On the asymptotic limit of flows past a ribbed boundary

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1 Introduction

Several physical as well as numerical experiments have shown that a fluid flowing between two parallel plates can be significantly influenced by the shape of the boundary. In particular, the presence of microscopic asperities parallel to the flow (riblets) can considerably reduce the drag experienced by the fluid (see Chu and Karniadakis [7], Savill et al. [11]). The main objective of the present paper is to discuss mathematical aspects of this phenomenon.

For the sake of simplicity, we consider a flow confined to a domain $\Omega \subset R^3$, periodic with respect to the first two spatial coordinates $y = (x_1, x_2)$:

$$\Omega = \{(x_1, x_2, x_3) \mid (x_1, x_2) \in \mathcal{T}^2,\ 0 < x_3 < 1 + \Phi(x_1, x_2)\}, \tag{1.1}$$

where $\mathcal{T}^2 = \Big((0,1)|_{\{0,1\}} \Big)^2$ is a two-dimensional torus, and $\Phi : \mathcal{T}^2 \to R^1$ is a given Lipschitz function.

With $\mathbf{u}$ denoting the Eulerian fluid velocity, the standard impermeability boundary condition reads

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0, \tag{1.2}$$

where $\mathbf{n}$ denotes the outer normal vector. For viscous fluids, condition (1.1) is usually accompanied by the no-slip boundary condition

$$[\mathbf{u}]_\tau|_{\partial\Omega} = 0, \tag{1.3}$$

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where $[\mathbf{u}]_\tau$ denotes the tangential component of the vector field $\mathbf{u}$. Viscous fluids adhere completely to rigid walls.

There have been several attempts to show that condition (1.1) itself produces (1.2) in the asymptotic limit provided the boundary is covered by microscopic asperities, the size of which tends to zero (see Amirat et al. [1], [2], Casado-Díaz et al. [6], Richardson [10]). In [6], the authors consider a family of domains $\{\Omega_\varepsilon\}_{\varepsilon>0}$ given through (1.1), with $\Phi = \Phi_\varepsilon$,

$$\Phi_\varepsilon(y) = \varepsilon\Phi\Big(\frac{y}{\varepsilon}\Big),\ y \in \mathcal{T}^2,$$

where Φ is a given (smooth) periodic function. They show, under certain non-degeneracy hypotheses imposed on Φ, that for any $\{\mathbf{u}_\varepsilon\}_{\varepsilon>0}$, bounded in $W^{1,2}(\Omega_\varepsilon, R^3)$, and such that

$$\mathbf{u}_\varepsilon \to \mathbf{u} \text{ weakly in } W^{1,2}(\Omega_0; R^3),\ \Omega_0 = \mathcal{T}^2 \times (0,1),$$

the limit function $\mathbf{u}$ satisfies *both* (1.2) and (1.3) on the upper part of the boundary $\Gamma_0 = \{x_3 = 1\}$ whenever $\mathbf{u}_\varepsilon$ satisfy merely the impermeability condition (1.2) on Γ_ε,

$$\Gamma_\varepsilon = \{(x_1, x_2, x_3) \mid (x_1, x_2) \in \mathcal{T}^2,\ x_3 = 1 + \Phi_\varepsilon(x_1, x_2)\}. \tag{1.4}$$

As already pointed out, such a result holds on condition that Φ is non-degenerate, more specifically, both $\partial_{x_1}\Phi$ and $\partial_{x_2}\Phi$ are not identically zero on $\mathcal{T}^2$. It was observed in [5] that such a phenomenon is intimately related to the character of oscillations of the vector fields $\{\nabla_y\Phi_\varepsilon\}_{\varepsilon>0}$ associated to the directional fluctuations of the normal vectors to Γ_ε. A sufficient condition for (1.2) to imply (1.2), (1.3) in the asymptotic limit can be expressed in terms of a *Young measure* $\{\mathcal{R}_y\}_{y\in\mathcal{T}^2}$ associated to the family $\{\nabla_y\Phi_\varepsilon\}_{\varepsilon>0}$. More specifically, condition (1.2) gives rise to (1.3), together with (1.2), in the asymptotic limit provided the support of each measure $\mathcal{R}_y$ contains at least two linearly independent vectors in R^2 (see Theorem 4.1 in [5]). As intuitively expected, the phenomenon of rugosity is of purely local nature and has nothing to do with the periodicity of the function Φ. Several specific examples may be found in [5].

In the present paper, we focus on the situation when rugosity is "degenerate" in one direction. Very roughly indeed, this is the case when the support of the Young measure $\{\mathcal{R}_y\}_{y\in\mathcal{T}^2}$ associated to the family $\{\nabla_y\Phi_\varepsilon\}_{\varepsilon>0}$ is contained in a linear subspace of R^2. A simple example reads

$$\Phi_\varepsilon(y_1, y_2) = \varepsilon\Phi\Big(\frac{y_1}{\varepsilon}\Big),\ (y_1, y_2) \in \mathcal{T}^2, \tag{1.5}$$

where Φ is a periodic function on R. In this simple case, the authors in [6] claim (without) proof that the limit satisfies a weak "partial slip" boundary condition

$$u_1|_{\partial\Omega_0} = u_3|_{\partial\Omega_0} = 0. \tag{1.6}$$

Ribbed boundaries prevent the fluid from slipping in the direction of asperities while the motion in the orthogonal (tangent) direction is allowed with no constraint.

We consider the Navier-Stokes system

$$\mathrm{div}_x(\mathbf{u} \otimes \mathbf{u}) + \nabla_x p = \mathrm{div}_x\mathbb{S} + \mathbf{f}, \tag{1.7}$$

$$\mathrm{div}_x\mathbf{u} = 0, \tag{1.8}$$

where the viscous stress tensor $\mathbb{S}$ is given through Newton's rheological law

$$\mathbb{S} = \mu(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u}),\ \mu > 0. \tag{1.9}$$

System (1.7 - 1.9) represents a standard mathematical model describing the motion of a viscous, incompressible fluid driven by a volumic force $\mathbf{f} = \mathbf{f}(x)$ at equilibrium. Here $\mathbf{u} = \mathbf{u}(x)$ is the fluid velocity, and $p = p(x)$ is the pressure.

