence of the eigenvalues for *smooth* domain perturbations (see [15, 7]) or, alternatively, the definition of the eigenvalues with the Rayleigh quotient, for every $c \in (0, \lambda_2(\Omega))$ and ε small enough, one can choose $\mu = \varepsilon$ and $\eta \in (0, \varepsilon)$ such that $\lambda_2(\Omega_\varepsilon) = c$.

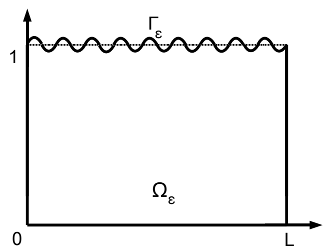
Consequently, when $\varepsilon \to 0$, the first nonzero eigenvalue of the Neumann-Laplacian on Ω_ε will converge to c, which is different from the first nonzero eigenvalue associated to Ω. The conclusion is that a "small" geometric perturbation of the square Ω leads to an uncontrollable behaviour of the Neumann-Laplacian spectrum (see [7] for details).

1.5 The rugosity effect

For simplicity, the Stokes equation with perfect slip boundary conditions (on a piece of the boundary) is considered in the 2D-rectangle $\Omega = (0, L) \times (0, 1)$. Roughly speaking, the rugosity effect is the following: a geometric perturbation of the boundary at a microscopic scale may transform perfect slip boundary conditions in total adherence. We refer the reader to [13] for a description of this phenomenon if the perturbation of the boundary has a periodic structure:

$$\Gamma_\varepsilon = \{(x, 1 + \varepsilon\varphi(\frac{x}{\varepsilon})) : x \in (0, L)\},$$

where $\varphi \in C^2[0, L]$, $\varphi(0) = \varphi(L)$, is extended by periodicity on $\mathbb{R}$.



Example of periodic rugosity. The amplitude ε of the perturbation vanishes.

This phenomenon occurs (in 2D) as soon as *some* rugosity is present (i.e. $\nabla\varphi \not\equiv 0$) in particular the boundary Γ_ε is not flat. This means for the periodic case above that $\varphi \not\equiv \varphi(0)$! It is a consequence of the oscillating normals in relationship with the non-penetration condition satisfied by the solutions $u_\varepsilon \cdot n_\varepsilon = 0$ on Γ_ε, where n_ε is the normal vector on the oscillating boundary.

Recent results in [9, 10, 11] give more hints on how arbitrary rugosity acts on the solution of a Stokes (or Navier-Stokes) equation, precisely by "driving" the flow on the boundary and by introducing some friction matrix.

In the next chapter we give some explanations of the rugosity effect, from the variational point of view. In particular one may use the results on the geometric perturbations for scalar elliptic equations with Dirichlet boundary conditions, since the perfect slip boundary conditions for vector valued PDEs can be seen as sort of partial Dirichlet boundary conditions for vector PDEs.

The influence of the rugosity in the presence of complete adherence is a different problem, and we refer the reader to [26]. In this case, the complete adherence is preserved in the limit, the challenge being to find better approximations of the solutions associated to the rough boundaries in a smooth domain where the complete adherence is replaced by a wall law (see also [4]).

2 Variational analysis of the rugosity effect

2.1 Scalar elliptic equations with Dirichlet boundary conditions

Let $D \subseteq \mathbb{R}^N$ be a bounded open set, $f \in H^{-1}(D)$ (one can consider $f \in L^2(D)$ for simplicity) and Ω_ε be a *geometrical* perturbation of $\Omega \subseteq D$. We consider the Dirichlet problem for the

Laplacian on the moving domain

$$\begin{cases} -\Delta u_\varepsilon = f \text{ in } \Omega_\varepsilon \\ u_\varepsilon \in H_0^1(\Omega_\varepsilon). \end{cases} \tag{6}$$

The question we deal with is whether the convergence $u_\varepsilon \to u$ holds, and *in which norm*?

The following abstract result can be found in [7]. It gives a first elementary approach to study whether or not the solution of the Dirichlet problem (6) is stable for an arbitrary geometric perturbation. The main drawback of this (abstract) result is that for particular geometric perturbations of non-smooth sets it does not give a clear answer whether or not the solution is stable.

