A FREE BOUNDARY APPROACH TO THE FABER-KRAHN INEQUALITY

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Abstract. The purpose of this survey article is to present a complete, comprehensive, proof of the Faber-Krahn inequality for the Dirichlet Laplacian from the perspective of free boundary problems. The proof is of a purely variational nature, proceeding along the following steps: proof of the existence of a domain which minimizes the first eigenvalue among all domains of prescribed volume, proof of (partial) regularity of the optimal domain and usage of a reflection argument in order to prove radiality. As a consequence, no rearrangement arguments are used and, although not the simplest of proofs of this statement, it has the advantage of its adaptability to study the symmetry properties of higher eigenvalues and also to other isoperimetric inequalities (Faber-Krahn or Saint-Venant) involving Robin boundary conditions.

1. Introduction

It was conjectured by Rayleigh in 1877 that among all fixed membranes with a given area, the ball would minimize the first eigenvalue [23]. This assertion was proved by Faber and Krahn in the nineteen twenties using a rearrangement technique, and since then several proofs have appeared in the literature.

The purpose of this survey article is to give a comprehensive proof of the Rayleigh–Faber–Krahn inequality in the context of free boundary problems. It may be argued that, in spite of the fact that this is a basic result in the theory, no more proofs of this statement are needed, particularly if they are not simpler than other existing proofs. However, it is our belief that because the proof we propose here is somehow of a different nature, has applications to other problems, and follows a natural sequence of intuitive (but mathematically not easy) steps, it deserves some attention. These steps are the following: existence of an optimal shape, proof of its (partial) regularity, and use of its optimality in order to conclude that the optimizer must be radially symmetric.

One has in mind the incomplete proof of the isoperimetic inequality by Steiner in 1836 (see [25] and the survey article [4]). Steiner proved (in two dimensions of the space) that if a *smooth* domain is not the ball, then there must exist another domain with the same area but lower perimeter. Of course, the missing step is precisely the proof of the existence of a sufficiently smooth set which minimizes the perimeter among all domains of fixed area!

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Moreover, existence *only* is not enough, since in order to use his argument, Steiner implicitly needed some *smoothness*. This proof became complete only in 1957, when De Giorgi proved that the Steiner argument could be carried out in the class where existence holds, precisely the sets which have a finite perimeter defined using functions of bounded variation (see [15]). We also refer the reader to the open problem on the isoperimetric inequality for the buckling load of a clamped plate, where the same couple of questions remain unsolved. Willms and Weinberger (see [17, Theorem 11.3.7]) noticed that if a smooth, simply connected set would minimise the buckling load among all domains with fixed area (in two dimensions of the space), then necessarily the minimizer is the ball.

In the proof we give of the Faber–Krahn inequality, we use only variational arguments developed in the context of free boundary problems (see for instance [1, 6, 26]). Our purpose is to present the basic tools which allow to complete the sequence : existence-regularity-radiality and give the lecturer the fundamental ideas hidden behind this scheme. This approach has the advantage to be adaptable to other isoperimetric inequalities where re-arrangement or mass transport techniques fail. We have in mind isoperimetric inequalities involving Robin boundary conditions, as for example the minimization of the first Robin eigenvalue or the maximization of the Robin torsional rigidity among domains of fixed volume [9, 10]. In the Robin case, the techniques to prove existence-regularity-radiality steps are definitely more involved and require finer analysis arguments on special functions with bounded variations, developed in the framework of free discontinuity problems.

Let $\Omega \subseteq \mathbb{R}^N$ be an open (or *quasi-open*) set of finite measure, but otherwise with no assumptions either on smoothness or on boundedness. Thanks to the compact embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$, the spectrum of Dirichlet-Laplacian on Ω is discrete and consists on a sequence of eigenvalues which can be ordered (counting multiplicities) as

$$0 < \lambda_1(\Omega) \leq \lambda_2(\Omega) \leq .. \leq \lambda_k(\Omega) \leq .. \to +\infty.$$

If Ω is open, $H_0^1(\Omega)$ is the classical Sobolev space consisting on the completion of $C_0^\infty(\Omega)$ for the L^2-norm of the gradients. If Ω is a quasi-open set, we refer the reader to the next section for some precisions.

The Faber–Krahn inequality asserts that

$$\lambda_1(\Omega) \geq \lambda_1(B),$$

where B is the ball having the same measure as Ω. Equality holds if and only if Ω is a ball (up to a negligible set of points, which may be expressed in terms of capacity).