The fluid is confined to a spatial domain Ω_ε determined through (1.1), with $\Phi = \Phi_\varepsilon$. System (1.7 - 1.9) is supplemented with the no-slip boundary conditions at the "bottom" part of the boundary:

$$\mathbf{u}|_{\{x_3=0\}} = 0, \tag{1.10}$$

and Navier's slip boundary conditions

$$\mathbf{u}\cdot\mathbf{n}|_{\Gamma_\varepsilon} = 0,\ (\mathbb{S}\mathbf{n}) \times \mathbf{n}|_{\Gamma_\varepsilon} = 0, \tag{1.11}$$

where Γ_ε is given by (1.4).

Under the uni-directional distribution of asperities in the spirit of (1.5), our main goal is to identify the limit problem when $\varepsilon \to 0$. Clearly, the main issue is to specify the boundary conditions to be satisfied by the limit velocity field $\mathbf{u}$ on the upper boundary $\{x_3 = 1\}$.

To begin with, it is worth noting that we have to consider the weak (distributional) solutions of the problem. Indeed the boundaries of Ω_ε are only equi-Lipschitz, while the standard higher order elliptic estimates require (uniformly) smooth boundaries. Accordingly, we expect to get uniform (independent of $\varepsilon \to 0$) estimates only in the framework of the associated *energy space.*

The second remark concerns the boundary conditions specified in (1.11). The former condition is of Dirichlet type, that means, at the level of weak solutions, it must be incorporated in the function space the limit velocity field $\mathbf{u}$ is to be looked for. On the other hand, the latter condition in (1.11) is a "natural" one to be satisfied implicitely through the choice of appropriate test functions. Accordingly, identifying the limit boundary conditions to be satisfies by $\mathbf{u}$ is not enough in order to characterize the limit problem; we have find the class of appropriate test functions as well.

Finally, let us point out that the main difficulty when dealing with the weak solutions of problems in mathematical fluid dynamics lies in the fact that the function space the solution belongs to, and the space of appropriate test functions used in the variational formulation may be *different.*

The paper is organized as follows. In Section 2, we introduce the variational formulation of the problem and state the main result. Section 3 is devoted to various concepts of rugosity and their implications in the particular situation considered in the present paper. In Section 4, we recall some basic facts about solvability of the equation $\mathrm{div}_x \mathbf{v} = g$ to be used in order to obtain uniform estimates on the pressure. The proof of the main result is completed in Section 5. Possible generalizations are briefly discussed in Section 6.

2 Variational formulation and main results

Let $\Omega_\varepsilon \subset R^3$ belong to the class of domains specified in (1.1), with $\Phi_\varepsilon \geq 0$. We set

$$V(\Omega_\varepsilon; R^3) = \{\mathbf{v} \in W^{1,2}(\Omega_\varepsilon; R^3) \mid \mathbf{v}|_{\{x_3=0\}} = 0, \tag{2.1}$$

$$\int_{\Omega_\varepsilon} \mathbf{v} \cdot \nabla_x \varphi \ \mathrm{d}\mathbf{x} = 0 \text{ for all } \varphi \in W^{1,2}(\Omega_\varepsilon; R^3)\}.$$

Note that

$$V(\Omega_\varepsilon; R^3) = \Big\{ \mathbf{v} \in W^{1,2}(\Omega_\varepsilon; R^3) \ \Big| \ \mathrm{div}_x \mathbf{v} = 0 \text{ a.a. in } \Omega_\varepsilon,$$

$$\mathbf{v}|_{\{x_3=0\}} = 0, \ \mathbf{v} \cdot \mathbf{n}|_{\{x_3 = 1 + \Phi_\varepsilon(x_1,x_2)\}} = 0 \Big\}$$

provided Φ_ε is a Lipschitz function on $\mathcal{T}^2$.

Definition 2.1 *We shall say that $\mathbf{u}_\varepsilon$, p_ε is a weak solution of problem (1.7 - 1.11) on Ω_ε if*

$$\mathbf{u}_\varepsilon \in V(\Omega_\varepsilon; R^3), \ p_\varepsilon \in L^2(\Omega), \tag{2.2}$$

and the integral identity

$$\int_{\Omega_\varepsilon} \Big(\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon : \nabla_x \vec{\varphi} + p_\varepsilon \mathrm{div}_x \vec{\varphi} \Big) \ \mathrm{d}\mathbf{x} = \int_{\Omega_\varepsilon} \Big(\mu (\nabla_x \mathbf{u}_\varepsilon + \nabla_x^t \mathbf{u}_\varepsilon) : \nabla_x \vec{\varphi} - \mathbf{f}_\varepsilon \cdot \vec{\varphi} \Big) \ \mathrm{d}\mathbf{x} \tag{2.3}$$

holds for any test function $\vec{\varphi} \in W^{1,2}(\Omega_\varepsilon; R^3)$ such that

$$\vec{\varphi}|_{\{x_3=0\}} = 0, \ \vec{\varphi} \cdot \mathbf{n}|_{\{x_3 = 1 + \Phi(x_1,x_2)\}} = 0.$$

In order to identify the limit problem, we set

$$\Omega = \mathcal{T}^2 \times (0,1),$$

and

$$V_{\mathbf{w}} = \Big\{ \mathbf{v} \in W^{1,2}(\Omega; R^3) \ \Big| \ \mathrm{div}_x \mathbf{v} = 0 \text{ a.a. in } \Omega, \tag{2.4}$$

$$\mathbf{v}|_{\{x_3=0\}} = 0, \ \mathbf{v} \cdot \mathbf{n}|_{\{x_3=1\}}, \quad \text{and } \mathbf{v} \times \mathbf{w}|_{\{x_3=1\}} = 0 \Big\},$$

where $\mathbf{w} = (w_1, w_2, 0)$ is a given vector field tangent to $\partial\Omega$. Thus, in addition, the space $V_{\mathbf{w}}$ contains the vector fields parallel to $\mathbf{w}$ on the upper part of the boundary $\{x_3 = 1\}$.