Theorem 2.1 *Assertions (1) to (4) below are equivalent:*

1. *For every $f \in H^{-1}(D)$, $u_\varepsilon \to u$ in $H_0^1(D)$-strong;*
2. *For $f = 1$, $u_\varepsilon \to u$ in $H_0^1(D)$-strong;*
3. *$H_0^1(\Omega_\varepsilon)$ converges in the sense of Mosco to $H_0^1(\Omega)$, i.e.*

   M1) *For all $\phi \in H_0^1(\Omega)$ there exists a sequence $\phi_\varepsilon \in H_0^1(\Omega_\varepsilon)$ such that ϕ_ε converges strongly in $H_0^1(D)$ to ϕ.*

   M2) *For every sequence $\phi_{\varepsilon_k} \in H_0^1(\Omega_{\varepsilon_k})$ weakly convergent in $H_0^1(D)$ to a function ϕ, then $\phi \in H_0^1(\Omega)$.*

4. *If $F_\varepsilon : L^2(D) \to \mathbb{R} \cup \{+\infty\}$,*

   $$F_\varepsilon(u) = \begin{cases} \int_D |\nabla u|^2 dx & \text{if } u \in H_0^1(\Omega_\varepsilon) \\ +\infty & \text{otherwise} \end{cases}$$

   then F_ε Γ-converges in $L^2(D)$ to F, i.e.

   - *$\forall \phi_\varepsilon \to \phi$ in $L^2(D)$ then*
     $$F(\phi) \leq \liminf_{\varepsilon \to 0} F_\varepsilon(\phi_\varepsilon)$$
   - *$\forall \phi \in L^2(D)$ there exists $\phi_\varepsilon \to \phi$ in $L^2(D)$ s.t.*
     $$F(\phi) \geq \limsup_{\varepsilon \to 0} F_\varepsilon(\phi_\varepsilon)$$

Remark 2.2 From the previous theorem, it appears clearly that the solution of the equations with the right hand side equal to 1 plays a crucial role. For simplicity, let us denote w_ε the solutions for $f \equiv 1$. Assume now that $(\Omega_\varepsilon)_\varepsilon$ is a sequence of arbitrary open subsets of D and that for some $f \in H^{-1}(D)$ $u_\varepsilon \rightharpoonup u$ and $w_\varepsilon \rightharpoonup w$ weakly in $H_0^1(D)$. Here the limit set Ω is not given, so we wonder whether u and w are solutions on some set Ω? If such set exists, its identification would not be complicated since by the maximum principle one should have $\Omega = \{x : w(x) > 0\}$. This set may be quasi-open, in general.

In practice, from the example of Cioranescu and Murat, one can notice that the set Ω may not exists because of the new term which appears: the strange term. In fact, one can formalise the emerging of this strange term (which in general will be a positive Borel measure, maybe infinite but absolute continuous with respect to capacity), and give a full interpretation through Γ-convergence arguments.

Let $\varphi \in C_0^\infty(D)$ and take $w_\varepsilon \varphi$ as test function in (6) on Ω_ε. Then (we integrate over D for simplicity)

$$\begin{aligned}\int_D f w_\varepsilon \varphi dx &= \int_D \nabla u_\varepsilon \nabla (w_\varepsilon \varphi) dx \\ &= \int_D \nabla u_\varepsilon \nabla \varphi w_\varepsilon dx + \int_D \nabla u_\varepsilon \nabla w_\varepsilon \varphi dx \\ &= \int_D \nabla u_\varepsilon \nabla \varphi w_\varepsilon dx - \int_D u_\varepsilon \nabla w_\varepsilon \nabla \varphi dx - \langle \Delta w_\varepsilon, \varphi u_\varepsilon \rangle_{H^{-1}\times H_0^1} \\ &= \int_D \nabla u_\varepsilon \nabla \varphi w_\varepsilon dx - \int_D u_\varepsilon \nabla w_\varepsilon \nabla \varphi dx + \int_D u_\varepsilon \varphi dx.\end{aligned}$$

Let $\varepsilon \longrightarrow 0$ and use

$$-\int_D u \nabla w \nabla \varphi dx = \int_D \nabla u \nabla w \varphi dx + \langle \Delta w, u\varphi \rangle_{H^{-1}(D)\times H_0^1(D)}.$$

Consequently,

$$\int_D \nabla u \nabla (\varphi w) dx + \langle \Delta w + 1, u\varphi \rangle_{H^{-1}\times H_0^1} = \int_D f \varphi w dx. \tag{7}$$

But $\nu = \Delta w + 1 \geq 0$ in $\mathcal{D}'(D)$ is a non-negative Radon measure belonging to $H^{-1}(D)$. In fact, the positivity can be easily proven for smooth sets, and then use the weak convergence in $H^{-1}(D)$: $\Delta w_\varepsilon + 1 \rightharpoonup \Delta w + 1$.

