Maximizing Neumann eigenvalues on rectangles

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Abstract

We obtain results for the spectral optimization of Neumann eigenvalues on rectangles in $\mathbb{R}^2$ with a measure or perimeter constraint. We show that the rectangle with measure 1 which maximizes the k'th Neumann eigenvalue converges to the unit square in the Hausdorff metric as $k \to \infty$. Furthermore, we determine the unique maximizer of the k'th Neumann eigenvalue on a rectangle with given perimeter.

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1 Introduction

Let Ω be an open or quasi-open set in Euclidean space $\mathbb{R}^m$ $(m = 2, 3, \cdots)$, with boundary $\partial\Omega$, and let $-\Delta_\Omega$ be the Dirichlet Laplacian acting in $L^2(\Omega)$. It is well known that if Ω has finite Lebesgue measure $|\Omega|$ then $-\Delta_\Omega$ has compact resolvent, and the spectrum of $-\Delta_\Omega$ is discrete and consists of eigenvalues $\lambda_1(\Omega) \le \lambda_2(\Omega) \le \cdots$ with $\lambda_j(\Omega) \to \infty$ as $j \to \infty$. The problem of minimizing the eigenvalues of the Dirichlet Laplacian over sets in $\mathbb{R}^m$ with a geometric constraint has been studied extensively. For example it was shown in [6] and [15] that for any $k \in \mathbb{N}$ the minimization problem

$$\inf\{\lambda_k(\Omega) : \Omega \text{ quasi-open in } \mathbb{R}^m,\ |\Omega| = c\} \tag{1.1}$$

has a bounded minimizer with finite perimeter. The celebrated Faber-Krahn and Krahn-Szegö inequalities assert that these minimizers are a ball with measure c for $k = 1$ or the union of two disjoint balls each with measure $c/2$ for $k = 2$ respectively [11]. It has been conjectured that if $m = 2, k = 3$ the disc with measure c is a minimizer. Little is known for higher values of k. Some bounds on the number of components of minimizers of (1.1) have been obtained in [5].

Other constraints than the measure have been considered in [9], [8] and [4]. For example, it was shown in [9] that a minimizer exists for the k'th Dirichlet eigenvalue under the constraint that the perimeter is fixed and the measure is finite. Existence in the planar case is particularly straightforward, since elements of minimizing sequences are convex and bounded uniformly in k. The latter fact allowed

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Bucur and Freitas to show in [8] that there exists a sequence of translates of these minimizers which converges to the disc in the Hausdorff metric. This phenomenon of an asymptotic shape has been established for a wide class of constraints in [4]. However, this class does not include the original measure constraint.

In a recent paper, [1], Antunes and Freitas proved the following asymptotic shape result with a measure constraint.

For $a \geq 1$, let

$$R_a = \{(x_1, x_2) : 0 < x_1 < a,\, 0 < x_2 < a^{-1}\} \tag{1.2}$$

be a rectangle with measure 1. The infimum of the variational problem

$$\lambda_k^* := \inf\{\lambda_k(R_a)\} \tag{1.3}$$

is achieved for some $a_k^* \geq 1$, and $\lim_{k\to\infty} a_k^* = 1$.

A heuristic explanation for this asymptotic shape result is the following (see [1]). For any rectangle in $\mathbb{R}^2$ with measure $|R|$ and perimeter $\mathrm{Per}(R)$ one has that

$$\lambda_k(R) = \frac{4\pi k}{|R|} + \frac{2\pi^{1/2}\mathrm{Per}(R)k^{1/2}}{|R|^{3/2}} + o(k^{1/2}),\ k \to \infty. \tag{1.4}$$

So if $|R| = 1$ then (1.4) suggests that the rectangle which minimizes $\lambda_k(R),\ k \to \infty$ is the one with minimal perimeter, i.e. the unit square. The main part of the proof in [1] is to show that the a_k^*'s are uniformly bounded. It is then possible to use well-known number theoretic results for the number of lattice points inside ellipses where the ratio of the axes remains bounded.

It is well known that the asymptotic formula (1.4) holds true for a wide class of planar domains with a smooth boundary which satisfy a billiard condition. This suggests that the asymptotic shape with fixed measure is a disc. The proof of this seems well beyond reach, even if an additional convexity constraint is imposed, [4].

In this paper we consider the maximization of Neumann eigenvalues. It is well known that if Ω is an open and bounded set in $\mathbb{R}^m$ with Lipschitz boundary then the spectrum of the Neumann Laplacian is discrete and consists of eigenvalues $0 = \mu_0(\Omega) < \mu_1(\Omega) \leq \mu_2(\Omega) \leq \cdots$ accumulating at infinity. Szegö and Weinberger showed that $\mu_1(\Omega) \leq \mu_1(\Omega^*)$, where Ω^* is the ball with the same measure as Ω, see [11]. It was shown in [10] that the union of two disjoint planar disks, each with measure $c/2$, achieves the supremum of $\mu_2(\Omega)$ in the class of simply connected sets in $\mathbb{R}^2$ with measure c. Nothing is known about the existence of maximizers for higher k (see for instance [7, Section 7.4]). In this paper we consider the problem of maximizing the k'th Neumann eigenvalue over all rectangles in $\mathbb{R}^2$ with fixed measure, and study the asymptotic behaviour as $k \to +\infty$.

Our main result is the following.

Theorem 1.1 (i) *Let $k \in \mathbb{N}$. The variational problem*

$$\mu_k^* := \sup\{\mu_k(R_b) : b \geq 1\}. \tag{1.5}$$

has a maximizing rectangle R_b with $b = b_k^*$.

(ii) *Any sequence of optimal rectangles $(R_{b_k^*})$ converges in the Hausdorff metric to the unit square as $k \to \infty$. Moreover there exists $\theta \in (\frac{1}{2}, 1)$ such that for $k \to \infty$,*

$$b_k^* = 1 + O(k^{(\theta-1)/4}). \tag{1.6}$$

(iii) *Let $\mu_k^* = \mu_k(R_{b_k^*})$. Then*

$$\mu_k^* = 4\pi k - 8(\pi k)^{1/2} + O(k^{(\theta+1)/4}),\ k \to \infty. \tag{1.7}$$

The exponent θ shows up in the remainder of Gauss' circle problem. It is known that for any $\epsilon > 0$, (see [13])

$$\theta = \frac{131}{208} + \epsilon. \tag{1.8}$$

The table below shows that the maximizing rectangles for $k = 4, 6, 10$ and $k = 15$ are not unique. The eigenvalues of the rectangle R_b are of the form

$$\mu_{p,q} = \frac{\pi^2 p^2}{b^2} + \pi^2 q^2 b^2, \tag{1.9}$$

for $p, q \in \mathbb{Z}^+ = \mathbb{N} \cup \{0\}$. The ordered list of real numbers $\{\mu_{p,q} : p \in \mathbb{Z}^+ \, q \in \mathbb{Z}^+\}$ are the eigenvalues $\{0 = \mu_0(R_b) < \mu_1(R_b) \leq \mu_2(R_b) \leq \cdots\}$ of the Neumann Laplacian on R_b. From the proof of Theorem 1.1(ii) we will see that the maximized k'th eigenvalue has multiplicity at least 2. In the table below we list the values of μ_k^* for $k = 1, \cdots, 15$ as well as the b_k^* and the pairs of maximizing modes which realize this maximum.