Definition 2.2 *We shall say that $\mathbf{u}$, p is a weak solution of problem (1.7 - 1.10) on $\Omega = \mathcal{T}^2 \times (0,1)$, supplemented with a partial slip boundary condition in the direction $\mathbf{w}$, if*

$$\mathbf{u} \in V_{\mathbf{w}}(\Omega; R^3), \ p \in L^2(\Omega), \tag{2.5}$$

and the integral identity

$$\int_{\Omega} \Big(\mathbf{u} \otimes \mathbf{u} : \nabla_x \vec{\varphi} + p \mathrm{div}_x \vec{\varphi} \Big) \ \mathrm{d}\mathbf{x} = \int_{\Omega} \Big(\mu (\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u}) : \nabla_x \vec{\varphi} - \mathbf{f} \cdot \vec{\varphi} \Big) \ \mathrm{d}\mathbf{x} \tag{2.6}$$

holds for any test function $\vec{\varphi} \in W^{1,2}(\Omega; R^3)$ such that

$$\vec{\varphi}|_{\{x_3=0\}} = 0, \ \vec{\varphi} \cdot \mathbf{n}|_{\{x_3=1\}} = \vec{\varphi} \times \mathbf{w}|_{\{x_3=1\}} = 0.$$

Formally, the partial slip boundary conditions in the direction $\mathbf{w}$ can be interpreted as

$$\mathbf{u} \cdot \mathbf{n}|_{\{x_3=1\}} = \mathbf{u} \times \mathbf{w}|_{\{x_3=1\}} = 0,\ (\mathbb{S}\mathbf{n}) \times \mathbf{n}|_{\{x_3=1\}} = (\mathbb{S}\mathbf{n}) \cdot \mathbf{w}|_{\{x_3=1\}} = 0. \tag{2.7}$$

provided all quantities appearing in (2.6) are smooth.

The slip direction $\mathbf{w}$ in Definition 2.2 need not be constant, however, we focus on the case of a ribbed boundary for which $\mathbf{w} = (0, 1, 0)$. More precisely, we assume that the functions Φ_ε depend on only one variable, say, $x_1 \in \mathcal{T}^1 = (0,1)|_{\{(0,1)\}}$. Under these conditions, it is easy to check that the boundary condition (2.7) reduces to

$$u_1|_{\{x_3=1\}} = u_3|_{\{x_3=1\}} = 0,\ \partial_{x_3} u_2|_{\{x_3=1\}} = 0, \tag{2.8}$$

in other words, the components u_1, u_3 satisfy the homogeneous Dirichlet boundary condition, while u_2 obeys the homogeneous Neumann boundary condition on the upper boundary $\{x_3 = 1\}$.

Our main result reads as follows.

Theorem 2.1 *Let a family of domains $\{\Omega_\varepsilon\}_{\varepsilon>0}$ be given through (1.1), where*

$$\Phi(x_1, x_2) = \Phi_\varepsilon(x_1),\ \Phi_\varepsilon \in W^{1,\infty}(\mathcal{T}^1),\ 0 \le \Phi_\varepsilon \le \varepsilon,\ |\Phi'_\varepsilon| \le L \tag{2.9}$$

uniformly for $\varepsilon \to 0$. In addition, assume that there exists $\lambda > 0$ such that

$$\liminf_{\varepsilon \to 0} \int_a^b |\Phi'_\varepsilon(z)|\ \mathrm{d}z \ge \lambda |a - b| \textit{ for arbitrary } a \le b,\ a, b \in \mathcal{T}^1. \tag{2.10}$$

Let $\mathbf{u}_\varepsilon$, p_ε be a family of weak solutions of problem (1.7 - 1.11) on Ω_ε in the sense of Definition 2.1, with

$$1_{\Omega_\varepsilon}\ \mathbf{f}_\varepsilon \to \mathbf{f} \textit{ weakly in } L^2(R^3; R^3). \tag{2.11}$$

Then, passing to a subsequence as the case may be, and normalizing the pressure as $\int_{\Omega_\varepsilon} p_\varepsilon\ \mathrm{d}\mathbf{x} = 0$, we have

$$\mathbf{u}_\varepsilon \to \mathbf{u} \textit{ in } W^{1,2}(\Omega; R^3),\ p_\varepsilon \to p \textit{ in } L^2(\Omega), \tag{2.12}$$

where $\mathbf{u}$, p solve problem (1.7 - 1.10) in Ω, together with the partial slip boundary condition in the direction $\mathbf{w} = (0, 1, 0)$, in the sense specified in Definition 2.2.

Hypothesis (2.10) admits a clear geometrical interpretation, namely the length of the curve represented by the graph of the function Φ_ε over (a, b) is (uniformly) larger than the length of the interval (a, b). Note that (2.10) is satisfied for periodically distributed asperities:

$$\Phi_{\varepsilon_n}(x_1, x_2) = \varepsilon_n \Phi\Big(\frac{x_1}{\varepsilon_n}\Big),\ \Phi \in W^{1,\infty}(\mathcal{T}^1),\ \frac{1}{\varepsilon_n} \text{ a positive integer}, \tag{2.13}$$

where

$$\lambda = \int_{\mathcal{T}^1} |\Phi'(z)|\ \mathrm{d}z;$$

as well as for the "crystalline" structure:

$$\Phi'_\varepsilon(z) \in F,\ F \subset R \text{ a finite set},\ 0 \notin F,\ \text{for a.a. } z \in \mathcal{T}^1 \text{ and all } \varepsilon, \tag{2.14}$$

where

$$\lambda = \min_{y \in F} |y| > 0.$$

The rest of the paper is devoted to the proof of Theorem 2.1. Possible generalizations with respect to the geometry of the domain are briefly discussed in Section 6.

