Monotonicity formula and regularity for general free discontinuity problems

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Abstract

We give a general monotonicity formula for local minimizers of free discontinuity problems which have a *critical* deviation from minimality, of order $d-1$. This result allows to prove partial regularity results (i.e. closure and density estimates for the jump set) for a large class of free discontinuity problems involving general energies associated to the jump set, as for example free boundary problems with Robin conditions. In particular, we give a short proof to the De Giorgi-Carriero-Leaci result for the Mumford-Shah functional.

1 Introduction

We give a monotonicity formula which describes the behaviour of the energy associated to *almost-quasi* local minimisers of general free discontinuity problems, near the points of the jump set. This formula is a key tool to prove both the closedenss of the jump set and of uniform density estimates of the associated Hausdorff measure. In this way we generalize the De Giorgi-Carriero-Leaci result on the Mumford-Shah functional [5] (see also [2, 4]) to a large class of functionals possibly involving geometric quantities of the jump set and the traces of the SBV functions on the jump set.

An important question is whether an SBV minimiser of a general free discontinuity problem is a "classical" one, i.e. its jump set is closed and so the function is locally smooth (in the complement of the jump set). A positive answer to this question was given by De Giorgi, Carriero and Leaci in [5] (see also [4] and [2]) for the Mumford-Shah functional. Precisely, they analyse properties of quasi minimisers of the Mumford-Shah functional, i.e. functions $u \in SBV_{loc}(\mathbb{R}^d)$ satisfying for some $\alpha > 0$, $c_\alpha \geq 0$ and $\mathbf{\Lambda} = \mathbf{1}$

$$\forall x \in \mathbb{R}^d, \forall \rho > 0, \forall v \in SBV_{loc}(\mathbb{R}^d) \text{ such that } \{u \neq v\} \subseteq B_\rho(x) \Longrightarrow$$

$$\int_{B_\rho(x)} |\nabla u|^2 dx + \mathcal{H}^{d-1}(J_u \cap \overline{B}_\rho(x)) \leq \int_{B_\rho(x)} |\nabla v|^2 dx + \mathbf{\Lambda} \mathcal{H}^{d-1}(J_v \cap \overline{B}_\rho(x)) + c_\alpha \rho^{d-1+\alpha}.$$

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The main purpose of this paper is to study the behaviour of *almost-quasi* minimisers, i.e. to consider $\Lambda \geq 1$. As the value of Λ is allowed to be greater than 1, the deviation from minimality becomes of the same order as the Hausdorff measure, $d-1$. The main consequence of this freedom is that one can deal with functionals involving quite general energies associated to the jump set, and still conclude that the solutions have a closed jump set with an associated Hausdorff measure satisfying upper and lower uniform density bounds.

The first result of the paper is a monotonicity formula (Theorem 2.2) which holds for *almost-quasi* minimisers. Roughly speaking, a quantity of the form

$$\Big[\frac{1}{\rho^{d-1}}\Big(\int_{B_\rho}|\nabla u|^2dx + \mathcal{H}^{d-1}(J_u\cap \overline{B}_\rho)\Big)\Big]\wedge c(d,\Lambda) + c(d,\alpha)\rho^\alpha \tag{1}$$

is non decreasing in ρ, for small ρ. For $\rho \to 0$, around a regular point of the jump set, the limit of the quantity above is bounded from below by a positive constant depending only on the dimension of the space and Λ. In this way, one can obtain key information for proving its closure, and latter uniform density bounds. The constants appearing in the density bounds are obtained by refining the monotonicity formula and are given by explicit estimates.

In the second part of the paper, we analyse a free boundary - free discontinuity problem with Robin conditions. A similar question for Dirichlet boundary conditions is treated by Alt and Caffarelli in [1]. In our example, the energy we minimize among all *admissible* functions u with values prescribed on a given compact subset of $\mathbb{R}^d$, is of the form

$$\int_{\mathbb{R}^d}|\nabla u|^2dx + \beta\int_{J_u}(|u^+|^2+|u^-|^2)d\mathcal{H}^{d-1} + |\{u>0\}|.$$

A variational framework based on SBV spaces which allows to deal with this type of problems was introduced in [3]. In order to prove that the solutions u of these free boundary problems are almost-quasi minimisers, and so they fit in the framework of our paper, the crucial difficulty is to establish uniform upper and lower positive bounds on their positivity set, i.e. to prove that they satisfy

$$\exists\ 0<\alpha_1<\alpha_2,\ \text{such that } \alpha_1 \leq u(x) \leq \alpha_2,\ \text{a.e. } x\in\{u>0\}.$$

If the jump set were closed and smooth, the lower bound would be a consequence of the Hopf lemma in relationship with the PDE satisfied by u. Nevertheless, these two properties are a priori unknown, being precisely the object of investigation.

2 The monotonicity formula

In the sequel $d \geq 2$, $B_\rho(x)$ stands for the open ball of $\mathbb{R}^d$ centered in x and of radius ρ (if ambiguity on the dimension occurs, the ball is denoted $B^d_\rho(x)$). If $x=0$, we simple denote it B_ρ. By $d_K(x)$ we denote the distance from the point x to set K in $\mathbb{R}^d$. For an open set $\Omega\subseteq\mathbb{R}^d$, by $SBV(\Omega)$ we denote the space of special functions with bounded variation and by J_u the jump set of a function $u\in SBV(\Omega)$ (see [2, Definition 3.67]). For every $m\in\mathbb{N}$, by $\mathcal{H}^m$ we denote the m-dimensional Hausdorff measure. The usual Lebesgue measure is denoted by $|\cdot|$.

Definition 2.1 *Let $\Lambda\geq 1$, $\alpha>0$, $c_\alpha\geq 0$. We say that a function $u\in SBV(\Omega)$ is a local $(\Lambda,\alpha,c_\alpha)$ almost-quasi minimizer of a free discontinuity problem at the point $x\in\Omega$, if*

$$\forall\, 0<\rho<d_{\partial\Omega}(x), \forall v\in SBV(\Omega)\ \textit{such that}\ \{u\neq v\}\subseteq B_\rho(x) \Longrightarrow$$

$$\int_{B_\rho(x)} |\nabla u|^2 dx + \mathcal{H}^{d-1}(J_u \cap \overline{B}_\rho(x)) \le \int_{B_\rho(x)} |\nabla v|^2 dx + \Lambda \mathcal{H}^{d-1}(J_v \cap \overline{B}_\rho(x)) + c_\alpha \rho^{d-1+\alpha}.$$

We say that a function $u \in SBV(\Omega)$ is a $(\Lambda, \alpha, c_\alpha)$ almost-quasi minimizer of a free discontinuity problem if it is a local almost quasi minimizer at every point of Ω.

For simplicity, in the sequel we drop the mention to the triplet $(\Lambda, \alpha, c_\alpha)$ and we simply refer to almost-quasi minimisers.

Here is the main result of the paper.

Theorem 2.2 (The monotonicity formula) *Let $u \in SBV(\Omega)$ be a local almost-quasi minimizer of a free discontinuity problem at 0. Then the mapping*

$$\rho \longmapsto E(\rho) := \Big[\frac{1}{\rho^{d-1}}\Big(\int_{B_\rho} |\nabla u|^2 dx + \mathcal{H}^{d-1}(J_u \cap \overline{B}_\rho)\Big)\Big] \wedge \frac{c_d \Lambda^{2-d}}{d-1} + (d-1)\frac{c_\alpha}{\alpha}\rho^\alpha \tag{2}$$

is non decreasing on $(0, d_{\partial\Omega}(0))$.

We give two lemmas relating SBV functions on spheres with the harmonic extensions of their H^1-approximations. The boundary of the ball $B_\rho(0)$ is denoted ∂B_ρ and by $SBV(\partial B_\rho)$ we denote the SBV space on the sphere. The tangential gradient is denoted ∇_τ.

Lemma 2.3 *For $d \ge 2$, there exists a constant c_d such that for every function $u \in SBV(\partial B_\rho)$ which satisfies*

$$\varepsilon(u) := \int_{\partial B_\rho} |\nabla_\tau u|^2 dx + \mathcal{H}^{d-2}(J_u \cap \partial B_\rho) \le c_d \Lambda^{2-d} \rho^{d-2} \tag{3}$$

there exists $w \in H^1(\partial B_\rho)$ such that

$$\int_{\partial B_\rho} |\nabla_\tau w|^2 dx + \frac{\Lambda(d-1)}{\rho}\mathcal{H}^{d-1}(\{u \ne w\} \cap \partial B_\rho) \le \varepsilon(u). \tag{4}$$

Lemma 2.4 *Under the hypotheses of Lemma 2.3, there exists $\tilde{w} \in H^1(B_\rho)$ such that $\Delta \tilde{w} = 0$ in B_ρ and*

$$\int_{B_\rho} |\nabla \tilde{w}|^2 dx + \Lambda \mathcal{H}^{d-1}(\{u \ne \tilde{w}\} \cap \partial B_\rho) \le \frac{\rho}{d-1}\varepsilon(u). \tag{5}$$

The proof is done simultaneously for Theorem 2.2 and Lemmas 2.3, 2.4, in four steps, using a cyclic inductive argument over the dimension of the space.