We formally write

$$\int_D \nabla u \nabla (\varphi w) dx + \int_D u \varphi w d\mu = \int_D f \varphi w dx, \tag{8}$$

where μ is the Borel measure defined by

$$\mu(B) = \begin{cases} +\infty & \text{if } cap(B \cap \{w = 0\}) > 0 \\ \displaystyle\int_B \frac{1}{w} d\nu & \text{if } cap(B \cap \{w = 0\}) = 0. \end{cases} \tag{9}$$

Using the density of $\{w\varphi : \varphi \in C_0^\infty(D)\}$ in $H_0^1(D) \cap L^2(D, \mu)$, it turns out that u solves in a weak sense the following problem

$$\begin{cases} -\Delta u + u\mu = f \text{ in } D \\ u \in H_0^1(D) \cap L^2(D, \mu). \end{cases} \tag{10}$$

i.e.

$$\forall \varphi \in H_0^1(D) \cap L^2(D, \mu) \quad \int_D \nabla u \nabla \varphi dx + \int_D u \varphi d\mu = \int_D f \varphi dx.$$

In the case of the example of Cioranescu-Murat, the measure μ equals $c\mathcal{L}\lfloor_\Omega$ and $+\infty$ on $S \setminus \Omega$, where $\mathcal{L}$ is the Lebesgue measure.

This phenomenon, called *relaxation*, plays a crucial role in optimal design problems (see [7]). It can be formalised as follows, in terms of Γ-convergence of the energy functionals (point (4) in Theorem 2.1).

Theorem 2.3 *Let $(\Omega_\varepsilon)_\varepsilon$ be an arbitrary sequence of open subsets of D. There exists a subsequence (still denoted using the same index) and a functional $F : L^2(D) \to \mathbb{R} \cup \{+\infty\}$ such that F_ε Γ-converges in $L^2(D)$ to F. Moreover, F can be represented as*

$$F(u) = \int_D |\nabla u|^2 dx + \int_D u^2 d\mu$$

where μ is a positive Borel measure, absolutely continuous with respect to capacity.

Remark 2.4 A way to prove this theorem (see Theorem 2.12 in the next paragraph for the vector case), is to prove in a first step the compactness result (which is of topological nature) and in a second step to use representation theorems in order to find the form of the Γ-limit functional.

Remark 2.5 The measure μ above, is precisely the measure computed with the help of the solutions w_ε for the right hand side $f \equiv 1$.

It is quite easy to notice that for every $f \in H^{-1}(D)$ we have that

$$F_\varepsilon(\cdot) - 2\langle f, \cdot\rangle_{H^{-1}(D)\times H_0^1(D)}$$

Γ-converges to

$$F(\cdot) - 2\langle f, \cdot\rangle_{H^{-1}(D)\times H_0^1(D)}.$$

As the Γ-convergence implies the convergence of the minimizers of the functionals, one gets the strong convergence $L^2(D)$ (and weak $H_0^1(D)$) of u_ε to the solution of (10) for every admissible right hand side f. Notice the very important fact, that the measure μ is independent on f, being only an effect of the geometric perturbation.

Remark 2.6 *When the measure is known*? The measure can be computed explicitly for very few geometric perturbations, often with periodic character. There are formulas giving in general the value of the measure in terms of the limits of local capacities of $\Omega_\varepsilon^c \cap B$ for a well chosen family of balls [16, 17]).

Remark 2.7 Also notice, that for some particular geometric perturbations, e.g. when one of the assertions of Theorem 2.1 holds, the relaxation process does not occur, and so the measure μ coming from Theorem 2.3 corresponds to a (quasi)-open set Ω, i.e. $\mu(A) = 0$ if $\text{cap}(A \cap \Omega^c) = 0$ and $\mu(A) = +\infty$ if $\text{cap}(A \cap \Omega^c) > 0$.

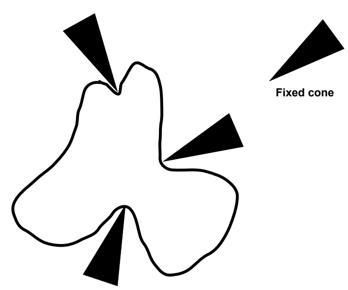
Remark 2.8 *When classical stability holds*? That means that in the limit no relaxation occurs and the (quasi)-open set Ω can be identified. Below are some situations when the geometric limit is identified.

- Increasing sequences of domains: this case is very easy, the geometric limit is the union of the open sets (direct use of Theorem 2.1).