3 Rugosity measures

3.1 Parametrized rugosity measures

Our aim is to identify the boundary behaviour of the weak limits of sequences $\{\mathbf{u}_\varepsilon\}_{\varepsilon>0}$, $\mathbf{u}_\varepsilon \in V(\Omega_\varepsilon; R^3)$. To this end, we consider a family $\{\mathbf{u}_\varepsilon\}_{\varepsilon>0}$, $\mathbf{u}_\varepsilon \in V(\Omega_\varepsilon; R^3)$ such that

$$\sup_{\varepsilon>0} \int_{\Omega_\varepsilon} \Big(|\nabla_x \mathbf{u}_\varepsilon|^2 + |\mathbf{u}_\varepsilon|^2 \Big) \ \mathrm{d}\mathbf{x} < \infty. \tag{3.1}$$

Accordingly, as $\Omega = \mathcal{T}^2 \times (0,1) \subset \Omega_\varepsilon$ for all $\varepsilon > 0$, we may suppose

$$\mathbf{u}_\varepsilon \to \mathbf{u} \text{ in } L^2(\Omega; R^3), \ \nabla_x \mathbf{u}_\varepsilon \to \nabla_x \mathbf{u} \text{ weakly in } L^2(\Omega; R^{3\times 3}), \tag{3.2}$$

at least for a suitable subsequence, where the limit vector field $\mathbf{u}$ vanishes on the "bottom" part of the boundary $\{x_3 = 0\}$.

To begin with, it is easy to see that $\mathbf{u}$ satisfies the impermeability boundary condition (1.2). Indeed we have

$$0 = \int_{\Omega_\varepsilon} \mathbf{u}_\varepsilon \cdot \nabla_x \varphi \ \mathrm{d}\mathbf{x} = \int_{\Omega_\varepsilon \setminus \Omega} \mathbf{u}_\varepsilon \cdot \nabla_x \varphi \ \mathrm{d}\mathbf{x} + \int_\Omega \mathbf{u}_\varepsilon \cdot \nabla_x \varphi \ \mathrm{d}\mathbf{x},$$

where

$$\Big| \int_{\Omega_\varepsilon \setminus \Omega} \mathbf{u}_\varepsilon \cdot \nabla_x \varphi \ \mathrm{d}\mathbf{x} \Big| \leq \sqrt{|\Omega_\varepsilon \setminus \Omega|} \sup_{x \in \mathcal{T}^2 \times R} |\nabla_x \varphi(x)| \ \|\mathbf{u}_\varepsilon\|_{L^2(\Omega_\varepsilon)} \to 0 \text{ for } \varepsilon \to 0$$

whenever $\varphi \in \mathcal{D}(\mathcal{T}^2 \times R)$. Thus we conclude that

$$\int_\Omega \mathbf{u}_\varepsilon \cdot \nabla_x \varphi \ \mathrm{d}\mathbf{x} \to \int_\Omega \mathbf{u} \cdot \nabla_x \varphi \ \mathrm{d}\mathbf{x} = 0 \text{ for any } \varphi \in \mathcal{D}(\mathcal{T}^2 \times R); \tag{3.3}$$

whence $\mathbf{u}$ belongs to $V(\Omega; R^3)$.

As observed in [5], the behavior of the tangential component $[\mathbf{u}]_\tau$ on $\{x_3 = 1\}$ is essentialy governed by the oscillations of the outer normal vector to $\partial\Omega_\varepsilon$. The analysis is based on two crucial observations:

- the trace $\mathbf{u}_\varepsilon(y, 1 + \Phi_\varepsilon(y))$ on the upper part of $\partial\Omega_\varepsilon$ is close to $\mathbf{u}_\varepsilon(y, 1)$ in the topology of $L^1(\mathcal{T}^2; R^3)$;
- the deviation of the outer normal vector to $\partial\Omega_\varepsilon$ from the vertical direction is described by $\nabla_y \Phi_\varepsilon$.

To see the former claim, we write

$$\int_{\mathcal{T}^2} |\mathbf{u}_\varepsilon(y, 1 + \Phi_\varepsilon(y)) - \mathbf{u}_\varepsilon(y, 1)| \ \mathrm{d}y \leq$$

$$\int_{\mathcal{T}^2} \int_1^{1+\Phi_\varepsilon(y)} |\partial_{x_3} \mathbf{u}_\varepsilon| \ \mathrm{d}x_3 \ \mathrm{d}y \leq \sqrt{|\Omega_\varepsilon \setminus \Omega|} \ \|\nabla_x \mathbf{u}_\varepsilon\|_{L^2(\Omega_\varepsilon; R^{3\times 3})} \to 0 \text{ for } \varepsilon \to 0;$$

as for the latter it is enough to observe that $-\mathbf{n} = (\nabla_y \Phi_\varepsilon, -1)$. On the basis of these two facts, one can deduce that

$$(u_1, u_2)(y, 1) \cdot \int_{R^2} G(\mathbf{Z})\mathbf{Z} \ \mathrm{d}\mathcal{R}_y(\mathbf{Z}) = 0 \text{ for any } G \in C(R^2), \text{ and for a.a. } y \in \mathcal{T}^2, \quad (3.4)$$

where $\mathcal{R}_y$, $y \in \mathcal{T}^2$ denotes a Young measure generated by the sequence $\{\nabla_y \Phi_\varepsilon\}_{\varepsilon>0}$ (see Lemma 7.1 in [5]). The Young measure $\{\mathcal{R}_y\}_{y\in\mathcal{T}^2}$ is termed *parametrized rugosity measure* associated to the family $\{\Omega_\varepsilon\}_{\varepsilon>0}$.

If the sequence satisfies hypothesis (2.9), we have

$$\mathrm{supp}[\mathcal{R}_y] \subset \{(y_1, 0) \mid y_1 \in R\} \text{ for a.a. } y \in \mathcal{T}^2. \quad (3.5)$$

On the other hand, hypothesis (2.10) yields

$$\overline{|\Phi'|} \geq \lambda > 0, \quad (3.6)$$

where $\overline{|\Phi'|}$ stands for a weak limit of $\{|\Phi'_\varepsilon|\}_{\varepsilon>0}$ in $L^1(\mathcal{T}^1)$.