Step 1. Proof of Lemma 2.3 for $d = 2$. It is enough to take $c_2 < 1$. Indeed, since $\mathcal{H}^{d-2}(J_u) \equiv \mathcal{H}^0(J_u)$ and by hypothesis $\mathcal{H}^0(J_u) < 1$, then $J_u = \emptyset$. So we can take directly $w := u$. □

Step 2. Proof of the implication: Lemma 2.3 in $\mathbb{R}^d \Longrightarrow$ Lemma 2.4 in $\mathbb{R}^d$, for every $d \ge 2$. Let $d \ge 2$ and assume Lemma 2.3 is true. Let $u \in SBV(\partial B_\rho)$ such that $\varepsilon(u) \le c_d \Lambda^{2-d} \rho^{d-2}$ and $w \in H^1(\partial B_\rho)$ be the function given by Lemma 2.3. We denote by $\tilde{w}$ the harmonic extension of w in B_ρ. Since

$$\frac{d-1}{\rho}\int_{B_\rho} |\nabla \tilde{w}|^2 dx \le \int_{\partial B_\rho} |\nabla_\tau w|^2 d\mathcal{H}^{d-1}, \tag{6}$$

relation (5) follows directly. □

Step 3. Proof of the implication: Lemma 2.4 in $\mathbb{R}^d \implies$ Theorem 2.2 in $\mathbb{R}^d$, for every $d \geq 2$.

Assume Lemma 2.4 is true. In order to prove the monotonicity formula, we introduce the function

$$\mathcal{E} : (0, d_{\partial\Omega}(0)) \to \mathbb{R}, \;\; \mathcal{E}(\rho) := \int_{B_\rho} |\nabla u|^2 dx + \mathcal{H}^{d-1}(J_u \cap \overline{B}_\rho).$$

Then $\mathcal{E}$ is a non decreasing function, and its distributional derivative can be written as

$$\partial_\rho \mathcal{E}(\rho) = e(\rho) d\rho + \mu,$$

where $e \in L^1(0, d_{\partial\Omega}(0))$ is a nonnegative function and μ is a postive singular measure. More precisely,

$$\forall 0 < a < b < d_{\partial\Omega}(0), \;\; \mu(a,b) = \mathcal{H}^{d-1}((B_b \setminus \overline{B}_a) \cap J_u \cap \{x : |\nu_u(x) \cdot x| = \|x\|\}).$$

As a consequence, from the Leibniz formula for BV functions, it is enough to estimate the absolutely continuous part of the derivative of E.

At a.e. point ρ where E is derivable and at which

$$\frac{\mathcal{E}(\rho)}{r^{d-1}} \geq \frac{c_d \Lambda^{2-d}}{d-1},$$

we have $\partial_\rho E(\rho) = (d-1) c_\alpha \rho^{\alpha-1} \geq 0$.

For a.e. point r at which

$$\frac{\mathcal{E}(\rho)}{\rho^{d-1}} < \frac{c_d \Lambda^{2-d}}{d-1}, \tag{7}$$

we have

$$\partial_\rho E(\rho) = \frac{\rho \mathcal{E}'(\rho) - (d-1)\mathcal{E}(\rho) + (d-1) c_\alpha \rho^{d-1+\alpha}}{\rho^d}. \tag{8}$$

Assuming for contradiction that for some point ρ we have $\partial_\rho E(\rho) < 0$, then from (8) we get

$$\mathcal{E}'(\rho) < \frac{d-1}{\rho} \mathcal{E}(\rho) - \frac{(d-1) c_\alpha}{\rho} \rho^{d-1+\alpha} \tag{9}$$

and using (7) we get

$$\mathcal{E}'(\rho) < c_d \rho^{d-2} \Lambda^{2-d}.$$

Since

$$\varepsilon(u) = \int_{\partial B_\rho} |\nabla_\tau u|^2 dx + \mathcal{H}^{d-2}(J_u \cap \partial B_\rho) \leq \mathcal{E}'(\rho)$$

we can use Lemma 2.4 to get the existence of a function $\tilde{w}$ such that

$$\int_{B_\rho} |\nabla \tilde{w}|^2 dx + \Lambda \mathcal{H}^{d-1}(\{u \neq \tilde{w}\} \cap \partial B_\rho) \leq \frac{r}{d-1} \mathcal{E}'(\rho).$$

Using (9), we get

$$\int_{B_\rho} |\nabla \tilde{w}|^2 dx + \Lambda \mathcal{H}^{d-1}(\{u \neq \tilde{w}\} \cap \partial B_\rho) \leq \mathcal{E}(\rho) - c_\alpha \rho^{d-1+\alpha}.$$

Consequently

$$\int_{B_\rho} |\nabla \tilde{w}|^2 dx + \Lambda \mathcal{H}^{d-1}(\{u \neq \tilde{w}\} \cap \partial B_\rho) + c_\alpha \rho^{d-1+\alpha} \leq \mathcal{E}(\rho), \tag{10}$$

and using the almost-quasi minimality of u we get that equality holds in (10) and thus in all preceding inequalities. This contradicts the hypothesis $\partial_\rho E(\rho) < 0$. □

Step 4. Proof of the implication: Theorem 2.2 in $\mathbb{R}^d \implies$ Lemma 2.3 in $\mathbb{R}^{d+1}$, for every $d \geq 2$. It is enough to prove Lemma 2.3 only for $\rho = 1$, and to rescale. To simplify our computations, we shall assume without restricting generality that $c_d \leq (d-1)\omega_{d-1}$, where ω_{d-1} is the volume of the unit ball in $\mathbb{R}^{d-1}$. We consider the unit ball $B_1 \subseteq \mathbb{R}^{d+1}$ and the d-dimensional unit sphere $S^d \subseteq \mathbb{R}^{d+1}$, $S^d = \partial B_1$. Let $u \in SBV(S^d)$ be fixed, such that

$$\varepsilon(u) := \int_{S^d} |\nabla_\tau u|^2 dx + \mathcal{H}^{d-1}(J_u) \leq c\Lambda^{1-d}. \tag{11}$$

The value of c will be specified below.

We introduce an auxiliary problem

$$\min_{w \in SBV(S^d)} \int_{S^d} |\nabla_\tau w|^2 d\mathcal{H}^d + \mathcal{H}^{d-1}(J_w) + \Lambda d \mathcal{H}^d(\{u \neq w\}), \tag{12}$$

and prove that for a sufficiently small constant c in (11), depending only on the dimension of the space, a minimizer w of (12) will satisfy $\mathcal{H}^{d-1}(J_w) = 0$. Note that here the hypothesis $\Lambda \geq 1$ can be relaxed. Consequently, for this minimizer w we have $w \in H^1(S^d)$ and

$$\int_{S^d} |\nabla_\tau w|^2 d\mathcal{H}^d + \Lambda d \mathcal{H}^d(\{u \neq w\}) \leq \int_{S^d} |\nabla_\tau u|^2 dx + \mathcal{H}^{d-1}(J_u)$$

so that we achieve the proof of Lemma 2.3 in $\mathbb{R}^{d+1}$.

Let us consider a minimizer w for the auxiliary problem (12) and assume for contradiction that $\mathcal{H}^{d-1}(J_w) \neq 0$, for some value of the constant c. We can further assume that the point $e_{d+1} = (0, 0, .., 0, 1)$ is a regular point of the jump set J_w and consider the orthogonal projection

$$\Pi : S^d \cap B_{\frac{1}{2}}(e_{d+1}) \longmapsto \mathbb{R}^d \times \{0\}.$$

Let us denote in the sequel the (d-dimensional) ball $B^d_{\frac{1}{2}}(0) \subseteq \mathbb{R}^d$ and introduce the functions $\overline{w}, \overline{u} \in SBV(B^d_{\frac{1}{2}}(0))$ defined by

$$\overline{w}(x) = w \circ \Pi^{-1}(x), \;\; \overline{u}(x) = u \circ \Pi^{-1}(x).$$

Then $\overline{w}$ is a minimizer for the functional

$$\int_{B^d_{\frac{1}{2}}} |\nabla \overline{w}|^2 dx + \mathcal{H}^{d-1}(J_{\overline{w}}) + \Lambda d \mathcal{H}^d(\{\overline{u} \neq \overline{w}\}) + R(\overline{w}, \Lambda, \frac{1}{2}),$$

where $0 \leq R(\overline{w}, \Lambda, r) \leq C_d(1+\Lambda)r^d$, for some constant C_d depending only on the dimension of the space. Consequently, $\overline{w}$ is a $(1, 1, c_1)$ almost-quasi minimizer of a free discontinuity problem for $\Lambda = 1, \alpha = 1$ and $c_1 = (\Lambda+1)(d + C_d)$, so that we can use the monotonicity formula of Theorem 2.2.