- Decreasing sequences of domains: this problem is not so simple. Yet, what is the limit domain? The intersection of a decreasing sequence of open sets is not, in general, an open set. One may suspect that the interior of the intersection is the right limit, but the answer is not always affirmative. Keldysh gave the answer to this problem in 1962 [27], and introduced a new regularity concept, called stability (see [7] for an interpretation through Γ-convergence).

- Perturbations satisfying some geometric constraints: if the domains satisfy a uniform geometric constraint forcing the boundary to avoid oscillations, or new holes to appear, than no relaxation occurs, and the limit set Ω can be identified by some geometric convergence, precisely in the Hausdorff complementary topology (see [7]).

  Here is an example of a domain satisfying a pointwise cone condition: there exists a non trivial cone C (of dimension N or $N-1$) such that for every point $x \in \partial\Omega_\varepsilon$ there exists a cone congruent to C with vertex at x and lying in Ω_ε^c. If every Ω_ε satisfy this condition with the cone C, then no relaxation occurs, and the geometric limit can be identified. In $\mathbb{R}^2$ a $2D$ cone is a triangle and a $1D$ cone is a segment. This condition is related to a uniformity property of the Wiener criterion (see Theorem 2.9 below).



Pointwise cone condition

- Perturbations satisfying some topological constraints: in two dimensions of the space provided the number of the connected components of the complements Ω_ε^c is uniformly bounded (roughly speaking there is a uniformly bounded number of holes) the relaxation process does not hold and the limit can be identified in the Hausdorff complementary topology. This result is due to Šverák [28] and opened the way of intensive use of potential theory in understanding the behaviour of the solutions u_ε near the oscillating boundaries. In fact, in any other dimension of the space the topological constraint is not relevant. The "equivalent" constraint is a density property in terms of capacity (see [7]).

The use of capacity estimates in terms of the Wiener criterion allows us to handle the local oscillations of the solutions (see [23], [7]). For the convenience of the reader we recall the definition of the capacity: let $E \subseteq D$ be two sets in R^N, such that D is open. The capacity of E in D is

$$\mathrm{cap}(E, D) = \inf\{\int_D |\nabla u|^2 + |u|^2 dx, \quad u \in \mathcal{U}_{E,D}\}$$

where $\mathcal{U}_{E,D}$ stand for the class of all functions $u \in H^1_0(D)$ such that $u \geq 1$ a.e. in an opens et containing E.

We recall the following result from [12] (see also [7]).

Theorem 2.9 *Assume that Ω_ε converges in the Hausdorff complementary topology to some open set Ω and that there exists a function $g : (0, 1] \times (0, 1] \to (0, +\infty)$ such that*

$$\lim_{r \to 0} g(r, R) = +\infty$$

and for every $\varepsilon > 0, x \in \partial\Omega_\varepsilon, 0 < r < R < 1$ we have

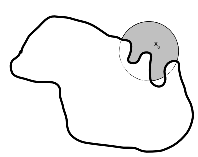
$$\int_r^R \frac{\mathrm{cap}(\Omega\varepsilon^c \cap B_{x,t}, B_{x,2t})}{\mathrm{cap}(B_{x,t}, B_{x,2t})} \frac{dt}{t} \geq g(r, R).$$

Then $u_\varepsilon \to u$ in $H^1_0(D)$.

Remark 2.10 Notice that this theorem involves a quantitative estimate of the complement of Ω^ε near the boundary and not its smoothness. A particular situation when this theorem can be applied, is the so called capacity density condition. For some positive constant c and for $t \in (0, r)$ independent on ε, the stronger estimate

$$\frac{\mathrm{cap}(\Omega_\varepsilon^c \cap B_{x,t}, B_{x,2t})}{\mathrm{cap}(B_{x,t}, B_{x,2t})} \geq c$$

holds for every $x \in \partial\Omega_\varepsilon$.



The uniform minoration of the local capacity of the complement

If ω is a smooth open subset of an $(N-1)$ dimensional manifold, such that $\{0\} \subseteq \omega \subseteq B(0, \frac{1}{2})$ and $F = \cup_{\alpha \in \mathbb{Z}^N} T_\alpha(\omega)$, then all the sets $(\varepsilon F)_\varepsilon$ satisfy uniformly a capacity density condition. Here $T_\alpha(\omega)$ is the translation of ω by the vector α.

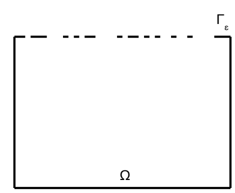
Remark 2.11 Recent advances on the stability question involve convergence of solutions in L^∞. Indeed, for right hand sides $f \in L^{\frac{N}{2}+\varepsilon}(D)$, the solutions u_ε belong to $L^\infty(\Omega_\varepsilon)$ so that a natural question is to seek if u_ε converges to u in $L^\infty(D)$. This problem is not anymore of variational type and relies on the study of the oscillations near the boundaries related to some geometric information. A characterization of the stability is given in [6]. We refer the reader to [1, 3, 20] for more results concerning this question.