As Φ'_ε tends weakly to zero, relation (3.6) implies that $\mathcal{R}_y$ does not reduce to a Dirac mass for any $y \in \mathcal{T}^2$. Consequently, relations (3.4), (3.5) give rise to

$$u_1|_{\{x_3=1\}} = 0 \quad (3.7)$$

Thus we have shown that the limit vector field $\mathbf{u}$ belongs to the space $V_{\mathbf{w}}(\Omega; R^3)$, with $\mathbf{w} = (0, 1, 0)$, specified through (2.4), provided the sequence $\{\mathbf{u}_\varepsilon\}_{\varepsilon>0}$ satisfies (3.1).

3.2 Directional rugosity measures

The parametrized rugosity measures $\{\mathcal{R}_y\}_{y\in\mathcal{T}^2}$ reflect oscillations of the boundaries and ignore sets of the 2d-Hausdorff measure zero on $\partial\Omega_\varepsilon$. Thus the procedure delineated above applies basically to any sequence $\{\mathbf{u}_\varepsilon\}_{\varepsilon>0}$ having traces in a suitable Lebesgue space on $\partial\Omega_\varepsilon$. An alternative refined approach sketched below and fully developed in [4] is based on the tools of $\Gamma-$convergence.

Let $\mathbf{v} \in C^1([0,1]^2 \times \{x_3 = 1\}, \mathbb{R}^3)$ be a restriction of a C^1-vector field on R^3. We introduce *directional rugosity measure* associated to $\mathbf{v}$, which is a capacitary measure supported on the set $\mathcal{T}^2 \times \{x_3 = 1\}$ expressing, in terms of energy, the "asymptotical roughness" of $\partial\Omega_\varepsilon$ when $\varepsilon \to 0$ with respect to the direction parallel to the vector field $\mathbf{v}$.

Let us set

$$B = \mathcal{T}^2 \times (-1, 2). \quad (3.8)$$

As Ω_ε are Lipschitz domains, there exist extension operators $P^B_\varepsilon : W^{1,2}(\Omega_\varepsilon) \to W^{1,2}_0(B)$, with norm bounded in terms of the Lipschitz constant L appearing in (2.9).

We consider a family $\mathcal{M}_{\mathbf{v}}$ consisting of all non-negative Borel measures ν (possibly infinite), absolutely continuous with respect to $W^{1,2}-$capacity on R^3, and such that the inequality

$$\int_B |\nabla(\mathbf{u} \cdot \mathbf{v})|^2 \mathrm{d}\mathbf{x} + \int_B (\mathbf{u} \cdot \mathbf{v})^2 \mathrm{d}\nu \leq \liminf_{k\to\infty} \int_B |\nabla(P^B_{\varepsilon_k} \mathbf{u}_{\varepsilon_k} \cdot \mathbf{v})|^2 \mathrm{d}\mathbf{x} \quad (3.9)$$

holds whenever

$$\mathbf{u}_{\varepsilon_k} \in W^{1,2}(\Omega_\varepsilon, R^3),\ \mathbf{u}_{\varepsilon_k} \cdot \mathbf{n} = 0 \text{ on } \{x_3 = \Phi_\varepsilon(x_1, x_2)\}, \tag{3.10}$$

$$P^B_{\varepsilon_k} \mathbf{u}_{\varepsilon_k} \to \mathbf{u} \text{ weakly in } W^{1,2}_0(B; R^3). \tag{3.11}$$

Definition 3.1 *The measure*

$$\nu_{\mathbf{v}} = \sup\{\nu : \nu \in \mathcal{M}_{\mathbf{v}}\}$$

is called directional rugosity measure associated to the field $\mathbf{v}$ *and to* $\{\Omega_\varepsilon\}_{\varepsilon>0}$.

Let us remark that the set $\mathcal{M}_{\mathbf{v}}$ is not empty as it contains $\nu = 0$; the supremum in the above definition is taken in the sense of measures.

Consequently, if $\nu_{\mathbf{v}}$ restricted to some Borel set $K \subset \mathcal{T}^2 \times \{x_3 = 1\}$ is infinite, then, by virtue of (3.9), $\mathbf{u} \cdot \mathbf{v} = 0$ quasi-everywhere on K. Therefore in order to understand the boundary condition satisfied by the weak limits in (3.11), one has to identify fields $\mathbf{v}$ and sets K for which the directional rugosity is infinite.

We claim that

$$\nu_{e_1}|_{\mathcal{T}^2\times\{x_3=1\}} = \infty,\ \nu_{e_2}|_{\mathcal{T}^2\times\{x_3=1\}} = 0,\ \text{where } e_1 = (1,0,0),\ e_2 = (0,1,0) \tag{3.12}$$

for both the periodically distributed asperities described in (2.13) and the "crystalline" structures introduced in (2.14).

In order to see that ν_{e_2} in (3.12) vanishes, it is enough to consider the sequence $\mathbf{u}_\varepsilon(x_1, x_2, x_3) = (0, 1, 0)$. The desired conclusion follows directly from (3.9).

A direct proof of the former claim in (3.12) is much more delicate, and we refer to [4] for a detailed discussion. Let us only remark that the statement $\nu_{e_1}|_{\mathcal{T}^2\times\{x_3=1\}} = \infty$ is in fact *equivalent* to saying that $u_1 = 0$ on $\{x_3 = 1\}$ for any sequence described in (3.10), (3.11). From this point of view, relation (3.12) may be seen as a consequence of (3.7). This observation provides an evidence that the piece of information provided by the directional rugosity measure is more complete than that one can obtain using the parametrized rugosity measure introduced in Section 3.1. This is indeed the case as the following example shows.