On the one hand, since 0 is a regular point of $J_{\overline{w}(x)}$ we have

$$\lim_{\rho\to 0_+} E(\rho) \geq \frac{c_d}{d-1} \wedge \omega_{d-1} = \frac{c_d}{d-1}.$$

Meanwhile, from the monotonicity formula we get for every $\rho \in (0, 1/2)$

$$\Big[\frac{1}{\rho^{d-1}}\Big(\int_{B_\rho} |\nabla \overline{w}|^2 dx + \mathcal{H}^{d-1}(J_{\overline{w}} \cap \overline{B}_\rho)\Big)\Big] \wedge \frac{c_d}{d-1} + (d-1)(\Lambda+1)(d+C_d)\rho \geq \frac{c_d}{d-1}. \tag{13}$$

Assume that for some ρ we have that

$$\frac{c\Lambda^{1-d}}{\rho^{d-1}} < \frac{c_d}{d-1}.$$

If ρ is fixed (its value will be given below), this assumption will be satisfied as soon as c will be chosen small enough.

Indeed, combining hypothesis (11) with (13) and the assumption above, we get that

$$\frac{c}{(\Lambda\rho)^{d-1}} + (d-1)(\Lambda+1)(d+C_d)\rho \geq \frac{c_d}{d-1},$$

or introducing suitable dimensional constants, for $\rho \leq \frac{c_d}{2(d-1)^2(d+C_d)}$ we get

$$\frac{c}{(\Lambda\rho)^{d-1}} + C'_d\Lambda\rho \geq \frac{c_d}{2(d-1)}.$$

Let us fix $\rho := \frac{C''_d c^{1/d}}{\Lambda}$. It is enough to take c small enough, e.g.

$$\frac{C''_d c^{1/d}}{\Lambda} < \frac{C''_d C_d}{(d-1)\Lambda} \wedge \frac{c_d}{2(d-1)^2(d+C_d)} \wedge \frac{1}{2},$$

to obtain a contradiction and finish the proof. □

3 Regularity properties for free discontinuity problems

Relaying on the monotonicity formula, in this section we prove the primary regularity property for almost-quasi minimizers of free discontinuity problems: their jump set is closed and satisfies density estimates from above and below (i.e. it is Ahlfors regular) and the function is smooth in the complement of the this set.

3.1 Closure of the jump set

Theorem 3.1 *Let u be an almost-quasi minimizer of a free discontinuity problem and denote J_u the family its jump points. Then $\mathcal{H}^{d-1}(\overline{J}_u \setminus J_u) = 0$.*

Proof Since J_u is rectifiable, it is contained in a countable union of C^1 hypersurfaces, up to a set of zero $\mathcal{H}^{d-1}$-measure. We denote J_u^r the set of regular points of J_u (i.e. which have density 1 in some of these C^1 manifolds). Let us consider $x_n \in J_u^r$ such that $x_n \to \overline{x} \in \overline{J}_u$. For every such point x_n, we have

$$\lim_{\rho\to 0} \frac{\mathcal{H}^{d-1}(J_u \cap B_\rho(x_n))}{\rho^{d-1}} \geq \omega_{d-1},$$

so that relaying on the monotonicity formula and comparing $E(\rho)$ to the behaviour of E near 0, we get for $\rho \in (0, d_{\partial\Omega}(x_n))$

$$\frac{1}{\rho^{d-1}}\Big(\int_{B_\rho(x_n)}|\nabla u|^2dx + \mathcal{H}^{d-1}(J_u\cap \overline{B}_\rho(x_n))\Big)\Big] + (d-1)\frac{c_\alpha}{\alpha}\rho^\alpha \geq \omega_{d-1}\wedge \frac{c_d\Lambda^{2-d}}{d-1} \tag{14}$$

There exists $\rho_0, C_0 > 0$ independent on n such that, provided $\rho \leq \rho_0 \wedge d_{\partial\Omega}(x_n)$,

$$\int_{B_\rho(x_n)}|\nabla u|^2dx + \mathcal{H}^{d-1}(J_u\cap \overline{B}_\rho(x_n)) \geq C_0\rho^{d-1}. \tag{15}$$

By continuity, inequality (15) holds also at the point $\overline{x}$

$$\int_{B_\rho(\overline{x})}|\nabla u|^2dx + \mathcal{H}^{d-1}(J_u\cap \overline{B}_\rho(\overline{x})) \geq C_0\rho^{d-1}. \tag{16}$$

In general, for every function $u \in SBV(\Omega)$, it is well known (see for instance [5, Theorem 3.6]) that for $\mathcal{H}^{d-1}$-almost every point $y \in \Omega\setminus J_u$ we have

$$\lim_{\rho\to 0}\rho^{1-d}\Big[\int_{B_\rho(y)}|\nabla u|^2dx + \mathcal{H}^{d-1}(J_u\cap \overline{B}_\rho(y))\Big] = 0. \tag{17}$$

Consequently, there exists a set A such that $\mathcal{H}^{d-1}(A) = 0$ and such that every point y satisfying inequality (16) belongs to $A \cup J_u$, and so the point $\overline{x}$. Consequently $\overline{J}^r_u \subseteq J_u \cup A$. Since $\mathcal{H}^{d-1}(J_u \setminus J^r_u) = 0$, we get that $\mathcal{H}^{d-1}(J_u \cup A \setminus \overline{J}^r_u) = 0$. As the following argument shows, the set $J_u \setminus \overline{J}^r_u$ is empty, which concludes the proof since this implies $\overline{J}_u \subseteq \overline{J}^r_u$ and so $\mathcal{H}^{d-1}(\overline{J}_u \setminus J_u) = 0$.

Indeed, since $\overline{J}^r_u$ is closed and $\mathcal{H}^{d-1}(J_u\setminus\overline{J}^r_u) = 0$, we get that $u \in H^1(\Omega\setminus\overline{J}^r_u)$. Since u is a local almost minimizer for the Dirichlet energy in the open set $\Omega\setminus\overline{J}^r_u$, we get that $u \in C^{0,\frac{1\wedge\alpha}{2}}$, so that u has no jump points outside $\overline{J}^r_u$ (see for instance [2, Remark 7.20]).

□

Remark 3.2 The proof of the result above shows in fact a slightly stronger result, precisely $\mathcal{H}^{d-1}(\overline{S}_u \setminus S_u) = 0$ where S_u stands for the family of singular points of u. The set S_u consists on all the points where the approximate limit of u does not exist (see [2, Definition 3.63]), and may be slightly larger than the set of jump points J_u. Nevertheless $\mathcal{H}^{d-1}(S_u \setminus J_u) = 0$.

3.2 Ahlfors regularity

The Ahlfors regularity of the jump set can be proved as a consequence of the monotonicity theorem 2.2 and of the *decay lemma* [2, Lemma 7.14], as we show below. In the next paragraph we shall give a more technical monotonicity type lemma, which not only gives in a direct way the Ahlfors regularity, but also contains implicit estimates of the density constants.

We start with the following density result for the energy which is a consequence of almost-quasi-minimality and of the monotonicity formula.

Corollary 3.3 *There exists constants $c = c(d,\Lambda)$, $\rho_0 = \rho_0(d,\alpha,c_\alpha,\Lambda)$ such that for every almost quasi-minimizer u, for every $x \in J_u$ and for every $0 < \rho < \rho_0$ such that $B_\rho(x) \subseteq \Omega$ we have*

$$\frac{1}{c} \geq \frac{1}{\rho^{d-1}}\Big(\int_{B_\rho}|\nabla u|^2dx + \mathcal{H}^{d-1}(J_u\cap\overline{B}_\rho)\Big) \geq c. \tag{18}$$

Remark 3.4 There exists $\rho_0' > 0, c' > 0$ such that for every $x \in J_u$ and for every $\rho < \rho_0'$ satisfying $B_\rho(x) \subseteq \Omega$, we have

$$c' \rho^{d-1} \leq \mathcal{H}^{d-1}(J_u \cap B_\rho(x)) \leq \frac{1}{c'} \rho^{d-1}. \tag{19}$$

Indeed, the right hand side of the inequality is a direct consequence of the minimality of u. To prove the inequality of the left hand side, one can relay on [2, Lemma 7.14], which roughly speaking asserts that if $\mathcal{H}^{d-1}(J_u \cap B_\rho) \leq \varepsilon \rho^{d-1}$ for ε small enough, then the decay of the energy is like ρ^d which is in contradiction with the monotonicity Theorem 2.2 which gives a density estimate of the energy.