2.2 The rugosity effect in fluid dynamics

2.2.1 The vector case: in a scalar setting...

The rugosity effect can be seen as the influence of partial Dirichlet boundary conditions on the behaviour of the solutions of vector valued PDEs. In order to make the relationship with the scalar case, we give below an example of scalar equation with partial Dirichlet boundary conditions. Here the word "partial" is understood in a geometric sense: there are small regions with perfect support of a membrane (homogeneous Dirichlet boundary conditions) and small regions with free membrane boundary conditions.

We consider a rectangle $\Omega \subseteq \mathbb{R}^2$, $f \in L^2(\Omega)$ and a sequence of closed sets $\Gamma_\varepsilon \subseteq \partial\Omega$ (for example located on the upper edge Γ of Ω). We consider the Laplace equation with mixed Dirichlet and Neumann homogeneous boundary conditions.



$$\begin{cases} -\Delta u_\varepsilon = f \text{ in } \Omega \\ u_\varepsilon = 0 \text{ on } \Gamma_\varepsilon \\ \frac{\partial u_\varepsilon}{\partial n} = 0 \text{ on } \Gamma \setminus \Gamma_\varepsilon \\ u_\varepsilon = 0 \text{ on } \partial\Omega \setminus \Gamma \end{cases} \tag{11}$$

When $\varepsilon \to 0$, for a subsequence one has $u_\varepsilon \to u$ weakly in $H^1(D)$ and the limit u solves the same equation on Ω but with Robin boundary conditions on the upper edge! There exists a positive measure μ such that u solves in a weak sense

$$\begin{cases} -\Delta u = f \text{ in } \Omega \\ \frac{\partial u}{\partial n} + \mu u = 0 \text{ on } \Gamma \\ u = 0 \text{ on } \partial\Omega \setminus \Gamma \end{cases} \tag{12}$$

This result fits precisely into the theory of the first section of this chapter. Indeed, one can formally reflect Ω and u_ε with respect to Γ, in Ω^r and u^r, respectively and obtain that u_ε together with its reflection, is solution of the Laplace equation with Dirichlet boundary conditions on $\Gamma_\varepsilon \cup \partial(\Omega \cup \Omega^r)$ in $\Omega \cup \Omega^r$. In this way, the Neumann b.c. can be ignored and all results of the previous section apply, thus the presence of the measure μ in the limit process.

2.2.2 The Stokes equation

For simplicity, we consider the following situation $\Omega = (0,1)^N \subseteq \mathbb{R}^N$, $N \geq 2$. Let us denote $T = (0,1)^{N-1}$ and a sequence of functions $\varphi_\varepsilon : T \to \mathbb{R}$ such that $\varphi_\varepsilon \in W^{1,\infty}(T)$, $\|\varphi_\varepsilon\|_\infty \leq \varepsilon$ and $\|\nabla\varphi_\varepsilon\|_\infty \leq M$, for some $M > 0$ independent on ε. If $x = (x_1, .., x_N) \in R^N$, by $\hat{x}$ we denote $\hat{x} = (x_1, .., x_{N-1})$. Then , we introduce the perturbed domains

$$\Omega_\varepsilon = \{x \in \mathbb{R}^N : \hat{x} \in T, 0 < x_N < 1 + \varphi_\varepsilon(\hat{x})\},$$

And denote $\Gamma_\varepsilon = \{x \in \mathbb{R}^N : \hat{x} \in T, x_N = 1 + \varphi_\varepsilon(\hat{x})\}$.

Let $f \in L^2_{loc}(\mathbb{R}^N)$. We consider the Stokes equation on Ω_ε with perfect slip boundary conditions on Γ_ε and total adherence boundary conditions on $\partial\Omega_\varepsilon \setminus \Gamma_\varepsilon$.

$$\left\{\begin{array}{rcl} -\mathrm{div}\ \mathbf{D}[\mathbf{u}_\varepsilon] + \nabla p_\varepsilon & = & \mathbf{f} \text{ in } \Omega_\varepsilon \\ \mathrm{div}\ \mathbf{u}_\varepsilon & = & 0 \text{ in } \Omega_\varepsilon \\ \mathbf{u}_\varepsilon \cdot \mathbf{n}_\varepsilon & = & 0 \text{ on } \Gamma_\varepsilon \\ (\mathbf{D}[\mathbf{u}_\varepsilon] \cdot \mathbf{n}_\varepsilon)_{tan} & = & 0 \text{ on } \Gamma_\varepsilon \\ \mathbf{u}_\varepsilon & = & 0 \text{ on } \partial\Omega_\varepsilon \setminus \Gamma_\varepsilon \end{array}\right. \tag{13}$$