Example 3.1
Let $q : \mathcal{T}^2 \to R$ *be a function defined by*

$$q(x_1, x_2) = \min\{|x_1|, |1 - x_1|\}.$$

We set

$$q_k(x_1, x_2) = \frac{1}{k} q(kx_1, kx_2) \text{ if } (x_1, x_2) \in [0, 1/k] \times [0, 1/k],$$

$$q_k(x_1, x_2) = 0 \text{ elsewhere on } \mathcal{T}^2.$$

In accordance with (3.12), the directional rugosity measures with respect to the two normals ξ_1, ξ_2 *to the graph of* q *associated to the family* $\Phi_\varepsilon = \varepsilon q_k(x_1/\varepsilon, x_2/\varepsilon)$ *are infinite for any fixed* $k > 0$. *On the other hand, a diagonal procedure based on metrizability of the* Γ*-convergence, there exists a sequence* $k_n \to \infty$ *and a sequence of functions* $\Phi_{\varepsilon_{k_n}}$ *such that the directional rugosity measures are still infinite in the directions* ξ_i. *Consequently* $\nu_{e_1}|_{\mathcal{T}^2\times\{x_3=1\}} = \infty$ *(see [4] for details). Finally, since* $k_n \to \infty$, *one gets that* $\nabla\Phi_{\varepsilon_{k_n}} \to 0$ *strongly in* L^1*; whence the parametrized rugosity measure introduced in Section 3.1 reduces to a family of Dirac masses centered at zero.*

4 On equation $\mathrm{div}_x \mathbf{v} = g$

This section contains a preliminary material to used to obtain uniform estimates on the pressure term p_ε in (2.3).

To this end, consider a family of Lipschitz domains defined through (1.1), where

$$\Phi_\varepsilon \in W^{1,\infty}(\mathcal{T}_2),\ 0 \leq \Phi_\varepsilon \leq \varepsilon,\ |\nabla_x \Phi_\varepsilon| \leq L,\ \text{with } L \text{ independent of } \varepsilon > 0. \tag{4.1}$$

By virtue of hypothesis (4.1), there exists $\omega > 0$ independent of ε such that the interior of the cone

$$(x_1, x_2, 1 + \Phi_\varepsilon(x_1, x_2)) + \mathcal{K},\ \mathcal{K} = \{(x_1, x_2, x_3) \mid x_3 \in (-1, 0),\ |(x_1, x_2)| < \omega |x_3|\}$$

is contained in Ω_ε for any $(x_1, x_2) \in \mathcal{T}^2$. Consequently, there is a finite number of domains Ω_ε^k, $k = 1, \ldots, m$, such that

$$\Omega_\varepsilon = \cup_{k=1}^m \Omega_\varepsilon^k,$$

and each Ω_ε^k is starshaped with respect to any point of a ball of a radius $r > 0$ contained in Ω_ε^k, where both m and r can be chosen independent of ε (for the relevant definition of a starshaped domain see Galdi [8, Chapter III.3]).

Consider an auxiliary problem: *Given*

$$g \in L^q(\Omega_\varepsilon),\ \int_{\Omega_\varepsilon} g \ \mathrm{d}\mathbf{x} = 0,\ 1 < q < \infty, \tag{4.2}$$

find a vector field $\mathbf{v} = \mathcal{B}_\varepsilon[g]$ *such that*

$$\mathbf{v} \in W_0^{1,q}(\Omega_\varepsilon; R^3),\ \mathrm{div}_x \mathbf{v} = g \ \textit{a.a. in } \Omega_\varepsilon. \tag{4.3}$$

We report the following result (see Theorem 3.1 in Chapter III.3 in Galdi [8]).

Proposition 4.1 *For each $\varepsilon > 0$ there is a solution operator $\mathcal{B}_\varepsilon$ associated to problem (4.2), (4.3) such that*

$$\| \mathcal{B}_\varepsilon[g] \|_{W_0^{1,q}(\Omega_\varepsilon; R^3)} \leq c(r, m, q) \|g\|_{L^q(\Omega_\varepsilon)}, \tag{4.4}$$

in particular, the norm of $\mathcal{B}_\varepsilon$ is independent of ε.

Remark 4.1 The construction of the operator $\mathcal{B}$ used in [8] is due to Bogovskii [3].

5 Proof of Theorem 2.1

5.1 Uniform estimates

First of all, we establish uniform estimates on the family of solutions $\{\mathbf{u}_\varepsilon\}_{\varepsilon>0}$, $\{p_\varepsilon\}_{\varepsilon>0}$ in the topology $W^{1,2} \times L^2$ in order to apply the results obtained in Section 3.1.

To begin with, we recall the following version of Korn's inequality (see Proposition 4.2 in Nitsche [9]).

Proposition 5.1 *Let Ω_ε be given by (1.1), with Φ_ε satisfying (4.1). Then there exist a family of extension operators*

$$E_{\Omega_\varepsilon} : V(\Omega_\varepsilon; R^3) \to W_0^{1,2}(B; R^3),\ B = \mathcal{T}^2 \times (0,2)$$

such that $E_{\Omega_\varepsilon}[\mathbf{v}]|_{\Omega_\varepsilon} = \mathbf{v}$, and

$$\| \ E_{\Omega_\varepsilon}[\mathbf{v}] \ \|_{W^{1,2}(B;R^3)} \leq c \|\nabla_x \mathbf{v} + \nabla_x^\perp \mathbf{v}\|_{L^2(\Omega_\varepsilon;R^{3\times 3})} \tag{5.1}$$

for any $\mathbf{v} \in V(\Omega_\varepsilon; R^3)$, where the constant c is independent of ε.

Taking $\vec{\varphi} = \mathbf{u}_\varepsilon$ in (2.3) we obtain

$$\frac{\mu}{2} \|\nabla_x \mathbf{u}_\varepsilon + \nabla_x^\perp \mathbf{u}_\varepsilon\|^2_{L^2(\Omega_\varepsilon;R^{3\times 3})} \leq \|\mathbf{f}_\varepsilon\|_{L^2(\Omega_\varepsilon;R^3)} \|\mathbf{u}_\varepsilon\|_{L^2(\Omega_\varepsilon;R^3)}; \tag{5.2}$$

whence, in accordance with Proposition 5.1, we deduce that

$$\{\mathbf{u}_\varepsilon\}_{\varepsilon>0} \text{ is bounded in } W^{1,2}(B;R^3) \tag{5.3}$$

where we have identified $\mathbf{u}_\varepsilon$ with $E_{\Omega_\varepsilon}[\mathbf{u}_\varepsilon]$ in B.