Roughly speaking, the proof will be split into the following four steps:

Step 1. Prove that there exists a domain Ω^* which minimizes the first Dirichlet eigenvalue among all domains of fixed volume, i.e.

$$\exists \Omega^* \subseteq \mathbb{R}^N, |\Omega^*| = m, \ \forall \Omega \subseteq \mathbb{R}^N, |\Omega| = m \quad \lambda_1(\Omega^*) \leq \lambda_1(\Omega).$$

At this first step, the existence result is carried in the family of *quasi-open sets*.

Step 2. Prove that the set Ω^* is *open* and connected. This result can be seen, by itself, as a first regularity result. No smoothness of the boundary is required by the next steps.

Step 3. Use a reflection argument in order to deduce that Ω^* is radially symmetric, hence it is one annulus.

Step 4. Prove that among all annuli of prescribed volume, the ball gives the lowest first eigenvalue.

It is clear that the proof of the Faber–Krahn inequality we propose in this note is not simple. The proof of Steps 1 and 2 requires some techniques developed in the context of free boundary problems, that we intend to present in a simplified way. Step 3 is done using an idea that may be traced to Steiner's original manuscript [25, 4] and which has also been used in the context of the minimization of integral functionals in H^1-Sobolev spaces [20, 21]. We show in detail how this can be adapted to shape optimization in a functional context. Step 4 consists in a one dimensional analysis argument for which a precise computation can be carried out.

We note that if in Step 2 one is able to prove the smoothness of the boundary of $\partial\Omega^*$, by the Hadamard argument relying on the vanishing of the shape derivative, one can extract an *overdetermined* boundary condition. Precisely, one would get that $|\nabla u^*|$ is constant on $\partial\Omega^*$ and come to a Serrin problem which could be solved by moving plane techniques (see the pioneering paper [24]). Nevertheless, proving the smoothness of $\partial\Omega^*$ requires definitely more involved regularity techniques, as references [1, 12, 26] show. Proving only the openness of Ω^* is quite elementary, as shown in the sequel (see [26, 12] for a complete analysis of the regularity question).

In the last section of the paper we discuss briefly the problem of minimizing the k-th eigenvalue of the Dirichlet Laplacian among all quasi-open sets of prescribed measure. We present some new results where, in particular, we focus on the symmetry of a minimizer of the k-th eigenvalue and show how our arguments may also be used there. We point out that although one might expect minimisers of this type of spectral problems to always have some symmetry, say at least for the reflection with respect to one hyperplane, recent numerical evidence on this problem has raised the issue of whether or not this is actually true [2]. More precisely, the planar domain found (numerically) in that paper which minimises the thirteenth Dirichlet eigenvalue of the Laplace operator does not have any such symmetry and, furthermore, if a restriction is imposed enforcing that the optimiser does have some symmetry, namely invariance under reflection with respect to an axis, the resulting value of the optimal eigenvalue is worse than the unrestricted case. Although a proof of such a statement, if true, is likely to be quite elusive, there has been some independent numerical confirmation of this observation given in [3, 5]. In any case, these results do beg the question as to what is the minimum symmetry which we can guarantee these optimal domains will have. In this sense, the results presented here are a first step in this direction.

2. Setting the variational framework

Solving Step 1 requires to set up a very large framework for the existence question. As one cannot impose *a priori* constraints on the competing sets, in order to achieve existence the largest class of admissible shapes should be considered. This is a classical principle in shape optimization.

The most natural framework where the Dirichlet-Laplacian operator is well defined is the family of *quasi-open* sets. More precisely, $\Omega \subseteq \mathbb{R}^N$ is called quasi-open if for all $\varepsilon > 0$ there exists an open set U_ε such that the set $\Omega \cup U_\varepsilon$ is open, where

$$\mathrm{cap}(U_\varepsilon) := \inf\{\int_{\mathbb{R}^N} |\nabla u|^2 + |u|^2 dx : u \in H^1(\mathbb{R}^N), u \geq 1 \text{ a.e. } U_\varepsilon\} < \varepsilon.$$

Roughly speaking, quasi-open sets are precisely the level sets $\{\tilde{u} > 0\}$ of the "most continuous" representatives of Sobolev functions $u \in H^1(\mathbb{R}^N)$, i.e. the ones given by

$$\tilde{u}(x) = \lim_{r \to 0} \frac{\displaystyle\int_{B_r(x)} u(y)dy}{|B_r(x)|}. \tag{1}$$

The limit above exists for all points except a set of capacity zero.