It is easy to notice that the solutions $\mathbf{u}_\varepsilon \in H^1(\Omega_\varepsilon)$ are uniformly bounded, as a consequence of the uniform Korn inequality in the equi-Lipschitz domains Ω_ε. For a subsequence (still denoted using the same index) we have that

$$1_{\Omega_\varepsilon}\mathbf{u}_\varepsilon \overset{L^2(\mathbb{R}^n)}{\to} 1_\Omega \mathbf{u}, \tag{14}$$

and

$$1_{\Omega_\varepsilon}\nabla\mathbf{u}_\varepsilon \overset{L^2(\mathbb{R}^n)}{\rightharpoonup} 1_\Omega \nabla\mathbf{u}. \tag{15}$$

The question is: *what is the equation satisfied by* $\mathbf{u}$?

It is not complicated to observe that $\mathbf{u}$ satisfies in a weak sense the equation (by multiplication with test functions with free divergence in $H^1_0(\Omega)$)

$$-\mathrm{div}\ \mathbf{D}[\mathbf{u}] + \nabla p = \mathbf{f} \text{ in } \Omega$$

and

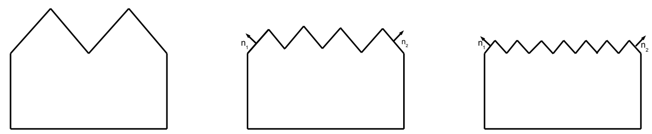
$$\mathrm{div}\ \mathbf{u} = 0 \text{ in } \Omega,$$

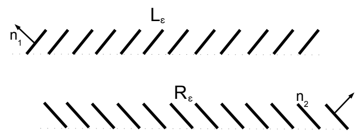
in the sense of distributions. As well, on the part of $\partial\Omega$ which is not oscillating, namely $\partial\Omega \setminus \Gamma$, one gets immediately $\mathbf{u} = 0$.

Several approaches are available in the literature in order to understand the behaviour of the solution on the upper boundary.

Below there is an intuitive justification of the rugosity phenomenon in $\mathbb{R}^2$. Let us consider the function $\varphi(x) = |x - \frac{1}{2}|$ defined on $[0, 1]$ and extended by periodicity on $\mathbb{R}$. Moreover, the upper boundaries Γ_ε of the two dimensional sets are given by the functions $\varphi_\varepsilon(x) = \varepsilon\varphi(\frac{x}{\varepsilon})$.

If we denote n_1 and n_2 the two normals at the boundaries, for every solution u_ε we have $u_\varepsilon \cdot n_1 = 0$ on L_ε and $u_\varepsilon \cdot n_2 = 0$ on R_ε (L_ε stands for the segments of Γ_ε which correspond to the locally increasing part of φ_ε and R_ε to the complement).



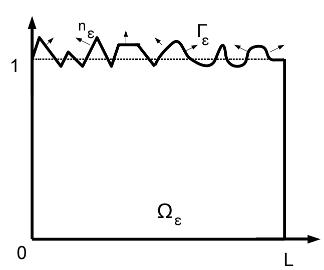


At this point, we use the vanishing information for the scalar H^1-functions $u_\varepsilon \cdot n_1$ and $u_\varepsilon \cdot n_2$. As pointed in the previous paragraph, both L_ε and R_ε satisfy a capacity density condition and converge in the Hausdorff metric to the segment $\Gamma = [0,1] \times \{1\}$. Consequently

$$u \cdot n_1 = 0 \quad \text{and } u \cdot n_2 = 0 \quad \text{on } \Gamma.$$

As n_1 and n_2 are linearly independent, we conclude with $u = 0$ on Γ.

For general rugosity it is more difficult to follow the normals. Below we briefly describe four methods.



Example of "arbitrary" rugosity. The amplitude ε of the perturbation vanishes.

Method 1: use of Young measures. In order to handle the oscillations of the boundaries, a very efficient way to describe the limit(s) of $\nabla\phi_\varepsilon$ is the use of Young measures. We refer the reader to [22] for an introduction to Young measures. The passage to the limit of the impermeability condition $u_\varepsilon \cdot (\nabla\phi_\varepsilon, -1) = 0$ may give a substantial information provided that the support of the Young measures associated to the sequence $(\nabla\phi_\varepsilon)_\varepsilon$ is large enough. We refer the reader to [9] for a description of this method.