k	μ_k^*	b_k^*	maximizing pair of modes
1	π^2	1	$(1,0),(0,1)$
2	$2\pi^2$	$\sqrt{2}$	$(2,0),(0,1)$
3	$3\pi^2$	$\sqrt{3}$	$(3,0),(0,1)$
4	$4\pi^2$	2 or 1	$(4,0),(0,1)$ or $(2,0),(0,2)$
5	$5\pi^2$	$\sqrt{5}$	$(5,0),(0,1)$
6	$6\pi^2$	$\sqrt{6}$ or $\frac{1}{2}\sqrt{6}$	$(6,0),(0,1)$ or $(3,0),(0,2)$
7	$7\pi^2$	$\sqrt{7}$	$(7,0),(0,1)$
8	$\frac{18}{\sqrt{5}}\pi^2$	$\frac{\sqrt{2}}{2}5^{1/4}$	$(2,2),(3,0)$
9	$\frac{16\sqrt{3}}{3}\pi^2$	$\frac{2}{3^{1/4}}$	$(4,1),(0,2)$
10	$10\pi^2$	$\frac{1}{2}\sqrt{10}$ or $\sqrt{10}$	$(5,0),(0,2)$ or $(10,0),(0,1)$
11	$12\pi^2$	$\frac{2}{3}\sqrt{3}$	$(4,0),(0,3)$
12	$\frac{77}{20}\sqrt{10}\pi^2$	$\left(\frac{8}{5}\right)^{1/4}$	$(1,3),(3,2)$
13	$8\sqrt{3}\pi^2$	$3^{1/4}\sqrt{2}$	$(6,1),(0,2)$
14	$15\pi^2$	$\frac{1}{3}\sqrt{15}$	$(5,0),(0,3)$
15	$16\pi^2$	2 or 1	$(8,0),(0,2)$ or $(4,0),(0,4)$

We also see in the table above that the unit square is a maximizer for $k = 1, 4$ and $k = 15$. We conjecture that the unit square is a maximizer if the maximizing pair of modes are given by $(2^n, 0), (0, 2^n) : n \in \mathbb{Z}^+$. This gives that the unit square is a maximizer for μ_k if

$$k = \sum_{l \in \mathbb{Z}^+} \lfloor (4^n - l^2)_+^{1/2} \rfloor + 2^n - 1, \, n \in \mathbb{Z}^+. \tag{1.10}$$

The heuristic explanation of (1.6) is that for Neumann eigenvalues on a rectangle $R \subset \mathbb{R}^2$,

$$\mu_k(R) = \frac{4\pi k}{|R|} - \frac{2\pi^{1/2}\mathrm{Per}(R)k^{1/2}}{|R|^{3/2}} + o(k^{1/2}), \; k \to \infty, \tag{1.11}$$

so that the maximizing rectangle with measure $|R|$ is the one which minimizes its perimeter, i.e. the square with measure $|R|$.

The key ingredient in the proof of (1.6) in Section 2 below is to show that $\limsup_{k\to\infty} b_k^* < \infty$. This is more involved than the corresponding proof of Antunes and Freitas that $\limsup_{k\to\infty} a_k^* < \infty$ for the minimizing rectangles of the Dirichlet eigenvalues. In particular, it requires an a-priori bound on $\limsup b_k^*/k^{1/2}$ with sufficiently sharp constant. This is achieved in Lemma 2.3. The number theoretical estimates are also more involved, and will be given in Lemma 2.2. Finally, in Section 3, we discuss some variational problems for Neumann eigenvalues on rectangles with a perimeter constraint.

2 Proof of Theorem 1.1

Proof of Theorem 1.1(i). Fix $k \in \mathbb{N}$. Suppose that $\{R_{b^{(\ell)}}\}_{\ell\in\mathbb{N}}$ is a maximizing sequence for μ_k such that $b^{(\ell)} \to \infty$ as $\ell \to \infty$. Then, for sufficiently large ℓ,

$$\mu_k(R_{b^{(\ell)}}) = \frac{\pi^2 k^2}{(b^{(\ell)})^2}, \quad k \in \mathbb{N}, \tag{2.1}$$

and so $\mu_k(R_{b^{(\ell)}}) \to 0$ as $\ell \to \infty$. On the other hand, we have that $b = 1$ for a square and so $\mu_k \ge \pi^2 > 0$. This contradicts the assumption that $\{R_{b^{(\ell)}}\}_{\ell\in\mathbb{N}}$ is a maximizing sequence for μ_k. Thus any maximizing sequence $\{R_{b^{(\ell)}}\}_{\ell\in\mathbb{N}}$ for μ_k is such that $b^{(\ell)}$ remains bounded. Hence there exists a convergent subsequence, again denoted by $b^{(\ell)}$, such that $b^{(\ell)} \to b_k^*$ for some $b_k^* \in [1,\infty)$. Since $b \mapsto \mu_k(R_b)$ is continuous, $\mu_k(R_{b^{(\ell)}}) \to \mu_k(R_{b_k^*})$ as $\ell \to \infty$. Hence $R_{b_k^*}$ is a maximizer.

In order to prove Theorem 1.1(ii) we need three lemmas which will be given below.

Lemma 2.1 *Let $\nu_k = \mu_k(R_1)$ be the k'th positive Neumann eigenvalue for the unit square in $\mathbb{R}^2$. Then*

$$\nu_k \ge 4\pi k - 16(\pi k)^{1/2},\ k = 3, 4, \cdots. \tag{2.2}$$

Proof. For the unit square we have by (2.7) that

$$N(\nu; 1) = 2\left\lfloor \frac{\nu^{1/2}}{\pi} \right\rfloor + \left| \left\{(x, y) \in \mathbb{N}^2 : x^2 + y^2 \le \nu/\pi^2\right\} \right|. \tag{2.3}$$