Assume for contradiction that there exist sequences $x_n \in J_u$, $\rho_n \to 0$, $C_n \to 0$ such that

$$\mathcal{H}^{d-1}(J_u \cap B_{\rho_n}(x_n)) \leq C_n \rho_n^{d-1}. \tag{20}$$

Inequalities (41)-(19)-(20) imply that there exist $\varepsilon_n \to 0$ such that

$$\varepsilon_n \frac{1}{c} \rho_n^{d-1} \geq \varepsilon_n \int_{B_{\rho_n}(x_n)} |\nabla u|^2 dx \geq \mathcal{H}^{d-1}(J_u \cap B_{\rho_n}(x_n)).$$

Consequently, the deviation from minimality (see [2, Definition 7.2] for its definition) of u for $\psi_\Lambda(u, B_\rho(x)) := \int_{B_\rho(x)} |\nabla u|^2 dx + \Lambda \mathcal{H}^{d-1}(J_u \cap B_\rho(x))$ in $B_{\rho_n}(x_n)$ satisfies

$$\text{Dev } (u, B_{\rho_n}(x_n)) \leq \left[(\Lambda - 1) \frac{\varepsilon_n}{c} + c_\alpha \rho_n^\alpha \right] \rho_n^{d-1} \leq \theta_n \psi_\Lambda(u, B_{\rho_n}(x_n)),$$

for some $\theta_n \to 0$. For every $\tau > 0$, we can apply [2, Lemma 7.14] so that there exists $C > 0$ such that for n large enough

$$\psi_\Lambda(u, B_{\tau\rho_n}(x_n)) \leq C \tau^d \psi_\Lambda(u, B_{\rho_n}(x_n)).$$

which contradicts the lower density estimate in (41), provided τ is chosen small enough.

3.3 Density estimates

In the sequel, we give a refined monotonicity type formula which allows a direct proof of the Ahlfors regularity and moreover gives explicit estimates for the density constants, depending on the dimension of the space, Λ, α, c_α.

We start with a technical result consisting in a refined inequality for harmonic extensions of H^1-functions.

Lemma 3.5 *There exists a constant $\alpha_d > 0$ such that for every harmonic function w in B_1, such that $w|_{\partial B_1} \in H^1(\partial B_1)$, we have*

$$\int_{\partial B_1} |\nabla_\tau w|^2 dx \geq (d-1) \int_{B_1} |\nabla w|^2 dx + \alpha_d \int_{\partial B_1} |\nabla_\tau (w - \overline{w})|^2 dx, \tag{21}$$

where $\overline{w}(y) = \fint_{B_1} w dx + \big(\fint_{B_1} \nabla w dx \big) \cdot y.$

Proof The proof is a consequence of the identity

$$\int_{\partial B_1} |\nabla_\tau w|^2 dx = (d-1) \int_{B_1} |\nabla w|^2 dx + \frac{1}{2} \int_{B_1} (1 - |x|^2) |D^2 w|^2 dx,$$

and of the following inequality

$$\frac{1}{2}\int_{B_1}(1-|x|^2)|D^2w|^2dx \geq \alpha_d \int_{\partial B_1}|\nabla_\tau(w-\overline{w})|^2dx,$$

which can be proved, for instance, by contradiction. Clearly, equality holds in the previous inequality if and only if $w=\overline{w}$. □

In the sequel, we assume that u is an almost quasi minimizer of a free discontinuity problem, satisfying the hypotheses of Theorem 2.2. We shall denote $u_\rho(x)=\dfrac{u(\rho x)}{\rho^{1/2}}$ and, as before,

$$\mathcal{E}(\rho)=\int_{B_\rho}|\nabla u|^2dx+\mathcal{H}^{d-1}(J_u\cap\overline{B}_\rho).$$

Lemma 3.6 *There exist constants $\rho_0=\rho_0(d,\alpha,c_\alpha,\Lambda)$ and δ,ε,β depending on d only, such that $\forall x\in\Omega$, $\forall 0<\rho<\rho_0$ with $B_\rho(x)\subseteq\Omega$ at least one of the following assertions holds:*

a) $\mathcal{H}^{d-2}(J_{u_\rho}\cap S^{d-1})\geq\delta$;

b) $\mathcal{H}^{d-2}(J_{u_\rho}\cap S^{d-1})<\delta$ *and* $\mathcal{E}(\rho)<\frac{\rho^{d-1}}{d-1}\int_{S^{d-1}}|\nabla_\tau u_\rho|^2dx+\beta\Lambda\rho^{d-1}\mathcal{H}^{d-2}(J_{u_\rho}\cap S^{d-1})+c_\alpha\rho^{d-1+\alpha}$, *and*

$$\frac{1-\varepsilon}{d-1}<\frac{\mathcal{E}(\rho)}{\rho^{d-1}\int_{S^{d-1}}|\nabla_\tau u_\rho|^2dx}<\frac{1+\varepsilon}{d-1},$$

and $\exists a_\rho\in\mathbb{R}$, $\exists\xi_\rho\in\mathbb{R}^d$ *such that*

$$\mathcal{H}^{d-1}(\{x\in S^{d-1}:|u_\rho(x)-a_\rho-\xi_\rho\cdot x|>\delta|\xi_\rho|\})<\delta \;\; and \;\; 2|\xi_\rho|^2\geq \fint_{S^{d-1}}|\nabla_\tau u_\rho|^2.$$

c) $\mathcal{E}(\rho)<\frac{1-\varepsilon}{d-1}\rho^{d-1}\int_{S^{d-1}}|\nabla_\tau u_\rho|^2dx+\beta\Lambda\rho^{d-1}\mathcal{H}^{d-2}(J_{u_\rho}\cap S^{d-1})$

Proof Notice that by almost quasi minimality of u, condition a) implies condition c) for some suitable constant β. Without restricting the generality, we assume $x=0$.
Step 1. From the minimality of u, we get for every admissible $\rho\leq 1$ and for a suitable constant K_1

$$\mathcal{E}(\rho)\leq\Lambda\rho^{d-1}d\omega_d+c_\alpha\rho^{d-1+\alpha}\leq K_1\rho^{d-1}. \tag{22}$$

For some constant $\tilde{c}>d-1$, if

$$\int_{S^{d-1}}|\nabla_\tau u_\rho|^2dx\geq\tilde{c}K_1, \tag{23}$$

then

$$\mathcal{E}(\rho)\leq\frac{1}{\tilde{c}}\rho^{d-1}\int_{S^{d-1}}|\nabla_\tau u_\rho|^2dx.$$

Introducing ε, given by

$$\frac{1}{\tilde{c}}=\frac{1-\varepsilon}{d-1},$$

assertion c) holds.
Step 2. We assume the contrary of (23), i.e. $\int_{S^{d-1}}|\nabla_\tau u_\rho|^2dx<\tilde{c}K_1$. For some constants c_1,c_2, we introduce the following auxiliary problem on S^{d-1}:

$$\min_{w\in SBV(S^{d-1})}\frac{c_1}{\Lambda}\int_{S^{d-1}}|\nabla_\tau w|^2dx+c_2\mathcal{H}^{d-1}(\{w\neq u_\rho\}\cap S^{d-1})+\mathcal{H}^{d-2}(J_w\cap S^{d-1}). \tag{24}$$

Following Lemma 2.3, there exists a constant c such that if

$$\frac{c_1}{\Lambda}\int_{S^{d-1}}|\nabla_\tau u_\rho|^2dx+\mathcal{H}^{d-2}(J_{u_\rho}\cap S^{d-1})<\frac{c}{(\frac{c_2}{d-1})^{d-2}} \tag{25}$$

then problem (24) has a solution w such that $w\in H^1(S^{d-1}))$, so $\mathcal{H}^{d-2}(J_w\cap S^{d-1})=0$.

We introduce

$$\delta<\frac{1}{2}\frac{c}{(\frac{c_2}{d-1})^{d-2}},$$

and fix c_1 such that

$$\frac{c_1}{\Lambda}\tilde{c}K_1<\frac{1}{2}\frac{c}{(\frac{c_2}{d-1})^{d-2}} \tag{26}$$

Consequently, either assertion a) holds with

$$\mathcal{H}^{d-2}(J_{u_\rho}\cap S^{d-1})\geq\delta,$$

or condition (25) is satisfied.