Here are some examples where the rugosity effect is produced under mild assumptions (see [9]).

- periodic boundaries of the form $\varphi_\varepsilon(x') = \varepsilon\varphi(\frac{x}{\varepsilon})$ for some Lipschitz function defined on $\mathcal{T}$;
- crystalline boundaries;
- riblets;
- etc.

Method 2: use of capacity estimates. This method relies on the previous paragraph on scalar functions. One may mimic the intuitive example above but, as normals vary, should work with cones of normals instead of discrete normals. Roughly speaking, as an intuitive possibility, one could define for every vector n a cone $C(n)$ of axis n and opening ω. Then, if for some point x we have $u_\varepsilon(x)\cdot n_\varepsilon(x) = 0$ and $n_\varepsilon(x) \in C(n)$ then we get

$$|u_\varepsilon(x)\cdot n| \le |n - n_\varepsilon(x)||u_\varepsilon(x)|.$$

Consequently, if $|u_\varepsilon(x)|_\infty \le M$, uniformly with respect to ε then the following information can be extracted

$$|u_\varepsilon(x)\cdot n| \le Mc(\omega),$$

where $c(\omega)$ depends only on the opening of the cone. In particular, this means that $(|u_\varepsilon \cdot n| - Mc(\omega))^+$ vanishes on the region where the normals n_ε belong to the cone $C(n)$. Consequently, for this scalar sequence of functions we can fully use the scalar setting for Dirichlet Laplacian and get information about the limit.

More precisely, let $V \in W^{1,\infty}(\mathbb{R}^N, \mathbb{R}^N)$. As in the scalar case, one can construct a measure supported on Γ which is associated to V and counts energy effect of the asymptotical rugosity of $\partial\Omega_\varepsilon$ when $\varepsilon \to 0$, into the direction of the field V. The fact that the field V is fixed *a priori* allows, roughly speaking, to use the previous results for scalar functions by considering the family of scalar functions $(v_\varepsilon \cdot V)_\varepsilon$. Typically, the argument above for $V = n$ can be used. Nevertheless, in order to give a general framework, using the relationship $v_\varepsilon \cdot n_\varepsilon = 0$ on Γ_ε one can formally consider energy functionals of the form $F_\varepsilon : L^2(\mathbb{R}^N) \to \mathbb{R}\cup\{+\infty\}$,

$$F_\varepsilon(u) = \begin{cases} \int_{\mathbb{R}^N} |\nabla(u\cdot V)|^2 dx & \text{if } u \in H^1(\Omega_\varepsilon), u\cdot n_\varepsilon = 0 \text{ on } G_\varepsilon, u = 0 \text{ on } \partial\Omega_\varepsilon \setminus \Gamma_\varepsilon \\ +\infty & \text{otherwise} \end{cases}$$

and to investigate their inferior Γ-limit.

Consequently, we consider the family $\mathcal{M}_V$ of positive Borel measures, absolutely continuous with respect to the capacity, such that for every sequence $v_{\varepsilon_k} \in H^1(\Omega_{\varepsilon_k}, \mathbb{R}^N)$, $v_{\varepsilon_k}\cdot n_{\varepsilon_k} = 0$ *on* Γ_{ε_k} $v_{\varepsilon_k} = 0$ on $\partial\Omega_{\varepsilon_k}\setminus\Gamma_{\varepsilon_k}$ and such that $v_{\varepsilon_k} \to v$ in the sense of relations (14)-(15), then

$$\int_D |\nabla(v\cdot V)|^2 dx + \int_D (v\cdot V)^2 d\mu \le \liminf_{k\to\infty} \int_D |\nabla(v_{\varepsilon_k}\cdot V)|^2 dx.$$

The equality $v_{\varepsilon_k}\cdot n_{\varepsilon_k} = 0$ is understood pointwisely where the normal exists and for a quasi continuous representative of v.

Since at least the zero measure can be considered above, $\mathcal{M}_V \neq \emptyset$ so that

$$\mu_V = \sup\{\mu : \mu \in \mathcal{M}_V\}$$

is well defined.

The measure μ_V is supported on Γ and takes into account precisley the rugosity effect on $\partial\Omega$ in the direction of the field V from an energetic point of view. If, as in the scalar case, one proves that $\mu = \infty_\Gamma$, then we get $u \cdot V = 0$ on Γ, so that the flow is orthogonal to V on Γ.