For each lattice point in $\mathbb{N}^2$ satisfying $x^2 + y^2 \le \nu/\pi^2$ there exists an open lower left-hand square with vertices $(x, y), (x-1, y), (x-1, y-1), (x, y-1)$ inside the quarter circle with radius $\nu^{1/2}/\pi$ in the first quadrant. Hence

$$N(\nu; 1) \le \frac{\nu}{4\pi} + \frac{2\nu^{1/2}}{\pi}. \tag{2.4}$$

So for $\nu = \nu_k$ we have that

$$k \le \frac{\nu_k}{4\pi} + \frac{2\nu_k^{1/2}}{\pi}. \tag{2.5}$$

We note that (2.5) also holds in case ν_k has multiplicity larger than 1. Since the unit square tiles $\mathbb{R}^2$, we have by Pólya's Inequality, [16], that $\nu_k \le 4\pi k$. Hence

$$k \le \frac{\nu_k}{4\pi} + \frac{2(4\pi k)^{1/2}}{\pi}. \tag{2.6}$$

This implies (2.2). ∎

Lemma 2.2 *Define for all $\mu > 0$, $b > 0$ the counting function*

$$N(\mu; b) = \left| \left\{(x, y) \in (\mathbb{Z}^+)^2 \setminus \{(0, 0)\} : \frac{\pi^2 x^2}{b^2} + \pi^2 b^2 y^2 \le \mu \right\} \right|. \tag{2.7}$$

Then for all $\mu > 0$, $b > 0$ with $\frac{\mu^{1/2}}{b\pi} \ge 2$ we have that

$$N(\mu; b) \ge \frac{\mu}{4\pi} + \frac{b\mu^{1/2}}{2\pi} - \frac{b^{3/2}\mu^{1/4}}{(2\pi)^{1/2}} - 1. \tag{2.8}$$

To prove Lemma 2.2, we obtain a lower bound for the number of integer lattice points in $\mathbb{N}^2$ which are inside or on the ellipse

$$E(\mu) = \left\{(x, y) \in \mathbb{R}^2 : \frac{\pi^2 x^2}{b^2} + \pi^2 b^2 y^2 \le \mu \right\}. \tag{2.9}$$

Proof. For each $(x, y) \in E(\mu)$, we have that

$$x \leq \frac{b}{\pi}(\mu - \pi^2 b^2 y^2)_+^{1/2} = b^2 \left(\frac{\mu}{\pi^2 b^2} - y^2 \right)_+^{1/2}. \tag{2.10}$$

Then

$$\begin{aligned} N(\mu; b) &= \left\lfloor \frac{\mu^{1/2}}{\pi b} \right\rfloor + \left\lfloor \frac{b\mu^{1/2}}{\pi} \right\rfloor + \sum_{y \in \mathbb{N}} \left\lfloor b^2 \left(\frac{\mu}{\pi^2 b^2} - y^2 \right)_+^{1/2} \right\rfloor \\ &= \left\lfloor \frac{\mu^{1/2}}{\pi b} \right\rfloor + \left\lfloor \frac{b\mu^{1/2}}{\pi} \right\rfloor + \sum_{y=1}^{\lfloor \mu^{1/2}/(\pi b) \rfloor} \left\lfloor b^2 \left(\frac{\mu}{\pi^2 b^2} - y^2 \right)^{1/2} \right\rfloor \\ &\geq \left\lfloor \frac{\mu^{1/2}}{\pi b} \right\rfloor + \left\lfloor \frac{b\mu^{1/2}}{\pi} \right\rfloor + \sum_{y=1}^{\lfloor \mu^{1/2}/(\pi b) \rfloor} b^2 \left(\frac{\mu}{\pi^2 b^2} - y^2 \right)^{1/2} - \left\lfloor \frac{\mu^{1/2}}{\pi b} \right\rfloor \\ &= \left\lfloor \frac{b\mu^{1/2}}{\pi} \right\rfloor + b^2 \sum_{y=1}^{\lfloor \mu^{1/2}/(\pi b) \rfloor} \left(\frac{\mu}{\pi^2 b^2} - y^2 \right)^{1/2}. \end{aligned} \tag{2.11}$$

Let $R = \frac{\mu^{1/2}}{\pi b}$ and define $f(y) := (R^2 - y^2)^{1/2}$, $0 \leq y \leq R$. Then $\sum_{y=1}^{\lfloor R \rfloor} (R^2 - y^2)^{1/2}$ is the area of the rectangles which are inscribed in the first quadrant of the circle of radius R. Hence, we can re-write this as

$$\sum_{y=1}^{\lfloor R \rfloor} (R^2 - y^2)^{1/2} = \frac{\pi R^2}{4} - A, \tag{2.12}$$

where

$$A = \sum_{n=0}^{\lfloor R \rfloor - 1} \int_n^{n+1} (f(y) - f(n+1))dy + \int_{\lfloor R \rfloor}^R f(y)dy. \tag{2.13}$$

Since $\lfloor R \rfloor \leq y \leq R$ we have that

$$\begin{aligned} \int_{\lfloor R \rfloor}^R f(y)dy &= \int_{\lfloor R \rfloor}^R (R - y)^{1/2}(R + y)^{1/2}\, dy \\ &\leq (2R)^{1/2} \int_{\lfloor R \rfloor}^R (R - y)^{1/2}\, dy \\ &= \frac{2}{3}(R - \lfloor R \rfloor)^{3/2}(2R)^{1/2}. \end{aligned} \tag{2.14}$$

Since f is decreasing and concave we have that

$$f(y) \leq f(n) + (y - n)f'(n), \; n \leq y \leq n + 1. \tag{2.15}$$

Hence

$$\sum_{n=0}^{\lfloor R \rfloor - 1} \int_n^{n+1} (f(y) - f(n+1))dy \leq f(0) - f(\lfloor R \rfloor) + \frac{1}{2} \sum_{n=0}^{\lfloor R \rfloor - 1} f'(n). \tag{2.16}$$

Since

$$f'(y) = -\frac{y}{(R^2 - y^2)^{1/2}}, \tag{2.17}$$

$f'(0) = 0$, and $y \mapsto -f'(y)$ is increasing, we have that

$$\frac{1}{2} \sum_{n=0}^{\lfloor R \rfloor - 1} f'(n) \leq \frac{1}{2} \int_0^{\lfloor R \rfloor - 1} f'(y)dy = \frac{1}{2}(f(\lfloor R \rfloor - 1) - f(0)). \tag{2.18}$$

By (2.12)-(2.14), (2.16), and (2.18)

$$\sum_{y=1}^{\lfloor R\rfloor}(R^2-y^2)^{1/2} \geq \frac{\pi R^2}{4} - \frac{R}{2} + (R^2-\lfloor R\rfloor^2)^{1/2} - \frac{1}{2}(R^2-(\lfloor R\rfloor-1)^2)^{1/2} - \frac{2}{3}(R-\lfloor R\rfloor)^{3/2}(2R)^{1/2}. \tag{2.19}$$

Next note that

$$\frac{1}{2}(R^2-(\lfloor R\rfloor-1)^2)^{1/2} \leq \frac{1}{2}(R-\lfloor R\rfloor+1)^{1/2}(2R)^{1/2}, \tag{2.20}$$

and that for $R \geq 2$,

$$(R^2-\lfloor R\rfloor^2)^{1/2} \geq (R-\lfloor R\rfloor)^{1/2}\left(\frac{R+\lfloor R\rfloor}{2R}\right)^{1/2}(2R)^{1/2} \geq \left(\frac{5}{6}\right)^{1/2}(R-\lfloor R\rfloor)^{1/2}(2R)^{1/2}. \tag{2.21}$$