Step 3. Assume in the sequel that $\mathcal{H}^{d-2}(J_{u_\rho}\cap S^{d-1})<\delta$ and condition (25) is satisfied. We know that

$$\frac{c_1}{\Lambda}\int_{S^{d-1}}|\nabla_\tau w|^2dx+c_2\mathcal{H}^{d-1}(\{w\neq u_\rho\}\cap S^{d-1})\leq\frac{c_1}{\Lambda}\int_{S^{d-1}}|\nabla_\tau u_\rho|^2dx+\mathcal{H}^{d-2}(J_{u_\rho}\cap S^{d-1}).$$

Using Lemma 3.5 and multiplying by $\frac{\Lambda}{c_1}$ we get:

$$(d-1)\int_{B_1}|\nabla w|^2dx+\alpha_d\int_{S^{d-1}}|\nabla_\tau(w-\overline{w})|^2dx+\frac{\Lambda c_2}{c_1}\mathcal{H}^{d-1}(\{w\neq u_\rho\}\cap S^{d-1}) \tag{27}$$

$$\leq\int_{S^{d-1}}|\nabla_\tau u_\rho|^2dx+\frac{\Lambda}{c_1}\mathcal{H}^{d-2}(J_{u_\rho}\cap S^{d-1}). \tag{28}$$

We introduce some $\varepsilon>0$ (which will be fixed later) and assume that c) does not hold for this ε. Then

$$\mathcal{E}(\rho)\geq\frac{1-\varepsilon}{d-1}\rho^{d-1}\int_{S^{d-1}}|\nabla_\tau u_\rho|^2dx+\beta\Lambda\rho^{d-1}\mathcal{H}^{d-2}(J_{u_\rho}\cap S^{d-1}). \tag{29}$$

We fix $\beta=\frac{1}{c_1(d-1)}$. Then, by simple computation and using (27)

$$(d-1)\mathcal{E}(\rho)+\varepsilon\rho^{d-1}\int_{S^{d-1}}|\nabla_\tau u_\rho|^2dx\geq\rho^{d-1}\int_{S^{d-1}}|\nabla_\tau u_\rho|^2dx+\frac{\Lambda}{c_1}\rho^{d-1}\mathcal{H}^{d-2}(J_{u_\rho}\cap S^{d-1})$$

$$\geq\rho^{d-1}\Big[(d-1)\int_{B_1}|\nabla w|^2dx+\alpha_d\int_{S^{d-1}}|\nabla_\tau(w-\overline{w})|^2dx+\frac{\Lambda c_2}{c_1}\mathcal{H}^{d-1}(\{w\neq u_\rho\}\cap S^{d-1})\Big].$$

We chose $c_2=2c_1(d-1)$ and use the almost quasi minimality of u. Consequently,

$$(d-1)\mathcal{E}(\rho)+\varepsilon\rho^{d-1}\int_{S^{d-1}}|\nabla_\tau u_\rho|^2dx$$

$$\geq(d-1)\mathcal{E}(\rho)+\alpha_d\rho^{d-1}\int_{S^{d-1}}|\nabla_\tau(w-\overline{w})|^2dx+(d-1)\rho^{d-1}\Lambda\mathcal{H}^{d-1}(\{w\neq u_\rho\}\cap S^{d-1})-(d-1)c_\alpha\rho^{d-1+\alpha},$$

thus

$$\varepsilon\rho^{d-1}\int_{S^{d-1}}|\nabla_\tau u_\rho|^2dx+(d-1)c_\alpha\rho^{d-1+\alpha} \tag{30}$$

$$\geq \alpha_d\rho^{d-1}\int_{S^{d-1}}|\nabla_\tau(w-\overline{w})|^2dx+(d-1)\rho^{d-1}\Lambda\mathcal{H}^{d-1}(\{w\neq u_\rho\}\cap S^{d-1}) \tag{31}$$

We notice in the same time (we successively apply the quasi-minimality, inequality (6) and the auxiliary problem)

$$\mathcal{E}(\rho)\leq \rho^{d-1}\int_{B_1}|\nabla w|^2dx+\Lambda\rho^{d-1}\mathcal{H}^{d-1}(\{w\neq u_\rho\}\cap S^{d-1})+c_\alpha\rho^{d-1+\alpha}$$

$$\leq \frac{\rho^{d-1}}{d-1}\Big[\int_{S^{d-1}}|\nabla_\tau w|^2dx+\Lambda(d-1)\mathcal{H}^{d-1}(\{w\neq u_\rho\}\cap S^{d-1})\Big]+c_\alpha\rho^{d-1+\alpha}$$

$$\leq \frac{\rho^{d-1}}{d-1}\Big[\int_{S^{d-1}}|\nabla_\tau u_\rho|^2dx+\frac{\Lambda}{c_1}\mathcal{H}^{d-2}(J_{u_\rho}\cap S^{d-1})\Big]+c_\alpha\rho^{d-1+\alpha}.$$

Finally

$$\mathcal{E}(\rho)\leq \frac{\rho^{d-1}}{d-1}\Big[\int_{S^{d-1}}|\nabla_\tau u_\rho|^2dx+\frac{\Lambda}{c_1}\mathcal{H}^{d-2}(J_{u_\rho}\cap S^{d-1})\Big]+c_\alpha\rho^{d-1+\alpha}. \tag{32}$$

Assume that $\rho_0\leq 1$ is such that

$$\varepsilon\tilde{c}\Lambda d\omega_d\geq (d-1)c_\alpha\rho_0^\alpha.$$

Then, from (30)

$$2\varepsilon\tilde{c}\Lambda d\omega_d\geq \alpha_d\int_{S^{d-1}}|\nabla_\tau(w-\overline{w})|^2dx+\Lambda\mathcal{H}^{d-1}(\{w\neq u_\rho\}\cap S^{d-1}).$$

For ε small enough, using the Poincaré inequality on S^{d-1} we get

$$\mathcal{H}^{d-1}(\{x\in S^{d-1}:|u_\rho(x)-\overline{w}|>\delta|\xi_\rho|\})<\delta.$$

From (29)

$$\mathcal{E}(\rho)\geq \frac{1-\varepsilon}{d-1}\rho^{d-1}\int_{S^{d-1}}|\nabla_\tau u_\rho|^2dx.$$

Since $\mathcal{H}^{d-2}(J_{u_\rho}\cap S^{d-1})<\delta$ for a suitable ε' we get

$$\mathcal{E}(\rho)\leq \frac{1+\varepsilon'}{d-1}\rho^{d-1}\int_{S^{d-1}}|\nabla_\tau u_\rho|^2dx.$$

Assume that

$$\int_{S^{d-1}}|\nabla_\tau w|^2dx\leq (1-\varepsilon')\int_{S^{d-1}}|\nabla\tau u_\rho|^2dx.$$

Then

$$\int_{B_1}|\nabla w|^2dx\leq \frac{1-\varepsilon}{d-1}\int_{S^{d-1}}|\nabla_\tau u_\rho|^2dx.$$

Using quasi-minimality and the auxiliary problem, from (30) one reduces this case to c).

Assume now that

$$\int_{S^{d-1}}|\nabla_\tau w|^2dx>(1-\varepsilon)\int_{S^{d-1}}|\nabla\tau u_\rho|^2dx$$

and

$$\int_{S^{d-1}} |\nabla_\tau (w-\overline{w})|^2 dx > \varepsilon \int_{S^{d-1}} |\nabla_\tau w|^2 dx.$$

Relying on (21) we arrive again in situation c).

If

$$\int_{S^{d-1}} |\nabla_\tau w|^2 dx > (1-\varepsilon) \int_{S^{d-1}} |\nabla \tau u_\rho|^2 dx$$

and

$$\int_{S^{d-1}} |\nabla_\tau (w-\overline{w})|^2 dx \leq \varepsilon \int_{S^{d-1}} |\nabla_\tau w|^2 dx,$$

then by direct computation

$$(1-\varepsilon) \int_{S^{d-1}} |\nabla_\tau w|^2 dx \leq \int_{S^{d-1}} |\nabla_\tau \overline{w}|^2 dx,$$

so that

$$(1-\varepsilon)^2 \int_{S^{d-1}} |\nabla_\tau u_\rho|^2 dx \leq \int_{S^{d-1}} |\nabla_\tau \overline{w}|^2 dx = |S^{d-1}||\xi_0|^2.$$

□

Lemma 3.7 (Decay Lemma) *There exists constants $c = c_d, \varepsilon = \varepsilon_d, \delta = \delta_d$ and $\rho_0 = \rho_0(d, \alpha, c_\alpha, \Lambda)$ such that for every jump point x and $\min\{\rho_0, \partial_{\Omega^c}(x)\} > \rho_2 > \rho_1 > 0$, one of the following situations holds:*

$$\frac{\rho_2^{-\tilde{d}}\, \mathcal{E}(\rho_2)}{\rho_1^{-\tilde{d}}\, \mathcal{E}(\rho_1)} \geq \frac{1}{c} \exp\Big(- c\Lambda \int_{\ln \rho_1}^{\ln \rho_2} \frac{e^\sigma \mathcal{H}^{d-2}(e^\sigma S^{d-1} \cap J_u)}{\mathcal{E}(e^\sigma)} d\sigma \Big), \tag{33}$$

where $\tilde{d} = \frac{d-1}{1-\varepsilon}$, or

$$\sum_{\rho_1 < \rho < \rho_2} \mathcal{H}^{d-1}(S^{d-1} \cap J_{u_\rho}) > \delta. \tag{34}$$

Proof As noticed, case a) in Lemma 3.6 can be seen as a particular case of c), possibly considering a slightly greater β. As a consequence, only issues b) and c) shall be considered in the sequel after a possible modification of β.