If Ω is a quasi-open set, then the Sobolev space $H_0^1(\Omega)$ associated to the quasi-open set Ω is defined as a subspace of $H^1(\mathbb{R}^N)$, by

$$H_0^1(\Omega) = \bigcap_{\varepsilon > 0} H_0^1(\Omega \cup U_\varepsilon).$$

If Ω is a quasi-open set of finite measure, the spectrum of the Dirichlet-Laplacian on Ω is defined in the same way as for open sets, being the inverse of the spectrum of the compact, positive, self-adjoint resolvent operator $R_\Omega : L^2(\Omega) \to L^2(\Omega)$, $R_\Omega f = u$, where $u \in H_0^1(\Omega)$ satisfies

$$\forall \varphi \in H_0^1(\Omega) \quad \int_\Omega \nabla u \nabla \varphi dx = \int_\Omega f \varphi dx. \tag{2}$$

In particular

$$\lambda_1(\Omega) := \min_{u \in H_0^1(\Omega)} \frac{\displaystyle\int_\Omega |\nabla u|^2 dx}{\displaystyle\int_\Omega |u|^2 dx}.$$

The minimizing function solves the equation

$$\begin{cases} -\Delta u = & \lambda_1(\Omega) u \ \text{ in } \ \Omega \\ u = & 0 \ \partial\Omega \end{cases}$$

in the weak sense (2). Clearly, if Ω is open we find the classical definition of the first eigenvalue.

The following inequality, which holds for every quasi-open set of finite measure (see for instance [14, Example 2.1.8]), will be useful for the local study of the optimal sets

$$\|u\|_\infty \le C_N \lambda_1(\Omega)^{\frac{N}{4}} \|u\|_{L^2}.$$

The problem we intend to solve is the following: given $m > 0$, prove that the unique solution of

$$\min\{\lambda_1(\Omega) : \Omega \subseteq \mathbb{R}^N \text{ quasi-open}, |\Omega| = m\} \tag{3}$$

is the ball. Uniqueness is understood up to a set of zero capacity, which is precisely the size of a set for which one cannot distinguish the "precise" values of a Sobolev function.

Since for every $t > 0$, one has $\lambda_1(t\Omega) = \frac{1}{t^2}\lambda_1(\Omega)$, there exists $C > 0$ such that problem (3) is equivalent to

$$\min\{\lambda_1(\Omega) + C|\Omega| : \Omega \subseteq \mathbb{R}^N \text{ quasi-open}\}. \tag{4}$$

Indeed, let us denote

$$\alpha(m) = \inf\{\lambda_1(\Omega) : \Omega \subseteq \mathbb{R}^N \text{ quasi-open}, |\Omega| = m\}.$$

From the Sobolev inequality, we know that $\alpha(m)$ is strictly positive. Indeed, we have

$$\int_{\mathbb{R}^N} |\nabla u^2| dx \ge C_N \Big(\int_{\mathbb{R}^N} |u|^{\frac{2N}{N-1}} \Big)^{\frac{N-1}{N}},$$

and so from Cauchy-Schwarz on the left hand side and Hölder on the righ hand side

$$2\|u\|_{L^2(\mathbb{R}^N)} \|\nabla u\|_{L^2(\mathbb{R}^N)} \ge C_N \|u\|^2_{L^2(\mathbb{R}^N)} \frac{1}{|\Omega|^{\frac{1}{N}}},$$

which gives that

$$\lambda_1(\Omega) \ge \frac{C_N^2}{|\Omega|^{\frac{2}{N}}}.$$

Moreover,

$$\alpha(t^N m) = t^{-2}\alpha(m). \tag{5}$$

On the other hand, for every C, the minimizer Ω in (4) satisifies $2\lambda_1(\Omega) = CN|\Omega|$, as a consequence of the fact that the function $t \mapsto \lambda_1(t\Omega) + C|t\Omega|$ attains its minimum at $t = 1$. Consequently, $2\alpha(m) = CNm$, and so problems (3) and (4) are equivalent, as soon as

$$m = \Big(\frac{2\alpha(1)}{CN} \Big)^{\frac{N}{N+2}}. \tag{6}$$

3. Proof of the Faber-Krahn inequality

Proposition 3.1 (Step 1)**.** *Problem* (3) *has a solution, i.e. there exists a quasi-open set* Ω^* *such that* $|\Omega^*| = m$ *such that for every quasi-open set* $\Omega \subseteq \mathbb{R}^N$, $|\Omega| = m$ *we have*

$$\lambda_1(\Omega^*) \leq \lambda_1(\Omega).$$

Proof. The idea is very simple and is based on the concentration-compactness principle of P.L. Lions [19]. Assume that $(\Omega_n)_n$ is a minimizing sequence and let us denote $(u_n)_n$ a sequence of L^2-normalized, non negative, associated first eigenfunctions. Then the sequence $(u_n)_n \subseteq H^1(\mathbb{R}^N)$ is bounded. Since $H^1(\mathbb{R}^N)$ is not compactly embedded in $L^2(\mathbb{R}^N)$, for a subsequence (still denoted using the same index) one of the three possibilities below occurs:

i) compactness: $\exists y_n \in \mathbb{R}^N$ such that $u_n(\cdot + y_n) \longrightarrow u$ strongly in $L^2(\mathbb{R}_n)$ and weakly in $H^1(\mathbb{R}^N)$.