Let $\theta = R-\lfloor R\rfloor \in [0,1]$ and define $f:[0,1]\mapsto \mathbb{R}$ by

$$f(\theta) = -\left(\frac{5}{6}\right)^{1/2}\theta^{1/2} + \frac{1}{2}(\theta+1)^{1/2} + \frac{2}{3}\theta^{3/2}. \tag{2.22}$$

Then

$$f''(\theta) = \frac{1}{4}\left(\frac{5}{6}\right)^{1/2}\theta^{-3/2} - \frac{1}{8}(\theta+1)^{-3/2} + \frac{1}{2}\theta^{-1/2} > 0, \tag{2.23}$$

and so f is convex. Hence $f(\theta) \leq \max\{f(0), f(1)\} = \frac{1}{2}$. So for $R = \frac{\mu^{1/2}}{\pi b} \geq 2$ we have that

$$\begin{aligned} N(\mu;b) &\geq \left\lfloor \frac{b\mu^{1/2}}{\pi}\right\rfloor + b^2\left(\frac{\pi R^2}{4} - \frac{R}{2} - \frac{R^{1/2}}{2^{1/2}}\right) \\ &= \left\lfloor \frac{b\mu^{1/2}}{\pi}\right\rfloor + \frac{\mu}{4\pi} - \frac{b\mu^{1/2}}{2\pi} - \frac{b^{3/2}\mu^{1/4}}{(2\pi)^{1/2}} \\ &\geq \frac{\mu}{4\pi} + \frac{b\mu^{1/2}}{2\pi} - \frac{b^{3/2}\mu^{1/4}}{(2\pi)^{1/2}} - 1. \end{aligned} \tag{2.24}$$

∎

In the lemma below we obtain an a-priori upper bound on the b_k^* in terms of k.

Lemma 2.3 *As before we denote the length of the longest edge of a maximizing rectangle solving* (1.5) *by* b_k^*. *Then*

$$\limsup_{k\to\infty} \frac{b_k^*}{k^{1/2}} \leq 0.46359. \tag{2.25}$$

Proof. Define $c_k := \frac{b_k^*}{k^{1/2}}$. We shall bound c_k using the maximality of μ_k^* at $R_{b_k^*}$. We first note that

$$\limsup_{k\to\infty} c_k \leq \left(\frac{\pi}{4}\right)^{1/2}. \tag{2.26}$$

Indeed, we know by (1.9) that the eigenvalues of $R_{b_k^*}$ are of the form

$$\mu = \frac{\pi^2 p^2}{c_k^2 k} + \pi^2 q^2 c_k^2 k,$$

for $p, q \in \mathbb{Z}^+$. Choosing the pairs (p,q) in (1.9) as $(0,0), (1,0), \ldots, (k,0)$ we get that

$$\frac{\pi^2 k^2}{c_k^2 k} \geq \mu_k^* \geq \nu_k.$$

This gives by Lemma 2.1 that for $k \geq 3$,

$$c_k^2 \leq \frac{\pi^2 k}{4\pi k - 16(\pi k)^{1/2}},$$

which passing to the limit leads to (2.26).

Assume now that for some k (large), all the eigenvalues of $R_{b_k^*}$ up to index k are given by the pairs $(p,q) = (0,0), (1,0), \ldots, (k,0)$. If this is the case, we see that μ_k^* has to be (at least) double, and hence equal to some value of the form

$$\frac{\pi^2 p^2}{c_k^2 k} + \pi^2 q^2 c_k^2 k$$

for some $q \geq 1$. Indeed, if it is not double, then being simple, for a small variation of b around b_k^* it continues to be simple and we can perform the derivative of the mapping

$$b \mapsto \mu_k(R_b),$$

in b_k^*. This derivative equals $-\frac{2\pi^2 k^2}{(b_k^*)^3}$ which is not vanishing, in contradiction with the optimality of b_k^*.

So, either the first $k+1$ eigenvalues are not given by $(p,q) = (0,0), (1,0), \ldots, (k,0)$, or the value of μ_k is equal to some $\frac{\pi^2 p^2}{c_k^2 k} + \pi^2 q^2 c_k^2 k$, for $q \geq 1$. In both cases there exists some p such that one of the first $k+1$ eigenvalues is given by $(p,1)$. Let $\overline{p}$ be the lowest number such that

$$\frac{\pi^2 \overline{p}^2}{c_k^2 k} + \pi^2 c_k^2 k \geq 4\pi k - 16(\pi k)^{1/2}, \tag{2.27}$$

and $(\overline{p}, 1)$ does not produce an eigenvalue of the list $\mu_0, \ldots, \mu_k$.

Then all eigenvalues given by the pairs $(0,1), \ldots, (\overline{p}-1, 1)$ belong to the list $\mu_0, \ldots, \mu_k$. Now, we consider the eigenvalues given by the pairs

$$(0,0), (1,0), \ldots, (k - \overline{p} + 1, 0).$$

We conclude that the eigenvalue given by the last pair $(k - \overline{p} + 1, 0)$ is not smaller than μ_k. Consequently

$$\frac{\pi^2 (k - \overline{p} + 1)^2}{c_k^2 k} \geq \mu_k \geq 4\pi k - 16(\pi k)^{1/2}. \tag{2.28}$$

From (2.27) and (2.28) we get, respectively

$$\pi \overline{p} \geq c_k (4\pi k^2 - 16\pi^{1/2} k^{3/2} - \pi^2 c_k^2 k^2)^{1/2}$$

$$\pi(k - \overline{p} + 1) \geq c_k (4\pi k^2 - 16\pi^{1/2} k^{3/2})^{1/2}.$$

Adding the two inequalities, dividing by k and passing to the limit for $k \to \infty$, we get for any limit point $\alpha \in [0, (\pi/4)^{1/2}]$ of the sequence $(c_k)_k$

$$\pi \geq \alpha((4\pi)^{1/2} + (4\pi - \pi^2 \alpha^2)^{1/2}).$$

A numerical evaluation, gives that $\alpha \in [0, 0.46359]$. ∎

We now prove that $\limsup b_k^* < \infty$. Since (2.24) holds for all pairs (μ, b), it must also hold for all optimal pairs (μ_k^*, b_k^*). Furthermore we note that $\mu \mapsto N(\mu; b)$ is increasing. Then, μ_k^* being optimal and having finite multiplicity, we have for all $\epsilon \in (0, \nu_1)$ that

$$k - 1 \geq N(\mu_k^* - \epsilon; b_k^*) \geq N(\nu_k - \epsilon; b_k^*). \tag{2.29}$$