We shall work for simplicity in logarithmic co-ordinates by setting $\sigma = \ln \rho$. We get after rescaling and denoting χ_b, χ_c the characteristic functions associated to cases b) and c) in the previous lemma

$$\partial_\sigma \ln \mathcal{E}(e^\sigma) \geq (d-1)\chi_b(\sigma) + \frac{d-1}{1-\varepsilon_d}\chi_c(\sigma) + \frac{\rho}{\mathcal{E}(e^\sigma)} \int_{\rho S^{d-1}} |\nabla u \cdot n|^2 dx - \beta\Lambda \frac{\rho}{\mathcal{E}(e^\sigma)} \mathcal{H}^{d-2}(J_u \cap \rho S^{d-1}). \tag{35}$$

If $\chi_b(\sigma_0) = 1$, let $a_0 = a_{\rho_0} \in \mathbb{R}$ and $\xi_0 = \xi = \xi_{\rho_0} \in \mathbb{R}^d$ be given by point b). We introduce the set

$$M_0 = \{x \in S^{d-1} : |u_\rho - a_0 - x \cdot \xi_0| \leq \delta |\xi_0|\}.$$

One can choose the set M_0 to be symmetric with respect to every axes hyperplane, so that

$$\forall i \neq j, \quad \int_{M_0} x_i x_j d\mathcal{H}^{d-1} = 0,$$

and $\mathcal{H}^{d-1}(S^{d-1}\setminus M_0)\le\delta$. We have

$$|S^{d-1}|\delta^2|\xi_0|^2\ge\int_{M_0}|u_{\rho_0}-a_0-x\cdot\xi_0|^2dx=\int_{M_0}|u_{\rho_0}-a_0|^2dx+\int_{M_0}|x\cdot\xi_0|^2dx-2\int_{M_0}(u_{\rho_0}-a_0)(x\cdot\xi_0)dx.$$

Using the symmetry property of M_0, we get that

$$\int_{M_0}|x\cdot\xi_0|^2dx=C_d|\xi_0|^2.$$

Provided δ is small enough, we get

$$\int_{M_0}(u_{\rho_0}-a_0)(x\cdot\xi_0)dx\ge C_d'|\xi_0|^2. \tag{36}$$

Let us introduce

$$f(\rho)=\rho^{d-1}\frac{\Big(\int_{M_0}|u_\rho-a_0|\ |x\cdot\xi_0|dx\Big)^2}{\Big(\int_{M_0}|x\cdot\xi_0|^2\Big)^2}.$$

From (36) and the density of the energy, if condition b) holds, we have for a suitable constant $c>0$

$$f(\rho)\ge c^2\mathcal{E}(\rho)\ \text{ if }\ \chi_b(\rho)=1. \tag{37}$$

If the second assertion of the Lemma does not hold, so that we can chose M_0, possibly decreasing its measure, such that

$$M_0\cap\bigcup_{\rho,\mathcal{H}^{d-1}(J_{u_\rho}\cap S^{d-1})\neq 0}\{J_{u_\rho}\cap S^{d-1}\}=\emptyset,$$

then for a suitable constant c'

$$\partial_\sigma[\ln f(e^\sigma)\vee(\ln(E(e^\sigma)-c')]\chi_b=\partial_\sigma\ln f(e^\sigma)\chi_b,$$

and

$$|\partial_\sigma\ln f(e^\sigma)-(d-2)|^2\le c''\rho\frac{\int_{\rho M_0}|\nabla u\cdot n|^2dx}{f(e^\sigma)}. \tag{38}$$

This inequality relies on the derivative of the boundary integral in the expression of f with respect to the parameter ρ.

In view of (38), inequality (35) becomes

$$\partial_\sigma\ln\mathcal{E}(e^\sigma)\ge(d-1)\chi_b(\sigma)+\frac{d-1}{1-\varepsilon_d}\chi_c(\sigma)+\frac{f(e^\sigma)}{c''\mathcal{E}(e^\sigma)}|\partial_\sigma\ln f(e^\sigma)-(d-2)|^2-\beta\Lambda\frac{\rho}{\mathcal{E}(e^\sigma)}\mathcal{H}^{d-2}(J_u\cap\rho S^{d-1}).$$

Let c_1 be a sufficiently small constant and $\gamma>0$ a constant that will be fixed later. Then

$$\partial_\sigma\ln\Big(\mathcal{E}(e^\sigma)[(\Big(\frac{\mathcal{E}(e^\sigma)}{f(e^\sigma)}\Big)^\gamma\vee\frac{1}{c_1})\wedge c_1]\Big)=\partial_\sigma\ln\mathcal{E}(e^\sigma),$$

if $\Big(\frac{\mathcal{E}(e^\sigma)}{f(e^\sigma)}\Big)^\gamma\notin[\frac{1}{c_1},c_1]$, so that we are not in situation b). Being in c) we get

$$\partial_\sigma\ln\mathcal{E}(e^\sigma)\ge\frac{d-1}{1-\varepsilon_d}-\beta\Lambda\frac{\rho}{\mathcal{E}(e^\sigma)}\mathcal{H}^{d-2}(J_u\cap\rho S^{d-1})$$

Otherwise,

$$\partial_\sigma \ln\Big(\mathcal{E}(e^\sigma)[(\Big(\frac{\mathcal{E}(e^\sigma)}{f(e^\sigma)}\Big)^\gamma \vee \frac{1}{c_1}) \wedge c_1]\Big) = \partial_\sigma \ln(\mathcal{E}(e^\sigma)^{\gamma+1}) - \partial_\sigma \ln(f(e^\sigma)^\gamma) \geq$$

$$(1+\gamma)(d-1) + \frac{1+\gamma}{c''c_1}|\partial_\sigma \ln f(e^\sigma) - (d-2)|^2 - (1+\gamma)\beta\Lambda\frac{\rho}{\mathcal{E}(e^\sigma)}\mathcal{H}^{d-2}(J_u \cap \rho S^{d-1}) - \gamma\partial_\sigma \ln f(e^\sigma).$$

Since

$$\frac{1+\gamma}{c''c_1}|\partial_\sigma \ln f(e^\sigma) - (d-2)|^2 - \gamma(\partial_\sigma \ln f(e^\sigma) - (d-2)) \geq -\frac{1}{c_4}\gamma^2,$$

for a constant c_4 small enough and independent on γ, we get

$$\partial_\sigma \ln\Big(\mathcal{E}(e^\sigma)[(\Big(\frac{\mathcal{E}(e^\sigma)}{f(e^\sigma)}\Big)^\gamma \vee \frac{1}{c_1}) \wedge c_1]\Big)$$

$$\geq (1+\gamma)(d-1) - \gamma(d-2+\frac{\gamma}{c_4} - (1+\gamma)\beta\Lambda\frac{\rho}{\mathcal{E}(e^\sigma)}\mathcal{H}^{d-2}(J_u \cap \rho S^{d-1}) - \gamma\partial_\sigma \ln f(e^\sigma).$$

For γ small enough, we get for some $\tilde{d} > d$

$$\partial_\sigma \ln\Big(\mathcal{E}(e^\sigma)[(\Big(\frac{\mathcal{E}(e^\sigma)}{f(e^\sigma)}\Big)^\gamma \vee \frac{1}{c_1}) \wedge c_1]\Big) \geq \tilde{d} - 1 - (1+\gamma)\beta\Lambda\frac{\rho}{\mathcal{E}(e^\sigma)}\mathcal{H}^{d-2}(J_u \cap \rho S^{d-1}).$$

Summing between $\rho_1 < \rho_2$, we conclude the proof. □

Corollary 3.8 *There exists $\tilde{d} = \tilde{d}_d > d-1$, $c = c(d,\Lambda)$ and $\rho_0 = \rho_0(d,\alpha,c_\alpha,\Lambda)$ such that $\forall x \in J_u$, $\forall 0 < \rho < \rho_0$, $B_{\tau\rho}(x) \subseteq \Omega$ and for every $\tau > 1$ we have either*

$$c\mathcal{H}^{d-1}((B_{\tau\rho}(x) \setminus \overline{B}_\rho(x)) \cap J_u) \geq \mathcal{E}(\rho), \tag{39}$$

or

$$c\mathcal{E}(\tau\rho) \geq \tau^{\tilde{d}}\mathcal{E}(\rho). \tag{40}$$

Proof For some $\tau > 1$, we consider $0 < \rho_1 < \rho_2$ such that $\rho_2 = \tau\rho_1$. Assume that assertion (34) in Lemma 3.7 holds. Then

$$\delta < \sum_{\rho_1<\rho<\rho_2} \mathcal{H}^{d-1}(S^{d-1} \cap J_{u_\rho}) = \sum_{\rho_1<\rho<\rho_2} \frac{\mathcal{H}^{d-1}(S_\rho \cap J_u)}{\rho^{d-1}} \leq \frac{\mathcal{H}^{d-1}((B_{\rho_2} \setminus \overline{B}_{\rho_1}) \cap J_u)}{\rho_1^{d-1}}.$$

From the density estimate of the energy (41), relation (39) holds.