ii) dichotomy: there exists $\alpha \in (0,1)$ and two sequences $\{u_n^1\}, \{u_n^2\} \in H^1(\mathbb{R}^N)$, $\operatorname{supp} u_n^1 \cup \operatorname{supp} u_n^2 \subseteq \operatorname{supp} u_n$, such that

$$\|u_n - u_n^1 - u_n^2\|_{L^2(\mathbb{R}^N)} \to 0\,,$$
$$\int_{\mathbb{R}^N} |u_n^1|^2 dx \to \alpha \quad \int_{\mathbb{R}^N} |u_n^2|^2 dx \to 1-\alpha\,,$$
$$\operatorname{dist}(\operatorname{supp} u_n^1, \operatorname{supp} u_n^2) \to +\infty\,,$$

and

$$\liminf_{n\to\infty} \int_{\mathbb{R}^N} |\nabla u_n|^2 - |\nabla u_n^1|^2 - |\nabla u_n^2|^2 dx \geq 0\,. \tag{7}$$

iii) vanishing: for every $0 < R < \infty$

$$\lim_{n\to+\infty} \sup_{y\in\mathbb{R}^N} \int_{B(y,R)} u_n^2 dx = 0\,.$$

Situations ii) and iii) cannot occur to a minimizing sequence.

Indeed, situation ii) leads to searching the minimizer in a class of domains of measure strictly lower than m. First, we notice that the measures of the sets $\{u_n^1 > 0\}, \{u_n^2 > 0\}$ can not vanish as $n \to +\infty$. One of the two sequences $(\{u_n^1 > 0\})_n, (\{u_n^2 > 0\})_n$ has also to be minimizing for problem (3), in view of the algebraic inequality

$$\frac{a+b}{c+d} \geq \min\left\{\frac{a}{c}, \frac{b}{d}\right\}, \tag{8}$$

for positive numbers a, b, c, d. This allows to select (for a suitable subsequence, still denoted with the same index) either $(\{u_n^1 > 0\})_n$ or $(\{u_n^2 > 0\})_n$ as minimizing sequence of measure not larger than $m - \varepsilon$, for some $\varepsilon > 0$, in contradiction to the strict monotonicity (5).

Situation iii) can be excluded by an argument due to Lieb [18] which asserts that if iii) occurs then $\lambda_1(\Omega_n) \to +\infty$, in contradiction with the choice of a minimizing sequence (seealso [11]). For the sake of the clearness, we shall provide a short argument to prove this fact. Without restricting the generality, we can assume that $\lambda_1(\Omega_n) \le M_1$, for some $M_1 > 0$ independent on n. As $u_n \ge 0$, we get

$$-\Delta u_n \le \lambda_1(\Omega_n) u_n \text{ in } \mathcal{D}'(\mathbb{R}^N),$$

and so

$$-\Delta u_n \le C_n \lambda_1(\Omega_n)^{\frac{N}{4}+1} \le M_2.$$

This implies that for every $x_0 \in \mathbb{R}^N$, the function

$$u_n + M_2 \frac{|x - x_0|^2}{2N}$$

is subharmonic in $\mathbb{R}^N$. By direct computation we get that for every ball $B_x(r)$

$$u_n(x) \le \frac{\displaystyle\int_{B_x(r)} u dx}{|B_x(r)|} + \frac{M_2 r^2}{2N}.$$

Choosing first r such that $\frac{M_2 r^2}{2N} = \frac{1}{3\sqrt{m}}$, and than n large enough such that for every $x \in \mathbb{R}^N$

$$\frac{\displaystyle\int_{B_x(r)} u dx}{|B_x(r)|} \le \frac{1}{3\sqrt{m}},$$

we get that $\displaystyle\int_{\Omega_n} u_n^2 dx \le \frac{4}{9}$, in contradiction with the L^2-normalization of u_n.

Only situation i) can occur and this leads to the existence of an optimal domain which is a quasi-open set. Indeed, we consider the set $\Omega := \{u > 0\}$. Then, if we choose the representative defined by (1), the set Ω is quasi-open and has a measure less than or equal to m. This latter assertion is a consequence of the strong L^2 convergence of u_n which has as a consequence that (at least for a subsequence) $1_\Omega(x) \le \liminf_{n\to+\infty} 1_{\Omega_n}(x)$ a.e. $x \in \mathbb{R}^N$.