By Lemma 2.1 and Lemma 2.3 we have that for all $\epsilon > 0$ sufficiently small

$$\limsup_{k \to \infty} \frac{(\nu_k - \epsilon)^{1/2}}{\pi b_k^*} \geq 2. \tag{2.30}$$

So invoking Lemma 2.2, we obtain for all k sufficiently large

$$k-1 \geq N(\nu_k-\epsilon;b_k^*) \geq \frac{\nu_k-\epsilon}{4\pi}+\frac{b_k^*(\nu_k-\epsilon)^{1/2}}{2\pi}-\frac{(b_k^*)^{3/2}(\nu_k-\epsilon)^{1/4}}{(2\pi)^{1/2}}-1. \tag{2.31}$$

Rearranging terms we have that

$$\frac{4\pi k-\nu_k+\epsilon}{(\nu_k-\epsilon)^{1/2}} \geq 2b_k^*(1-(2\pi b_k^*)^{1/2}(\nu_k-\epsilon)^{-1/4}). \tag{2.32}$$

By Lemma 2.1 we conclude that

$$\limsup_{k\to\infty}(\nu_k-\epsilon)^{-1/2}(4\pi k-\nu_k+\epsilon) \leq 8. \tag{2.33}$$

On the other hand Lemma 2.3 gives that

$$\liminf_{k\to\infty}(1-(2\pi b_k^*)^{1/2}(\nu_k-\epsilon)^{-1/4}) \geq 1-\pi^{1/4}(0.46359)^{1/2}. \tag{2.34}$$

Putting (2.32),(2.33) and (2.34) together gives that

$$\limsup_{k\to\infty} b_k^* \leq \frac{4}{1-\pi^{1/4}(0.46359)^{1/2}} \leq 43. \tag{2.35}$$

Proof of Theorem 1.1(ii). Let

$$N_0(\mu;b) = \left|\left\{(x,y)\in\mathbb{Z}^2 : \frac{\pi^2x^2}{b^2}+\pi^2b^2y^2 \leq \mu\right\}\right|. \tag{2.36}$$

Then

$$N(\mu;b) = \frac{1}{4}N_0(\mu;b)+\frac{1}{2}\left\lfloor\frac{b\mu^{1/2}}{\pi}\right\rfloor+\frac{1}{2}\left\lfloor\frac{\mu^{1/2}}{\pi b}\right\rfloor-\frac{1}{4}. \tag{2.37}$$

We apply the identity above to the optimal pair (b_k^*,μ_k^*), and obtain that if μ_k^* has multiplicity Θ_k then

$$\begin{aligned} k+\Theta_k-1 = N(\mu_k^*;b_k^*) &= \frac{1}{4}N_0(\mu_k^*;b_k^*)+\frac{1}{2}\left\lfloor\frac{b_k(\mu_k^*)^{1/2}}{\pi}\right\rfloor+\frac{1}{2}\left\lfloor\frac{(\mu_k^*)^{1/2}}{b_k^*\pi}\right\rfloor-\frac{1}{4} \\ &\geq \frac{1}{4}N_0(\mu_k^*;b_k^*)+\frac{b_k^*(\mu_k^*)^{1/2}}{2\pi}+\frac{(\mu_k^*)^{1/2}}{2\pi b_k^*}-\frac{5}{4}. \end{aligned} \tag{2.38}$$

By (2.35) we have that the b_k^* are bounded uniformly in k. It is known by [13] that there exist constants $C<\infty$ and, for any $\epsilon>0$, $\frac{1}{2}<\theta<\frac{131}{208}+\epsilon$ such that

$$\frac{\mu}{\pi}+C\mu^{\theta/2}+1 \geq N_0(\mu;b) \geq \frac{\mu}{\pi}-C\mu^{\theta/2}. \tag{2.39}$$

So by (2.38) and (2.39) we conclude that

$$b_k^*+\frac{1}{b_k^*} \leq \frac{4\pi k-\mu_k^*}{2(\mu_k^*)^{1/2}}+\frac{\pi C}{2}(\mu_k^*)^{(\theta-1)/2}+\frac{2\pi\Theta_k}{(\mu_k^*)^{1/2}}+\frac{1}{2}, \tag{2.40}$$

where we have used that $\mu_k^* \geq \mu_1^* = \pi^2$. We observe that $\mu \mapsto \frac{4\pi k-\mu}{2\mu^{1/2}}+\frac{\pi C}{2\mu^{(1-\theta)/2}}+\frac{2\pi\Theta_k}{\mu^{1/2}}$ is decreasing. By the optimality of μ_k^* we have that

$$b_k^*+\frac{1}{b_k^*} \leq \frac{4\pi k-\nu_k}{2\nu_k^{1/2}}+\frac{\pi C}{2\nu_k^{(1-\theta)/2}}+\frac{2\pi\Theta_k}{\nu_k^{1/2}}+\frac{1}{2}. \tag{2.41}$$

By (2.37) and (2.39) we have that

$$k \le N(\nu_k;1) \le \frac{\nu_k}{4\pi} + \frac{\nu_k^{1/2}}{\pi} + \frac{C\nu_k^{\theta/2}}{4}. \tag{2.42}$$

It follows that

$$b_k^* + \frac{1}{b_k^*} \le 2 + O(k^{(\theta-1)/2}), \tag{2.43}$$

and

$$b_k^* = 1 + O(k^{(\theta-1)/4}). \tag{2.44}$$

This completes the proof of Theorem 1.1(ii). ∎

Proof of Theorem 1.1(iii). First we obtain a lower bound for μ_k^*. By its maximality we have that $\mu_k^* \ge \nu_k$, and so it suffices to obtain a lower bound for the latter. By (2.42) we have that

$$k \le \frac{\nu_k}{4\pi} + \left(\frac{4k}{\pi}\right)^{1/2} + O(k^{\theta/2}), \tag{2.45}$$

where we have used Pólya's Inequality $\nu_k \le 4\pi k$. This proves the lower bound in (1.7) since $(1+\theta)/4 > \theta/2$.