Assume that (39) does not hold. Consequently, assertion (33) in Lemma 3.7 holds so we get

$$\frac{\rho_2^{-\tilde{d}}}{\rho_1^{-\tilde{d}}}\frac{\mathcal{E}(\rho_2)}{\mathcal{E}(\rho_1)} \geq C_1,$$

where C_1 depends on d and Λ. Thus (40) holds. □

We conclude with the following corollary, as a consequence of the Corollaries 3.3 and 3.8.

Corollary 3.9 *There exist constants $c = c(d,\Lambda)$, $\rho_0 = \rho_0(d,\alpha,c_\alpha,\Lambda)$ such that for every almost quasi-minimizer u, for every $x \in J_u$ and for every $0 < \rho < \rho_0$ such that $B_\rho(x) \subseteq \Omega$ we have*

$$\frac{1}{c}\rho^{d-1} \geq \mathcal{H}^{d-1}(J_u \cap B_\rho) \geq c\rho^{d-1}. \tag{41}$$

4 A free boundary problem with Robin conditions

Let D be a smooth, bounded open subset of $\mathbb{R}^d$, $d \geq 2$ such that $\overline{B}_1 \subseteq D$. Let $0 < \alpha_1 < \alpha_2$, $0 < \beta$ and $g \in H^1(B_1)$ such that a.e. $x \in B$ $\alpha_1 \leq g(x) \leq \alpha_2$. We consider the following free boundary problem, the unknown being a (sufficiently smooth) open set Ω and a function $u \in H^1(\Omega)$:

$$\min_{B_1 \subseteq \Omega \subseteq D} \min_{u \in H^1(\Omega), u=g \text{ on } B_1} \int_\Omega |\nabla u|^2 dx + \beta \int_{\partial\Omega} u^2 d\mathcal{H}^{d-1} + |\{u > 0\}|. \tag{42}$$

Formally, if $\beta = +\infty$, this is a free boundary problem of Bernoulli type (see for instance the seminal paper of Alt-Caffarelli [1]). If β is finite, (42) becomes a free discontinuity problem, with Robin boundary conditions on the unknown jump set.

Indeed, if Ω is fixed and smooth enough, the boundary trace term in the formulation above is well defined and the minimizer $u \in H^1(\Omega)$ solves in $\Omega \setminus \overline{B}_1$ the following elliptic equation with Robin conditions on the free boundary $\partial\Omega$:

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \setminus \overline{B}_1, \\ \frac{\partial u}{\partial n} + \beta u = 0 & \text{on } \partial\Omega \setminus \partial B_1 \\ u = g & \text{on } \partial B_1. \end{cases} \tag{43}$$

In order to deal with problem (42), we write the relaxed formin the SBV framework (see [3])

$$\min_{u \in SBV^{\frac{1}{2}}(D), u=g \text{ on } B_1} \int_D |\nabla u|^2 dx + \beta \int_{J_u} [(u^+)^2 + (u^-)^2] d\mathcal{H}^{d-1} + |\{u > 0\}|. \tag{44}$$

Above, $SBV^{\frac{1}{2}}(D)$ stands for all nonnegative measurable functions u such that $u^2 \in SBV(D)$. The existence of a solution of the relaxed problem (44) is a direct consequence of [3, Theorem 3.3], once we have remarked that the minimization process has to be carried in the class of functions satisfying a.e. $u(x) \leq \alpha_2$. In order to apply the regularity result for almost-quasi minimizers, the key point is to prove that any solution u is minorated a.e. on its positivity region by a positive constant.

Notice, that if the set $\{u > 0\}$ were a priori smooth, the fact that u has a strictly positive lower bound on $\{u > 0\}$ is a consequence of the Hopf's lemma and of the pointwise equality $\frac{\partial u}{\partial n}(x) + \beta u(x) = 0$ on $\partial\Omega$, given by the Robin boundary condition.

Theorem 4.1 *For every solution u of the free discontinuity problem* (44)*, there exists $\alpha > 0$ such that*

$$u(x) \geq \alpha, \quad a.e. \ x \in \{u > 0\}.$$

In particular $u \in SBV(D)$ and $\mathcal{H}^{d-1}(J_u) < +\infty$.

Proof We introduce the following notations:

- $f(s) := \mathcal{H}^{d-1}(\partial^*\{u > s\} \setminus J_u)$
- $\Omega(\varepsilon, \delta) := \{\delta < u < \varepsilon\}$, $\Omega(\varepsilon) = \Omega(\varepsilon, 0)$
- $E(\varepsilon, \delta) = \displaystyle\int_{\Omega(\varepsilon,\delta)} |\nabla u|^2 dx$, $E(\varepsilon) = E(\varepsilon, 0)$
- $\gamma(\varepsilon, \delta) = \mathcal{H}^{d-1}(\partial^*\Omega(\varepsilon, \delta) \cap J_u)$

For $0 < \delta < \varepsilon < \alpha_1$, we get estimates for $f(s), \Omega(\varepsilon,\delta), E(\varepsilon,\delta), \gamma(\varepsilon,\delta)$ relying on the co-area formula, the isoperimetric inequality and the optimality of u. All the estimates hold for almost every ε and δ. For the simplicity of the exposition, up to the end of the proof we omit the words *almost everywhere* and work only with ε and δ in the complement of a zero measure set.

We notice first that $u \cdot 1_{\Omega(\varepsilon,\delta)} \in SBV(D)$. By the co-area formula we have

$$\int_{\Omega(\varepsilon,\delta)} |\nabla u| dx = \int_\delta^\varepsilon \mathcal{H}^{d-1}(\partial^*\{u > s\} \setminus J_u) ds,$$

and by the Cauchy inequality

$$\int_\delta^\varepsilon f(s) ds = \int_{\Omega(\varepsilon,\delta)} |\nabla u| dx \le |\Omega(\varepsilon,\delta)|^{\frac{1}{2}} E(\varepsilon,\delta)^{\frac{1}{2}}. \tag{45}$$

Denoting by γ_d the isoperimetric constant in $\mathbb{R}^d$, from the isoperimetric inequality we get

$$|\Omega(\varepsilon,\delta)|^{\frac{d-1}{d}} \le \gamma_d\big(f(\varepsilon) + f(\delta) + \mathcal{H}^{d-1}(\partial^*\Omega(\varepsilon,\delta) \cap J_u)\big),$$

so

$$|\Omega(\varepsilon,\delta)|^{\frac{d-1}{d}} \le \gamma_d\big(f(\varepsilon) + f(\delta) + \gamma(\varepsilon,\delta)\big). \tag{46}$$

For almost every $\varepsilon > \delta > 0$, the optimality condition written for u and the test function $u \cdot 1_{\{u \ge \varepsilon\}}$ gives

$$E(\varepsilon) + \delta^2 \gamma(\varepsilon,\delta) + |\Omega(\varepsilon)| \le \varepsilon^2 f(\varepsilon). \tag{47}$$

The main idea of the proof relies on a sort of reverse asymptotic method and can be summarized as follows: prove that if for some interval $I_1 \subseteq (0,\alpha_1)$ the sum $\int_{I_1} f(s)ds$ is small enough, then there exists a smaller non trivial interval $I_2 \subseteq I_1$ such that $\int_{I_2} f(s)ds = 0$. The optimality of u will directly give that $u(x) \ge \sup I_2$ a.e. $\{u > 0\}$.