Moreover, we have

$$\lambda_1(\Omega) \le \frac{\displaystyle\int_\Omega |\nabla u|^2 dx}{\displaystyle\int_\Omega |u|^2 dx} \le \liminf_{n\to\infty} \frac{\displaystyle\int_\Omega |\nabla u_n|^2 dx}{\displaystyle\int_\Omega |u_n|^2 dx} = \liminf_{n\to\infty} \lambda_1(\Omega_n).$$

If necessary, taking a suitable dilation $\Omega^* = t\Omega$, for some $t \ge 1$ such that $|\Omega^*| = m$ and using the rescaling properties of λ_1, we conclude that Ω^* is a minimizing domain. $\square$

The proof of existence of a solution can be repeated in the same way for problem (4), the relationship between the minimizers of (3) and (4) being given by (6).

Proposition 3.2 (Step 2)**.** *The optimal set Ω^* is open and connected.*

Proof. Assuming one knows that Ω^* is open, the proof of the connectedness is immediate. Indeed, assume $\Omega^* = \Omega_1 \cup \Omega_2$ with Ω_1, Ω_2 open, disjoint and non-empty. Then $\lambda_1(\Omega^*)$ is either equal to $\lambda_1(\Omega_1)$ or to $\lambda_1(\Omega_2)$, hence one could find a set, say Ω_1, with the first Dirichlet eigenvalue equal to $\lambda_1(\Omega^*)$ but with measure strictly less than m. This is in contradiction to the strict monotonicity (5).

The proof of the openness has a technical issue and is using a local perturbation argument developed by Alt and Caffarelli in the context of free boundary problems [1, Lemma 3.2]. The idea is to prove that u^* is at least continuous, so that $\Omega^* = \{u^* > 0\}$ is an open set. We refer the reader to [26, Theorem 3.2] and [12, Proposition 1.1] for a complete description of the method below.

Let $R > 0$ and $x_0 \in \mathbb{R}^N$. We introduce the harmonic extension of u^* in $B_R(x_0)$, by

$$\tilde{u}(x) = u^*(x) \text{ in } \Omega \setminus B_R(x_0),$$
$$\Delta\tilde{u}(x) = 0 \text{ in } B_R(x_0).$$

In view of (4), we have

$$\lambda_1(\Omega^*) + C|\Omega^*| \leq \frac{\int |\nabla\tilde{u}|^2 dx}{\int |\tilde{u}|^2 dx} + C|\{\tilde{u} > 0\}|,$$

or

$$\int_{\Omega^*} |\nabla u^*|^2 dx \leq \frac{\int_{\Omega^*\setminus B_R(x_0)} |\nabla u|^2 dx + \int_{B_R(x_0)} |\nabla\tilde{u}|^2 dx}{1 + \int_{B_R(x_0)} (|\tilde{u}|^2 - |u^*|^2) dx} + C|B_R(x_0) \setminus \Omega^*|.$$

From the L^∞ bound of the eigenfunction, we get $|\int_{B_R(x_0)} (|\tilde{u}|^2 - |u^*|^2) dx| \leq M_3 R^N$ so that after easy computation and for R smaller than a suitable constant independent on the point, we get

$$\int_{B_R(x_0)} |\nabla u^*|^2 dx \leq \int_{B_R(x_0)} |\nabla\tilde{u}|^2 dx + M_4 R^N.$$

Since $\tilde{u}$ is harmonic and equal to u^* on ∂B_R, we get

$$\int_{B_R(x_0)} |\nabla u^*|^2 dx - \int_{B_R(x_0)} |\nabla\tilde{u}|^2 dx = \int_{B_R(x_0)} |\nabla(u^* - \tilde{u})|^2 dx.$$

Consequently, for every $0 < r < R$ we have

$$\int_{B_r(x_0)} |\nabla u^*|^2 dx \leq \int_{B_r(x_0)} |\nabla(u^* - \tilde{u})|^2 dx + \int_{B_r(x_0)} |\nabla\tilde{u}|^2 dx \leq$$
$$\leq \int_{B_R(x_0)} |\nabla(u^* - \tilde{u})|^2 dx + \left(\frac{r}{R}\right)^N \int_{B_R(x_0)} |\nabla\tilde{u}|^2 dx.$$

The last inequality is due to the fact that $|\nabla\tilde{u}|^2$ is subharmonic in $B_R(x_0)$, as a consequence of the harmonicity of $\tilde{u}$ in $B_R(x_0)$.