To prove the upper bound we have by (2.24) and (1.6) that

$$\begin{aligned} N(\mu_k^*; b_k^*) &\ge \frac{\mu_k^*}{4\pi} + \frac{b_k^*(\mu_k^*)^{1/2}}{2\pi} - O(k^{1/4}) \\ &\ge \frac{\mu_k^*}{4\pi} + (1 + O(k^{(\theta-1)/4}))\nu_k^{1/2}. \end{aligned} \tag{2.46}$$

By Lemma 2.1 we have that $\nu_k^{1/2} \ge (4\pi k)^{1/2} + O(1)$. This shows that

$$N(\mu_k^*; b_k^*) \ge \frac{\mu_k^*}{4\pi} + (4\pi k)^{1/2} + O(k^{(\theta+1)/4}). \tag{2.47}$$

We note that the multiplicity Θ_k of μ_k^* is equal to the number of lattice points in the first quadrant lying on the curve

$$\frac{\pi^2 x^2}{(b_k^*)^2} + \pi^2 (b_k^*)^2 y^2 = \mu_k^*. \tag{2.48}$$

The latter multiplicity is bounded by Theorem 1 in [14], and is of order $O(\ell^{2/3})$, where ℓ is the length of the curve defined in (2.48), which in turn equals $O((\mu_k^*)^{1/2}) = O(k^{1/2})$. So the multiplicity of μ_k^* is bounded by $O(k^{1/3})$. It follows by (2.47) that

$$O(k^{1/3}) + k \ge \frac{\mu_k^*}{4\pi} + (4\pi k)^{1/2} + O(k^{(\theta+1)/4}). \tag{2.49}$$

This completes the proof of Theorem 1.1(iii) since $\frac{1}{3} < (1+\theta)/4$.

3 Neumann eigenvalues with a perimeter constraint

In general, the problems of maximizing or minimizing μ_k under a perimeter constraint are ill-posed. In fact, it is not difficult to see that for every $c > 0$

$$\inf\{\mu_k(\Omega) : \Omega \text{ open, bounded with } \mathrm{Per}(\Omega) = c\} = 0, \tag{3.1}$$

$$\sup\{\mu_k(\Omega) : \Omega \text{ open, bounded with } \mathrm{Per}(\Omega) = c\} = +\infty. \tag{3.2}$$

Indeed, the k-th eigenvalue of a set Ω which is the disjoint union of $k+1$ balls is equal to 0, so that the infimum under (3.1) is attained trivially. One can also construct a minimizing sequence of connected sets where the k-th eigenvalue tends to zero. For example by connecting $k+1$ fixed disjoint balls with k tubes of vanishing width (see [2]), while controlling the overall perimeter by rescaling.

For the maximization problem, we construct the following example in $\mathbb{R}^2$. Let $\Lambda > 0$ be arbitrary, and let $l > 0$ be such that $l < \frac{c}{4}$, and $\frac{\pi^2}{l^2} \geq \Lambda$. Let Ω be the square with vertices $(0,0),(l,0),(l,-l),(0,-l)$. Then $\mu_1(\Omega) = \frac{\pi^2}{l^2}$. Consider the function $\phi : \mathbb{R} \to \mathbb{R}$ defined by $\phi(x) = C\sin(\frac{2\pi}{l}x)$, where C is such that $\int_0^l \sqrt{1+(\phi'(x))^2}dx = c-3l$. We replace the edge between the first two vertices by the graph of the function $\frac{1}{n}\phi(nx)$. In this way we construct a set $\Omega_{n,l}$ with $\mathrm{Per}(\Omega_{n,l}) = c$. The sets $\Omega_{n,l}$ satisfy a uniform cone condition so that $\mu_1(\Omega_{n,l}) \to \mu_1(\Omega) = \frac{\pi^2}{l^2}$ as $n \to +\infty$. Hence for all n sufficiently large $\mu_1(\Omega_{n,l}) \geq \frac{\pi^2}{2l^2} \geq \frac{\Lambda}{2}$. Since $\Lambda > 0$ was arbitrary the supremum under (3.2) is $+\infty$. The above example is easily extended to dimensions larger than 2. We refer to [3] for related constructions.

In this section we obtain some results for the following variational problems with a perimeter constraint.

$$\sup\{\mu_k(R) : R \text{ rectangle}, \mathrm{Per}(R) = 4\}, \tag{3.3}$$

and

$$\inf\{\mu_k(R) : R \text{ rectangle}, \mathrm{Per}(R) = 4\}. \tag{3.4}$$

We let $R_{a,b}$ denote a rectangle in $\mathbb{R}^2$ of side-lengths $a, b > 0$ so that $\mathrm{Per}(R_{a,b}) = 2(a+b)$.

3.1 Analysis of the maximization problem (3.3).

Our main theorem is the following.

Theorem 3.1 *For $k \in \mathbb{N}$, there is a unique maximizing rectangle $R_{a_k^*,b_k^*}$ with $a_k^* = \frac{2}{k+1} \in (0,1]$ and $b_k^* = 2 - a_k^*$ such that*

$$\mu_k(R_{a_k^*,2-a_k^*}) = \frac{\pi^2 k^2}{(2-a_k^*)^2} = \frac{\pi^2}{(a_k^*)^2} = \frac{\pi^2(k+1)^2}{4}, \tag{3.5}$$

i.e. $\mu_k^* = \mu_k(R_{a_k^*,2-a_k^*})$ is realized by the modes $(k,0)$ and $(0,1)$.

Proof. We first show that for every $k \geq 0$, problem (3.3) has a solution.

Fix $k \in \mathbb{Z}^+$ and let $(R_{a_n,2-a_n})_n$, $a_n \in (0,1]$, be a maximizing sequence of rectangles for μ_k. By taking a subsequence if necessary we may assume that $(a_n)_n$ is monotone. Let $a_k^* = \lim_{n\to\infty} a_n$. Now, we claim that

$$a_k^* \geq \frac{2}{k+1}. \tag{3.6}$$

Suppose to the contrary that $a_k^* < \frac{2}{k+1}$. Then we have that

$$\frac{\pi^2 k^2}{(2-a_k^*)^2} < \frac{\pi^2}{(a_k^*)^2}, \tag{3.7}$$

where the right-hand side above is $+\infty$ in the case that $a_k^* = 0$. Hence, the k eigenvalues which are given by the pairs $(1,0),(2,0),\cdots,(k,0)$ are smaller than the eigenvalue which is given by the pair $(0,1)$. So $\mu_k^* = \frac{\pi^2k^2}{(2-a_k^*)^2}$. However, if we consider $\tilde{a}_k \in (a_k^*, \frac{2}{k+1})$, then

$$\mu_k(\tilde{a}_k) = \frac{\pi^2 k^2}{(2-\tilde{a}_k)^2} > \frac{\pi^2 k^2}{(2-a_k^*)^2}, \tag{3.8}$$

which contradicts the maximality of μ_k^*. This proves (3.6).

For $a_k = \frac{2}{k+1}$, we have that

$$\mu_k(R_{a_k,2-a_k}) = \frac{\pi^2 k^2}{(2-\frac{2}{k+1})^2} = \frac{\pi^2(k+1)^2}{4}. \tag{3.9}$$

So, by maximality, we deduce that

$$\mu_k(R_{a_k^*,2-a_k^*}) \geq \frac{\pi^2(k+1)^2}{4}. \tag{3.10}$$

Let

$$\mu_k(R_{a_k^*,2-a_k^*}) = \frac{\pi^2 p^2}{(2-a_k^*)^2} + \frac{\pi^2 q^2}{(a_k^*)^2}, \tag{3.11}$$

for some $(p,q) \in (\mathbb{Z}^+)^2$, $p+q \leq k$.