Method 3: uniform estimates. Let us denote $U_\varepsilon = (0,1)^{N-1} \times \{1 - 2\varepsilon\}$. Provided some uniformity on the rugosities φ_ε, one can prove the existence of a constant $C > 0$, independent on ε such that for every solution of the Stokes equation (13), we have

$$\int_{U_\varepsilon} |u_\varepsilon|^2 d\sigma \leq C\varepsilon \int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 dx.$$

Of course, if such an estimate holds and since the solutions $(u_\varepsilon)_\varepsilon$ have uniformly bounded energy, then as $\varepsilon \to 0$ one gets $u = 0$ on Γ.

We refer to [8, 13] for estimates of this kind in the periodic case, and to [5] for improvements of the periodic case, if the Lipschitz hypothesis is removed.

Method 4: representation by Γ-convergence. In order to find the general form of the limit problem, in [11] it is used an approach based on Γ-convergence.

Theorem 2.12 *Let $\varepsilon \to 0$ and let $\mathbf{f} \in L^2_{loc}(\mathbb{R}^N, \mathbb{R}^N)$ be given. Let $\{\mathbf{u}_\varepsilon\}_{\varepsilon>0}$ be the family of (weak) solutions to the Stokes equation (13) in Ω_ε.*

Then, at least for a suitable subsequence we have

$$1_{\Omega_\varepsilon}\mathbf{u}_\varepsilon \to 1_\Omega \mathbf{u} \textit{ (strongly) in } L^2(\mathbb{R}^N, \mathbb{R}^N),$$

$$1_{\Omega_\varepsilon}\nabla\mathbf{u}_\varepsilon \to 1_\Omega \nabla\mathbf{u} \textit{ weakly in } L^2(\mathbb{R}^N, \mathbb{R}^{N\times N}),$$

and there exists a suitable trio $\{\mu, A, \mathcal{V}\}$ independent of the driving force $\mathbf{f}$ such that

- *μ is a capacitary measure concentrated on Γ*
- *$\{\mathcal{V}\}_{x\in\Gamma}$ is a family of vector subspaces in $\mathbb{R}^{N-1}$*
- *A is a positive symmetric matrix function A defined on Γ*

and $\mathbf{u}$ is a solution in Ω of the Stokes equation with friction-driven b.c.

$$\begin{cases} -div\, \mathbf{D}[\mathbf{u}] + \nabla p &= \mathbf{f} \textit{ in } \Omega \\ div\, \mathbf{u} &= 0 \textit{ in } \Omega \\ \mathbf{u}(x) &\in V(x) \textit{ for q.e. } x \in \Gamma \\ \Big[\mathbf{D}[\mathbf{u}] \cdot \mathbf{n} + \mu A \mathbf{u}\Big] \cdot \mathbf{v} &= 0 \textit{ for any } \mathbf{v} \in V(x),\ x \in \Gamma \\ \mathbf{u}(x) &= 0 \textit{ for q.e. } x \in \partial\Omega \setminus \Gamma. \end{cases} \tag{16}$$

The sense in which u solves the equation (16) is the following: u is solution of the minimization of

$$\mathcal{J}(\mathbf{v}) := \frac{1}{2}\int_{\Omega}\Big(|\mathbf{D}[\mathbf{v}]|^2 + |\mathbf{v}|^2\Big)dx + \frac{1}{2}\int_{\partial\Omega}\mathbf{v}^T A\mathbf{v}d\mu - \int_{\Omega}\mathbf{f}\cdot\mathbf{v}dx, \tag{17}$$

on

$$\Big\{\mathbf{v}\in H^1(\Omega, R^N)\ \Big|\ \operatorname{div}\mathbf{v}=0 \text{ in } \Omega,\ \mathbf{v}(x)\in V(x) \text{ for q. e. } x\in\Gamma, \mathbf{v}=0 \text{ on } \partial\Omega\setminus\Gamma\Big\}.$$

Proof The main steps of the proof are the following:

Step 1. introduce energy functionals involving the boundary constraint: $\mathbf{u}_\varepsilon\cdot n_\varepsilon = 0$ and remove incompressibility condition;

Step 2. use representation results of the Γ-limit for vector valued functionals (see [18] and also [16, 17] for scalar or vector equations for Dirichlet boundary conditions);

Step 3. prove that the measure is concentrated on the surface;

Step 4. use a diagonal argument in order to handle the incompressibility condition.

□

This theorem gives the general form of the limit problem, but in any particular situation, specific computations should be carried out in order to identify the trio $\{\mu, A, \mathcal{V}\}$.

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