Finally, $\forall x_0 \in \mathbb{R}^N$, $\forall 0 < r < R \leq R_0$ we have

$$\int_{B_r(x_0)} |\nabla u^*|^2 dx \leq M_4 R^N + \left(\frac{r}{R}\right)^N \int_{B_R(x_0)} |\nabla u^*|^2 dx. \tag{9}$$

This inequality implies in a classical way that u^* is Hölder continuous. Indeed, one can prove in a first step that

$$\int_{B_r(x_0)} |\nabla u^*|^2 dx \leq M_5 r^{N-1}. \tag{10}$$

The proof of the passage from (9) to (10) is classical, see for instance [16, Lemma 2.1, Chapter 3]. The fact that u^* is Hölder continuous of order $\frac{1}{2}$ in $\mathbb{R}^N$ is a consequence of the Dirichlet growth theorem (see [16, Theorem 1.1, Chapter3]). For a detailed proof of all steps, we refer to [26] and the references therein.

□

Analyzing the optimality of Ω^* by comparison with the sets $\Omega^* \setminus \overline{B}_R(x_0)$, one can deduce that Ω^* satsfies an inner density property, so that has to be bounded. As well, by comparision with the set $\{u^* > \varepsilon\}$, with vanishing ε, one can find an upper bound of the generalized perimeter. Further analysis, leads to the smoothness of the boundary. The precise smoothness depends on the dimension, but we do not insist on this point since *openness* alone is enough for our purposes.

Proposition 3.3 (Step 3)**.** *The optimal set Ω^* has radial symmetry.*

Proof. As mentioned in the Introduction, the idea to prove radial symmetry for minimizers of integral functionals in $H^1(\mathbb{R}^N)$ has been used before (see, for instance, [20, 21]), while its usage in a geometrical setting appears in Steiner's arguments [25, 4]. Roughly speaking, if Ω^* is a minimizer then one can cut it by a hyperplane $\mathcal{H}$ in two pieces of equal measure. Up to a translation of Ω^*, we can assume that $\mathcal{H}$ is given by the equation $x_1 = 0$. Then both the left and right parts together with their respective reflections are admissible (they have the correct measure) and are also minimizers. Indeed, let u be a non-zero first eigenfunction on Ω^*. We put the indices l, r to the corresponding quantities on the left, right parts of Ω^*, respectively. We denote $\Omega_l = \Omega \cap \{x \in \mathbb{R}^N : x_1 \leq 0\}$.

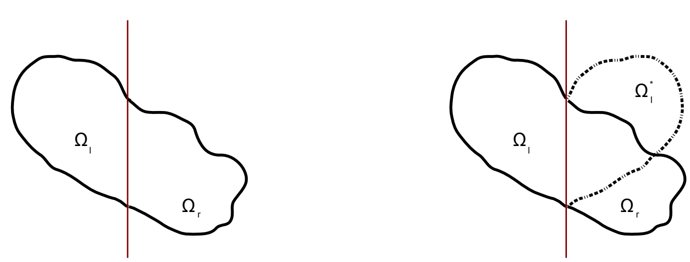

FIGURE 1. A region and one of its symmetrised counterparts with respect to a hyperplane

Then,

$$\lambda_1(\Omega^*) = \frac{\int_{\Omega_l^*} |\nabla u_l|^2 dx + \int_{\Omega_r^*} |\nabla u_r|^2 dx}{\int_{\Omega_l^*} u_l^2 dx + \int_{\Omega_r^*} u_r^2 dx}.$$

Using the algebraic inequality (8), we get (assuming for instance the left part is minimal)

$$\lambda_1(\Omega^*) \geq \frac{\int_{\Omega_l^*} |\nabla u_l|^2 dx}{\int_{\Omega_l^*} u_l^2 dx}.$$

Defining the reflection transformation $\mathcal{R} : \mathbb{R}^N \to \mathbb{R}^N$ $\mathcal{R}(x) = (-x_1, x_2, .., x_N)$, we introduce the reflected domain

$$\Omega^l := \Omega_l^* \cup \mathcal{R}\Omega_l^*,$$

together with the reflected test function

$$\overline{u}(x) = u_l(x) \text{ if } x_1 \leq 0 \text{ and } \overline{u}(x) = u_l(\mathcal{R}x) \text{ if } x_1 \geq 0.$$

Then $|\Omega^l| = m$, $u^l \in H_0^1(\Omega^l)$ (this is immediate using the density of C_0^∞-function in H^1) and we get

$$\lambda_1(\Omega^*) \geq \frac{\int_{\Omega_l^*} |\nabla u_l|^2 dx}{\int_{\Omega_l^*} u_l^2 dx} = \frac{\int_{\Omega^l} |\nabla \overline{u}|^2 dx}{\int_{\Omega^l} |\overline{u}|^2 dx} \geq \lambda(\Omega^l).$$