In order to obtain uniform estimates for the pressure, we have only to realize that, by virtue of Proposition 4.1, any function $g \in L^2(\Omega)$ of zero integral mean can be represented as $\mathrm{div}_x \vec{\varphi}$, where $\vec{\varphi}$ belongs to the space $W_0^{1,2}(\Omega_\varepsilon; R^3)$. Consequently, by means of (4.4), (5.3), we deduce from (2.3) that

$$\{p_\varepsilon\}_{\varepsilon>0} \text{ is bounded in } L^2(B) \tag{5.4}$$

provided $\int_{\Omega_\varepsilon} p_\varepsilon \ \mathrm{d}\mathbf{x} = 0$, where p_ε were extended to be zero outside Ω_ε.

5.2 Identifying the asymptotic limit

In accordance with (5.3), the family $\{\mathbf{u}_\varepsilon\}_{\varepsilon>0}$ admits the uniform bound (3.1). Consequently, identifying $\mathbf{u}_\varepsilon$ with their extensions on B we may assume that

$$\mathbf{u}_\varepsilon \to \mathbf{u} \text{ weakly in } W^{1,2}(B;R^3) \text{ and strongly in } L^2(B;R^3), \tag{5.5}$$

where, by means of (3.3), (3.7), the limit vector field $\mathbf{u}$ belongs to the space $V_{\mathbf{w}}(\Omega; R^3)$, with $\mathbf{w} = (0,1,0)$. In other words,

$$\mathbf{u}|_{\{x_3=0\}} = 0,\ u_1|_{\{x_3=1\}} = u_3|_{\{x_3=1\}} = 0. \tag{5.6}$$

Similarly, by virtue of (5.4),

$$1_{\Omega_\varepsilon} p_\varepsilon \to p \text{ weakly in } L^2(B) \tag{5.7}$$

at least for a suitable subsequence.

In accordance with Definition 2.2, the class of test functions for the limit problem (2.6) consists of functions $\vec{\varphi}$,

$$\vec{\varphi} \in W^{1,2}(\Omega;R^3),\ \varphi_1, \varphi_3 \in W_0^{1,2}(\Omega;R^3),\ \varphi_2|_{\{x_3=0\}} = 0.$$

Consequently, by means of a density argument, it is enough to show that $\mathbf{u}$, p satisfy (2.6) for any test function $\vec{\varphi}$ such that

$$\varphi_1, \varphi_3 \in \mathcal{D}(\Omega;R^3),\ \varphi_2 \in \mathcal{D}(B;R^3). \tag{5.8}$$

However, such a φ is clearly an admissible test function in (2.3). Letting $\varepsilon \to 0$ in (2.3) for a fixed $\vec{\varphi}$ we easily conclude that $\mathbf{u}$, p solve the limit problem (2.6).

5.3 Strong convergence

In order to complete the proof of Theorem 2.1, we have to show that $\nabla_x \mathbf{u}_\varepsilon$, p_ε converge strongly as claimed in (2.12).

As $\mathbf{u}_\varepsilon$ is an admissible test function in (2.3), we get

$$\int_{\Omega_\varepsilon} \frac{\mu}{2} |\nabla_x \mathbf{u}_\varepsilon + \nabla_x^t \mathbf{u}_\varepsilon|^2 \ \mathrm{d}\mathbf{x} = \int_{\Omega_\varepsilon} \mathbf{f}_\varepsilon \cdot \mathbf{u}_\varepsilon \ \mathrm{d}\mathbf{x};$$

therefore, letting $\varepsilon \to 0$,

$$\int_\Omega \frac{\mu}{2} |\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u}|^2 \ \mathrm{d}\mathbf{x} \le \liminf_{\varepsilon \to 0} \int_\Omega \frac{\mu}{2} |\nabla_x \mathbf{u}_\varepsilon + \nabla_x^t \mathbf{u}_\varepsilon|^2 \ \mathrm{d}\mathbf{x} \le \tag{5.9}$$

$$\lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} \frac{\mu}{2} |\nabla_x \mathbf{u}_\varepsilon + \nabla_x^t \mathbf{u}_\varepsilon|^2 \ \mathrm{d}\mathbf{x} = \int_\Omega \mathbf{f} \cdot \mathbf{u} \ \mathrm{d}x.$$

On the other hand, $\mathbf{u}$ is an admissible test function in (2.6); whence

$$\int_\Omega \frac{\mu}{2} |\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u}|^2 \ \mathrm{d}\mathbf{x} = \int_\Omega \mathbf{f} \cdot \mathbf{u} \ \mathrm{d}\mathbf{x} \tag{5.10}$$

Relations (5.9), (5.10) give rise to the strong convergence of the symmetric parts of the gradients $\nabla_x \mathbf{u}_\varepsilon$, therefore we conclude that

$$\mathbf{u}_\varepsilon \to \mathbf{u} \text{ in } W^{1,2}(\Omega; R^3). \tag{5.11}$$

Finally, let $\mathcal{B}$ be the Bogovskii operator associated to Ω, the existence of which is guaranteed by Proposition 4.1. Consider

$$\vec{\varphi} = \mathcal{B}\Big[p_\varepsilon - \frac{1}{|\Omega|} \int_\Omega p_\varepsilon \ \mathrm{d}x \Big].$$

The function $\vec{\varphi}$ belonging to the space $W_0^{1,2}(\Omega; R^3)$ can be extended by zero outside Ω to be used as a test function in (2.3). Seeing that

$$\int_\Omega p_\varepsilon \ \mathrm{d}x \to 0 \text{ for } \varepsilon \to 0,$$

and taking (5.11) into account, we obtain

$$\lim_{\varepsilon \to 0} \int_\Omega p_\varepsilon^2 \ \mathrm{d}x = \tag{5.12}$$

$$\int_\Omega \Big(\mu(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u}) : \nabla_x(\mathcal{B}[p]) - \mathbf{f} \cdot \nabla_x(\mathcal{B}[p]) - \mathbf{u} \otimes \mathbf{u} : \nabla_x(\mathcal{B}[p]) \Big) \ \mathrm{d}\mathbf{x}.$$

On the other hand, as the quantity $\mathcal{B}[p]$ is an admissible test function in (2.6), we conclude that the expression on the right-hand side of (5.12) equals $\int_\Omega p^2 \ \mathrm{d}x$. Thus

$$p_\varepsilon \to p \text{ in } L^2(\Omega),$$

which completes the proof of Theorem 2.1.