Below we show that $q \leq 2$. Suppose to the contrary that $q \geq 3$. Then, by Pólya's Inequality and since $a_k^* \in (0,1]$, we have that

$$\frac{9\pi^2}{(a_k^*)^2} \leq \mu_k(R_{a_k^*,2-a_k^*}) \leq \frac{4\pi k}{a_k^*(2-a_k^*)} \leq \frac{4\pi k}{a_k^*}, \tag{3.12}$$

which implies that

$$a_k^* \geq \frac{9\pi}{4k}. \tag{3.13}$$

Hence, we have that

$$\mu_k^* \leq \frac{4\pi k}{a_k^*} \leq \frac{16k^2}{9} < \frac{\pi^2(k+1)^2}{4}. \tag{3.14}$$

This contradicts (3.10). So, for all $k \in \mathbb{Z}^+$, μ_k^* has $q \leq 2$.

Now we consider the case where $q = 2$, and note that

$$\frac{4\pi^2}{(a_k^*)^2} > \frac{\pi^2}{(2-a_k^*)^2} + \frac{\pi^2}{(a_k^*)^2}, \tag{3.15}$$

since $a_k^* \in (0,1]$. This shows that the eigenvalues given by the pairs $(0,1)$ and $(1,1)$ are strictly smaller than the one given by the pair $(0,2)$. Below we will show that the eigenvalues given by the pairs $(0,0),(1,0),\cdots,(k-2,0)$ are also strictly smaller than the eigenvalue given by the $(0,2)$ pair. By (3.10) and by Pólya's Inequality, we have that

$$\frac{\pi^2(k+1)^2}{4} \leq \mu_k(R_{a_k^*,2-a_k^*}) \leq \frac{4\pi k}{a_k^*(2-a_k^*)}, \tag{3.16}$$

which implies that

$$a_k^*(2-a_k^*) \leq \frac{16}{\pi(k+1)} \tag{3.17}$$

Since $a_k^*(2-a_k^*) \leq 1$, we see that (3.17) does not give any information about a_k^* for $k = 1,2,3,4$. We first consider the case $k \geq 5$. By solving (3.17), and taking into account that $a_k^* \leq 1$, we have that

$$a_k^* \leq 1 - \sqrt{1-(16/\pi(k+1))}. \tag{3.18}$$

We wish to show that

$$\frac{4\pi^2}{(a_k^*)^2} > \frac{\pi^2(k-2)^2}{(2-a_k^*)^2}. \tag{3.19}$$

This is equivalent to showing that $a_k^* < \frac{4}{k}$. The latter is clearly satisfied if $1-\sqrt{1-(16/\pi(k+1))} < \frac{4}{k}$. After elementary arithmetic, we see that this is equivalent to

$$k > \frac{\pi}{\pi-2} + \frac{2\pi}{(\pi-2)k}. \tag{3.20}$$

Since $k \geq 5$, we have that the right-hand side of (3.20) is bounded from above by $\frac{7\pi}{5(\pi-2)} < 5$. So (3.19) holds for $k \geq 5$. So the eigenvalues which are given by the pairs $(0,0),(1,0),\cdots,(k-2,0),(0,1),(1,1)$

are all strictly smaller than the one which is given by the pair $(0,2)$, and there are $k+1$ of them. Hence μ_k^* cannot have $q=2$. Thus $q=0$ or $q=1$.

Either $q=0$ and $p=k$, $\mu_k^*=\frac{\pi^2k^2}{(2-a_k^*)^2}$ and the first $k+1$ eigenvalues are given by the pairs $(0,0),(1,0),\cdots,(k,0)$. In this case, μ_k^* cannot be simple. Otherwise the derivative of the mapping $a\mapsto\mu_k(R_{a,2-a})$ with respect to a would be non-vanishing as before, thus contradicting the maximality of μ_k^*. Hence $\mu_k^*=\frac{\pi^2}{(a_k^*)^2}$, i.e. μ_k^* is realized by the modes $(k,0)$ and $(0,1)$.

Or one of the first $k+1$ eigenvalues is given by a pair $(p,1),p\in\mathbb{Z}^+$. Let $\bar{p}$ be the smallest number such that

$$\frac{\pi^2\bar{p}^2}{(2-a_k^*)^2}+\frac{\pi^2}{(a_k^*)^2}>\mu_k^*. \tag{3.21}$$

Then all eigenvalues given by the pairs $(0,1),(1,1),\cdots,(\bar{p}-1,1)$ are in the list

$$\mu_0(R_{a_k^*,2-a_k^*}),\mu_1(R_{a_k^*,2-a_k^*}),\cdots,\mu_k(R_{a_k^*,2-a_k^*}).$$

By considering the eigenvalues given by the pairs $(0,0),(1,0),\cdots,(k-\bar{p}+1,0)$, we deduce that

$$\frac{\pi^2(k-\bar{p}+1)^2}{(2-a_k^*)^2}\geq\mu_k^*\geq\frac{\pi^2(k+1)^2}{4}. \tag{3.22}$$

Thus we have that

$$\bar{p}\leq\frac{1}{2}(k+1)a_k^*, \tag{3.23}$$

which, together with (3.18), gives that

$$\begin{aligned}\bar{p}&\leq\frac{1}{2}(k+1)(1-\sqrt{1-(16/\pi(k+1))})\\&=\frac{8}{\pi}\left(1+\left(1-\frac{16}{\pi(k+1)}\right)^{1/2}\right)^{-1}.\end{aligned} \tag{3.24}$$

The right-hand side of (3.24) is decreasing in k. So for $k\geq5$ have that the right-hand side of (3.24) is bounded from above by $\frac{8}{\pi}\left(1+\left(1-\frac{8}{3\pi}\right)^{1/2}\right)^{-1}<2$. Hence $\bar{p}=1$, since $\bar{p}\in\mathbb{N}$. Therefore $p=0$. So $p=0$ and the first $k+2$ eigenvalues are given by the pairs $(0,0),(1,0),\cdots,(k,0),(0,1)$. Then, as before, $\mu_k^*=\frac{\pi^2k^2}{(2-a_k^*)^2}=\frac{\pi^2}{(a_k^*)^2}$, since in either case μ_k^* cannot be simple, i.e. μ_k^* is realized by the modes $(k,0)$ and $(0,1)$.

It remains to deal with the cases $k=1,2,3,4$.

Let $a_1\in(0,1]$. Then

$$\mu_1(R_{a_1,2-a_1})=\frac{\pi^2p^2}{(2-a_1)^2}+\frac{\pi^2q^2}{a_1^2} \tag{3.25}$$

for either the pair $(1,0)$ or the pair $(0,1)$. Since $a_1\in(0,1]$, $\mu_1(R_{a_1,2-a_1})=\frac{\pi^2}{(2-a_1)^2}$. This is maximal for $a_1=1$. Hence $\mu_1^*=\pi^2$ with $a_1^*=1$ and corresponding modes $(1,0),(0,1)$.