More precisely, let $\eta > 0$. We construct two sequences $(\varepsilon_n)_n$ and $(\delta_n)_n$ such that

$$\varepsilon_0 = \eta, \delta_0 = \frac{\eta}{2}, \varepsilon_i = (1-\alpha_i)\varepsilon_{i-1}, \delta_i = (1+\alpha_i)\delta_{i-1},$$

where $\alpha_i > 0$ is chosen such that $\varepsilon_i \to \varepsilon_\infty, \delta_i \to \delta_\infty$, with $\delta_\infty < \varepsilon_\infty$. We shall prove that for η small enough, there exists a suitable choice of α_i such that

$$\int_{\delta_\infty}^{\varepsilon_\infty} f(s) ds = 0. \tag{48}$$

For simplicity, we denote

$$a_i = \int_{\delta_i}^{\varepsilon_i} f(s) ds \;\text{ and }\; b_i = |\Omega(\varepsilon_i,\delta_i)|.$$

We start with the following estimates: for $\eta \ge \varepsilon > \delta \ge \frac{\eta}{2}$, from (47) we get

$$\delta^2 \gamma(\varepsilon,\delta) \le \varepsilon^2 f(\varepsilon) \text{ so } \gamma(\varepsilon,\delta) \le 4 f(\varepsilon).$$

From (46) we get

$$|\Omega(\varepsilon,\delta)|^{\frac{d-1}{d}} \le \gamma_d(5f(\varepsilon) + f(\delta)) \le 5\gamma_d(f(\varepsilon) + f(\delta)).$$

Summing between $[\varepsilon_i, \varepsilon_{i-1}]$ and taking into account the range of ε_i, δ_i we get

$$|\Omega(\varepsilon_i,\delta_i)|^{\frac{d-1}{d}}\alpha_i\varepsilon_i \leq 5\gamma_d\Big[\int_{\varepsilon_i}^{\varepsilon_{i-1}} f(s)ds + f(\delta)\alpha_i\varepsilon_i\Big],$$

and further summing on $[\delta_{i-1}, \delta_i]$

$$|\Omega(\varepsilon_i,\delta_i)|^{\frac{d-1}{d}}\alpha_i^2\varepsilon_i\delta_i \leq 5\gamma_d\Big[\alpha_i\delta_i\int_{\varepsilon_i}^{\varepsilon_{i-1}} f(s)ds + \alpha_i\varepsilon_i\int_{\delta_{i-1}}^{\delta_i} f(s)ds\Big] \leq 5\gamma_d\varepsilon\alpha_i(a_{i-1}-a_i).$$

Consequently

$$b_i^{\frac{d-1}{d}} \leq 20\gamma_d\frac{a_{i-1}-a_i}{\eta\alpha_i}. \tag{49}$$

On the other hand, from (45) and (47) we get

$$\int_\delta^\varepsilon f(s)ds \leq |\Omega(\varepsilon,\delta)|^{\frac{1}{2}}\varepsilon f(\varepsilon)^{\frac{1}{2}} \leq |\Omega(\varepsilon,\delta)|^{\frac{1}{2d}}|\Omega(\varepsilon,\delta)|^{\frac{d-1}{2d}}\varepsilon f(\varepsilon)^{\frac{1}{2}}$$

$$\leq |\Omega(\varepsilon,\delta)|^{\frac{1}{2d}}\big[5\gamma_d(f(\varepsilon)+f(\delta))\big]^{\frac{1}{2}}\varepsilon f(\varepsilon)^{\frac{1}{2}} \leq |\Omega(\varepsilon,\delta)|^{\frac{1}{2d}}\varepsilon(5\gamma_d)^{\frac{1}{2}}(f(\varepsilon)+f(\delta)).$$

Finally, we have

$$\int_\delta^\varepsilon f(s)ds \leq (5\gamma_d)^{\frac{1}{2}}|\Omega(\varepsilon,\delta)|^{\frac{1}{2d}}\varepsilon(f(\varepsilon)+f(\delta)). \tag{50}$$

We sum again (50) between $[\varepsilon_i, \varepsilon_{i-1}]$ and $[\delta_{i-1}, \delta_i]$ with respect to ε and δ, respectively, and using the monotonicity of a_i, b_i and the range of ε and δ

$$\frac{\alpha_i\eta}{4}a_i \leq (5\gamma_d)^{\frac{1}{2}}b_{i-1}^{\frac{1}{2d}}\eta(a_{i-1}-a_i),$$

so

$$a_i \leq \frac{4(5\gamma_d)^{\frac{1}{2}}b_{i-1}^{\frac{1}{2d}}}{\alpha_i}(a_{i-1}-a_i). \tag{51}$$

Assume that the choice $\alpha_i = 4(5\gamma_d)^{\frac{1}{2}}b_{i-1}^{\frac{1}{2d}}$ is valid, in the sense that $\varepsilon_\infty > \delta_\infty$. Then we get that

$$a_i \leq a_{i-1} - a_i \implies a_i \leq \frac{a_0}{2^i}, \tag{52}$$

which leads to the conclusion (48).

It remains to prove that the choice of α_i is valid, i.e. $\varepsilon_\infty > \delta_\infty$. This will be ensured by a suitable choice of η. In an equivalent way, this can be re-written as

$$\prod_{i=1}^\infty(1-\alpha_i) > \frac{1}{2}\prod_{i=1}^\infty(1+\alpha_i) \text{ or } 2 > \prod_{i=1}^\infty\big(1+\frac{2\alpha_i}{1-\alpha_i}\big).$$

If $\alpha_i \leq \frac{1}{3}$, it is enough to have

$$2 > exp\big(\sum_{i=1}^\infty(3\alpha_i)\big)$$

since

$$exp\big(\sum_{i=1}^\infty(3\alpha_i)\big) > \prod_{i=1}^\infty(1+3\alpha_i) > \prod_{i=1}^\infty\big(1+\frac{2\alpha_i}{1-\alpha_i}\big).$$

We need that

$$\sum_{i=1}^{\infty} \alpha_i < \frac{\ln 2}{3} \text{ or, equivalently, by the definition of } \alpha_i\text{: } \sum_{i=1}^{\infty} b_{i-1}^{\frac{1}{2d}} < \frac{\ln 2}{12(5\gamma_d)^{\frac{1}{2}}}.$$

From (49) and (52) we get

$$b_i^{\frac{d-1}{d}} \leq 20\gamma_d \frac{a_0}{2^{i-1}} \frac{1}{\eta} \frac{1}{4(5\gamma_d)^{\frac{1}{2}}} \frac{1}{b_{i-1}^{\frac{1}{2d}}},$$

so

$$b_i \leq (5\gamma_d)^{\frac{1}{2}} \frac{a_0}{\eta 2^{i-1}} \frac{b_i^{\frac{1}{d}}}{b_{i-1}^{\frac{1}{2d}}}.$$

But $b_i \leq b_{i-1}$ and if η is small enough, $b_i \leq 1$, so that we have

$$b_i \leq (5\gamma_d)^{\frac{1}{2}} \frac{a_0}{\eta 2^{i-1}}.$$

Consequently, we get

$$\sum_{i=1}^{\infty} \alpha_i \leq \frac{1}{4(5\gamma_d)^{\frac{1}{2}}} \sum_{i=1}^{\infty} (5\gamma_d)^{\frac{1}{4d}} \Big(\frac{a_0}{\eta}\Big)^{\frac{1}{2d}} \Big(\frac{1}{2^{i-2}}\Big)^{\frac{1}{2d}} = C_d \Big(\frac{a_0}{\eta}\Big)^{\frac{1}{2d}},$$

where C_d is a constant depending only on the dimension of the space.

It remains to prove that we can find some η such that

$$C_d \Big(\frac{a_0}{\eta}\Big)^{\frac{1}{2d}} < \frac{\ln 2}{3}.$$

From (47) we get

$$E(\varepsilon) + |\Omega(\varepsilon)| \leq \varepsilon^2 f(\varepsilon).$$

Summing from ε to 2ε and using the monotonicity of E and Ω, we get

$$E(\varepsilon) + |\Omega(\varepsilon)| \leq \frac{1}{\varepsilon} \int_{\varepsilon}^{2\varepsilon} s^2 f(s) ds \leq 4\varepsilon \int_{\varepsilon}^{2\varepsilon} f(s) ds \leq 4\varepsilon \big[E(2\varepsilon) + |\Omega(2\varepsilon)|\big].$$

The last inequality above comes from (45) on $[\varepsilon, 2\varepsilon]$ and the arithmetic/geometric inequality. So we get

$$E(\varepsilon) + |\Omega(\varepsilon)| \leq 4\varepsilon \big[E(2\varepsilon) + |\Omega(2\varepsilon)|\big].$$

In the same way

$$\int_{\frac{\varepsilon}{2}}^{\varepsilon} f(s) ds \leq \big[E(\varepsilon) + |\Omega(\varepsilon)|\big],$$

hence

$$\frac{\int_{\frac{\varepsilon}{2}}^{\varepsilon} f(s) ds}{\varepsilon} \leq 4\big[E(2\varepsilon) + |\Omega(2\varepsilon)|\big].$$

We choose η such that

$$4\big[E(2\eta) + |\Omega(2\eta)|\big] \leq \frac{1}{C_d^{2d}} \Big(\frac{\ln 2}{3}\Big)^d$$

so that

$$\frac{a_0}{\eta} = \frac{\displaystyle\int_{\frac{\eta}{2}}^{\eta} f(s)ds}{\eta} \leq \frac{1}{C_d^{2d}} \Big(\frac{\ln 2}{3}\Big)^d,$$

and the proof is complete. □

Theorem 4.2 *Every solution u of the free discontinuity problem* (44) *satisfies $\mathcal{H}^{d-1}(\overline{J}_u \setminus J_u) = 0$ and the density property* (41).

Proof After a suitable rescaling in space, u turns out to be an almost-quasi minimizer, as a consequence of its optimality and of the inequality proved in Theorem 4.1. The conclusion then follows is a direct consequence of Theorem 3.1 and Corollary 3.9. □

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