Relying on the minimality of Ω^* and on the inequalities above, we conclude that Ω^l is also a minimizer and that $\overline{u}$ is an eigenfunction on Ω^l. Finally, this means that u_l has two analytic extensions: one in the open set Ω_r which is u_r and another one in $\mathcal{R}\Omega_l$ which is $\overline{u}$. Using the maximum principle, there cannot be a point of the complement of Ω_r where u_r is vanishing which is interior for $\mathcal{R}\Omega_l$, and vice-versa. Finally, this implies that $\Omega_r = \mathcal{R}\Omega_l$ and so Ω^* is symmetric with respect to $\mathcal{H}$.

We continue the procedure with the hyperplanes $\{x_i = 0\}$, for $i = 2, \ldots, N$. At this point, up to a translation, we know that the optimal domain Ω^* is symmetric with respect to all the hyperplanes $\{x_i = 0\}$, for $i = 1, \ldots, N$. Now, we can continue the procedure with an arbitrary hyperplane passing thorough the origin (without any translation), since such a hyperplane divides Ω^* in two pieces of equal measure. We conclude that Ω^* has to be radially symmetric. ☐

Proposition 3.4 (Step 4)**.** *The optimal set Ω^* is the ball.*

Proof. Since the optimal set is connected, using the proposition above, we know it is an annulus. Precisely, for some $t \geq 0$ this annulus can be written

$$\Omega^* = K(0, t, r(t)) := \{x \in \mathbb{R}^N : t < |x| < r(t)\},$$

where $\omega_{N-1}(r^N(t) - t^N) = m$. In order to prove that the solution of the Faber-Krahn inequality is the ball, it is enough to study the mapping $t \mapsto \lambda_1(K(0, t, r(t)))$ and to prove it is increasing. Using the the shape derivative formula for λ_1 (and denoting u an L^2-normalized eigenfunction on $K(0,t,r(t))$, we get

$$\frac{d}{dt}\lambda_1(K(0,t,r(t)))) = \int_{\partial B(0)_t}\left(\frac{\partial u}{\partial n}\right)^2 d\mathcal{H}^{N-1} - \frac{t^{N-1}}{r^{N-1}(t)}\int_{\partial B(0)_{r(t)}}\left(\frac{\partial u}{\partial n}\right)^2 d\mathcal{H}^{N-1}.$$

Since u is radially symmetric, proving that $\frac{d}{dt}\lambda_1(K(0,t,r(t)))) > 0$ is equivalent to proving that $|u'(t)|^2 > |u'(r(t))|^2$ (in this notation u depends only on the radius).

The equation satisfied by the radial function u is

$$-u''(s) - \frac{N-1}{s}u'(s) = \lambda_1 u(s), \text{ on } (t, r(t)),$$

$$u(t) = u(r(t)) = 0.$$

Denoting $v(s) = |u'(s)|^2$, we get that

$$v'(s) = 2u'(s)u''(s) = -\frac{2(N-1)}{s}|u'(s)|^2 - 2\lambda_1 u'(s)u(s),$$

and summing between t and $r(t)$ we get

$$v(r(t)) - v(t) = -\frac{2(N-1)}{s}\int_t^{r(t)}|u'(s)|^2 ds < 0.$$

The last inequality is obvious since u is not constant, hence the mapping $t \mapsto \lambda_1(K(0,t,r(t)))$ is strictly increasing on $(0,+\infty)$. Consequently the ball, corresponding to $t = 0$, is the global minimizer. $\square$

4. Further remarks: higher order eigenvalues

For $k \geq 2$ one can also consider the isoperimetric problem

$$\min\{\lambda_k(\Omega) : \Omega \subseteq \mathbb{R}^N \text{ quasi-open}, |\Omega| = m\}. \tag{11}$$

When $k = 2$ the minimiser consists of two equal and disjoint balls of measure $m/2$, this being a direct consequence of the inequality for the first eigenvalue and one which was already considered by Krahn. For $k \geq 3$, and apart from numerical results (see, for instance, [2]) only few facts are known:

- A solution to problem (11) exists (let us call it Ω_k^*), it is a bounded set and has finite perimeter (see [6] and [22]).
- There exists an eigefunction u_k^* of the the optimal set Ω_k^*, corresponding to the k-th eigenvalue $\lambda_k(\Omega_k^*)$ which is a Lipschitz function (see [8]). Note that this information does not imply that the optimal set Ω_k^* is open, because Ω_k^* contains both the open set $\{u_k^* \neq 0\}$ and the nodal set $\{u_k^* = 0\}$, which has a structure, not yet completely understood.