6 Conclusion

A specific feature of the situation examined in Theorem 2.1 is the fact that the slip direction $\mathbf{w} = (0,1,0)$ is the same for all ε, more precisely,

$$\mathbf{w} \cdot \mathbf{n} = 0 \text{ whenever } \mathbf{n} = (\partial_{x_1}\Phi_\varepsilon(x_1,x_2), \partial_{x_2}\Phi_\varepsilon(x_1,x_2), -1)$$

$$\text{for a.a. } (x_1,x_2) \in \mathcal{T}^2, \ \varepsilon > 0.$$

Accordingly, the test functions for the limit problem (2.6) can be easily extended outside Ω so that the resulting quantities are admissible in (2.3) (cf. (5.8)).

In order to attack a more general setting, we introduce a concept of *slip direction* as follows:

Definition 6.1 *We shall say that* $\mathbf{w} \in C^1(\mathcal{T}^2; R^3)$ *is a slip direction associated to* $\{\Omega_\varepsilon\}_{\varepsilon>0}$ *if for any* $\vec{\varphi} \in C^1(\overline{\Omega}; R^3) \cap V_{\mathbf{w}}(\Omega; R^3)$ *there exists a family* $\{\vec{\varphi}_\varepsilon\}_{\varepsilon>0}$, $\vec{\varphi}_\varepsilon \in V(\Omega_\varepsilon; R^3)$ *such that*

$$\{\vec{\varphi}_\varepsilon\}_{\varepsilon>0} \text{ is precompact in } W^{1,2}(B; R^3),$$

and

$$\vec{\varphi}_\varepsilon \to \vec{\varphi} \text{ in } W^{1,2}(\Omega; R^3).$$

Possible generalizations of Theorem 2.1 for the case when $\mathbf{w}$ is not constant can be done in the spirit of the following observation.

Lemma 6.1 *Let a family of domains* $\{\Omega_\varepsilon\}_{\varepsilon>0}$ *be given through (1.1), with* Φ_ε *satisfying (4.1). Suppose that* $\mathbf{w} \in C^1(R^3, R^3)$ *satisfies*

$$\frac{\mathbf{w} \cdot \nabla_x \Phi_\varepsilon}{1 + |\nabla_x \Phi_\varepsilon|^2} (\nabla_x \Phi_\varepsilon, -1) \to 0 \text{ in } W^{1,2}(\mathcal{T}^2; R^3). \tag{6.1}$$

Then $\mathbf{w}$ *is a complete slip vector field associated to* $\{\Omega_\varepsilon\}_{\varepsilon>0}$ *in the sense of Definition 6.1.*

Proof:
Let $\vec{\varphi} \in C^1(\overline{\Omega}; R^3) \cap V_{\mathbf{w}}(\Omega; R^3))$, in particular,

$$\vec{\varphi}(x_1,x_2,1) = \Lambda(x_1,x_2)\mathbf{w}(x_1,x_2,1) \text{ for } (x_1,x_2) \in \mathcal{T}^2.$$

We define an extension $E[\vec{\varphi}]$ on the set $B = \mathcal{T}^2 \times (0,2)$ as

$$E[\vec{\varphi}](x_1,x_2,x_3) = \begin{cases} \vec{\varphi}(x_1,x_2,x_3) \text{ for } 0 \le x_3 \le 1, \\ \\ \Lambda(x_1,x_2)\mathbf{w}(x_1,x_2) \text{ if } x_3 > 1. \end{cases}$$

Clearly, $\vec{\varphi}$ belongs to the space $W^{1,\infty}(B; R^3)$.

We set

$$\vec{\varphi}_\varepsilon = E[\vec{\varphi}] - \psi(x_3)\Lambda(x_1,x_2)\frac{\mathbf{w} \cdot \nabla_x \Phi_\varepsilon}{1 + |\nabla_x \Phi_\varepsilon|^2}(\nabla_x \Phi_\varepsilon, -1),$$

where $\psi \in C^\infty(R)$ satisfies

$$\psi(x_3) = \begin{cases} 0 \text{ for } x_3 < 1/4, \\ \\ 1 \text{ for } x_3 > 3/4. \end{cases}$$

It follows from (6.1) that

$$\psi(x_3)\Lambda(x_1, x_2)\frac{\mathbf{w} \cdot \nabla_x \Phi_\varepsilon}{1 + |\nabla_x \Phi_\varepsilon|^2}(\nabla_x \Phi_\varepsilon, -1) \to 0 \text{ in } W^{1,2}(B; R^3);$$

whence $\vec{\varphi}_\varepsilon$ can be taken as the approximate sequence required by Definition 6.1

q.e.d.

Example 6.1 *Assume* $\mathbf{w}(y)$ *is orthogonal on* $(\nabla \Phi_\varepsilon(y), -1)$ *for any* $y \in \mathcal{T}^2$ *and any* $\varepsilon > 0$. *Then* $\mathbf{w}$ *is a complete slip vector field associated to* $\{\Omega_\varepsilon\}_{\varepsilon > 0}$.

Example 6.2 *Let*

$$\Phi_\varepsilon(x_1, x_2) = \varepsilon \Psi\Big(\frac{x_1}{\varepsilon}\Big) + \varepsilon^m \Gamma\Big(\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}\Big),\ m > 2,$$

where $\Psi \in C^2(\mathcal{T}^1)$, $\Gamma \in C^2(\mathcal{T}^2)$. *Then* $\mathbf{w} = (0, 1, 0)$ *is a complete slip vector field associated to* $\{\Omega_\varepsilon\}_{\varepsilon > 0}$.

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