Relying on the reflection argument, one can get some more information on the symmetry of Ω_k^*, depending on the dimension of the space. Roughly speaking, the larger the space dimension, the more symmetric must the minimizer be.

Assume that $u_1, .., u_k$ are L^2-normalized eigenfunctions corresponding to $\lambda_1(\Omega_k^*), .., \lambda_k(\Omega_k^*)$ such that

$$\forall i, j = 1, .., k, i \neq j, \quad \int u_i u_j dx = 0, \int \nabla u_i \nabla u_j = 0.$$

Problem (11) can be re-written as

$$\min_{|\Omega|=m} \min_{S_k \subset H_0^1(\Omega)} \max_{u \in S_k} \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega |u|^2 dx},$$

where S_k denotes any subspace of dimension k.

Assume $\mathcal{H}$ is a hyperplane splitting Ω_k^* into Ω^l and Ω^r such that

$$|\Omega^l| = |\Omega^r|,$$

$$\forall i, j = 1, .., k, i \neq j, \int_{\Omega^l} u_i u_j dx = 0,$$

$$\forall i, j = 1, .., k, i \neq j, \int_{\Omega^l} \nabla u_i \nabla u_j dx = 0,$$

$$\forall i = 1, .., k-1, \quad \frac{\int_{\Omega^l} |\nabla u_i|^2 dx}{\int_{\Omega^l} |u_i|^2 dx} = \lambda_j(\Omega_k^*).$$

Notice that the number of constraints equals k^2.

Assuming that

$$\frac{\int_{\Omega^l} |\nabla u_k|^2 dx}{\int_{\Omega^l} |u_k|^2 dx} \leq \frac{\int_{\Omega^r} |\nabla u_k|^2 dx}{\int_{\Omega^r} |u_k|^2 dx},$$

and reflecting Ω^l together with the functions $u_1|_{\Omega^l}, .., u_k|_{\Omega^l}$, we get

$$\lambda_k(\Omega^l \cup \mathcal{R}\Omega^l) \leq \max\{\lambda_{k-1}(\Omega_k^*), \frac{\int_{\Omega^l} |\nabla u_k|^2 dx}{\int_{\Omega^l} |u_k|^2 dx}\} \leq \lambda_k(\Omega_k^*).$$

Consequently, the set $\Omega^l \cup \mathcal{R}\Omega^l$ is also a minimizer and is symmetric with respect to $\mathcal{H}$.

If Ω_k^* were open, then we could conclude that $1_{\Omega_k^*} = 1_{\Omega^l \cup \mathcal{R}\Omega^l}$. Following the same arguments as [21, Theorem 2], we would get that the set Ω_k^* would be symmetric with respect to an affine subspace of dimension $k^2 - 1$. As Ω_k^* is not known to be open, we can only assert the existence of a minimizer which has $N - (k^2 - 1)$ hyperplanes of symmetry. In either case, we see that it is only for k equal to one that *full symmetry* may be obtained in this way.

Remark 4.1. Similar questions can be raised for more general functions of eigenvalues. Let $F : \mathbb{R}^k \to \mathbb{R}$ be a lower semicontinuous function, non decreasing in each variable. Then, the following problem

$$\min\{F(\lambda_1(\Omega), \ldots, \lambda_k(\Omega)) : \Omega \subseteq \mathbb{R}^N \text{ quasi-open}, |\Omega| = m\}. \tag{12}$$

has a solution. If moreover F is strictly increasing in at least one variable, then any optimal set has to be bounded with finite perimeter. We refer the reader to [13, 6, 22, 7]. A key argument is contained in the following surgery result, for which we refer the reader to [7].

Surgery result for the spectrum: for every $K > 0$, there exists $D, C > 0$ depending only on K and the dimension N, such that for every open (quasi-open, measurable) set $\tilde{\Omega} \subset \mathbb{R}^N$ there exists an open (quasi-open, measurable, respectively) set Ω satisfying

- the measure of Ω equals the measure of $\tilde{\Omega}$,
- diam $(\Omega) \leq D|\Omega|^{\frac{1}{N}}$ and $Per(\Omega) \leq \min\{Per(\tilde{\Omega}), C|\Omega|^{\frac{N-1}{N}}\}$,
- if for some $k \in \mathbb{N}$ it holds $\lambda_k(\tilde{\Omega}) \leq K|\tilde{\Omega}|^{-\frac{2}{N}}$, then $\lambda_i(\Omega) \leq \lambda_i(\tilde{\Omega})$ for all $i = 1, \ldots, k$.

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