Let $a_2\in(0,1]$. Then

$$\mu_2(R_{a_2,2-a_2})=\frac{\pi^2p^2}{(2-a_2)^2}+\frac{\pi^2q^2}{a_2^2}, \tag{3.26}$$

with $p\leq2,q\leq2$ and $p+q\leq2$. The possible pairs which give $\mu_2(R_{a_2,2-a_2})$ are

$$(2,0),(1,0),(2,1),(1,1),(0,1),(0,2).$$

Now $\mu_1(R_{a_1,2-a_1})=\frac{\pi^2}{(2-a_1)^2}$ is given by the pair $(1,0)$. So $\mu_2(R_{a_2,2-a_2})$ must be given by either $(2,0)$ or $(0,1)$. We have that

$$\frac{4\pi^2}{(2-a_2)^2}\leq\frac{\pi^2}{a_2^2}\iff a_2\leq\frac{2}{3}, \tag{3.27}$$

hence

$$\mu_2(R_{a_2,2-a_2}) = \begin{cases} \frac{4\pi^2}{(2-a_2)^2}, & 0 < a_2 \le \frac{2}{3}, \\ \frac{\pi^2}{a_2^2}, & \frac{2}{3} \le a_2 \le 1. \end{cases} \tag{3.28}$$

Thus we obtain that $\mu_2^* = \frac{9\pi^2}{4}$ with $a_2^* = \frac{2}{3}$ and corresponding modes $(2,0),(0,1)$.

Let $a_3 \in (0,1]$. Then

$$\mu_3(R_{a_3,2-a_3}) = \frac{\pi^2 p^2}{(2-a_3)^2} + \frac{\pi^2 q^2}{a_3^2}, \tag{3.29}$$

with $p \le 3, q \le 2$ and $p+q \le 3$. The possible pairs which give $\mu_3(R_{a_3,2-a_3})$ are

$$(3,0),(2,0),(1,0),(2,1),(1,1),(0,1),(1,2),(0,2).$$

For $0 < a_2 \le \frac{2}{3}$, $\mu_2(R_{a_2,2-a_2}) = \frac{4\pi^2}{(2-a_2)^2}$ is given by the pair $(2,0)$. So for $0 < a_3 \le \frac{2}{3}$, $\mu_3(R_{a_3,2-a_3})$ must be given by either $(3,0)$ or $(0,1)$. We have that

$$\frac{9\pi^2}{(2-a_3^2)^2} \le \frac{\pi^2}{a_3^2} \iff a_3 \le \frac{1}{2}. \tag{3.30}$$

In addition, for $\frac{2}{3} \le a_2 \le 1$, $\mu_2(R_{a_2,2-a_2}) = \frac{\pi^2}{a_2^2}$ is given by the pair $(0,1)$. So for $\frac{2}{3} \le a_3 \le 1$, $\mu_3(R_{a_3,2-a_3})$ must be given by either $(2,0)$ or $(1,1)$. We have that

$$\frac{4\pi^2}{(2-a_3)^2} \le \frac{\pi^2}{(2-a_3)^2} + \frac{\pi^2}{a_3^2} \iff a_3 \le \sqrt{3}-1. \tag{3.31}$$

Thus, we obtain that

$$\mu_3(R_{a_3,2-a_3}) = \begin{cases} \frac{9\pi^2}{(2-a_3)^2}, & 0 < a_3 \le \frac{1}{2} \\ \frac{\pi^2}{a_3^2}, & \frac{1}{2} \le a_3 \le \frac{2}{3} \\ \frac{4\pi^2}{(2-a_3)^2}, & \frac{2}{3} \le a_3 \le \sqrt{3}-1 \\ \frac{\pi^2}{(2-a_3)^2} + \frac{\pi^2}{a_3^2}, & \sqrt{3}-1 \le a_3 \le 1. \end{cases} \tag{3.32}$$

We deduce that $\mu_3^* = 4\pi^2$ with $a_3^* = \frac{1}{2}$ and corresponding modes $(3,0),(0,1)$.

Let $a_4 \in (0,1]$. Then

$$\mu_4(R_{a_4,2-a_4}) = \frac{\pi^2 p^2}{(2-a_4)^2} + \frac{\pi^2 q^2}{a_4^2}, \tag{3.33}$$

with $p \le 4, q \le 2$ and $p+q \le 4$. The possible pairs which give $\mu_4(R_{a_4,2-a_4})$ are

$$(4,0),(3,0),(2,0),(1,0),(3,1),(2,1),(1,1),(0,1),(2,2),(1,2),(0,2).$$

For $0 < a_3 \le \frac{1}{2}$, $\mu_3(R_{a_3,2-a_3}) = \frac{9\pi^2}{(2-a_3)^2}$ is given by the pair $(3,0)$. So for $0 < a_4 \le \frac{1}{2}$, $\mu_4(R_{a_4,2-a_4})$ must be given by either $(4,0)$ or $(0,1)$. We have that

$$\frac{16\pi^2}{(2-a_4)^2} \le \frac{\pi^2}{a_4^2} \iff a_4 \le \frac{2}{5}. \tag{3.34}$$

In addition, for $\frac{1}{2} \le a_2, a_3 \le \frac{2}{3}$, $\mu_3(R_{a_3,2-a_3}) = \frac{\pi^2}{a_3^2}$ is given by the pair $(0,1)$, and $\mu_2(R_{a_2,2-a_2}) = \frac{4\pi^2}{(2-a_2)^2}$ is given by the pair $(2,0)$. So for $\frac{1}{2} \le a_4 \le \frac{2}{3}$, $\mu_4(R_{a_4,2-a_4})$ must be given by either $(3,0),(1,1)$ or $(0,2)$. We have that

$$\frac{9\pi^2}{(2-a_4)^2} \le \frac{\pi^2}{(2-a_4)^2} + \frac{\pi^2}{a_4^2} \iff a_4 \le \frac{2}{7}(\sqrt{8}-1), \tag{3.35}$$

$$\frac{9\pi^2}{(2-a_4)^2} \le \frac{4\pi^2}{a_4^2} \iff a_4 \le \frac{4}{5} \tag{3.36}$$

$$\frac{\pi^2}{(2-a_4)^2} + \frac{\pi^2}{a_4^2} \le \frac{4\pi^2}{a_4^2} \Leftarrow \quad a_4 \in (0,1]. \tag{3.37}$$

For $\frac{2}{3} \le a_3 \le \sqrt{3}-1$, $\mu_3(R_{a_3,2-a_3}) = \frac{4\pi^2}{(2-a_3)^2}$ is given by the pair $(2,0)$. Similarly to the above, for $\frac{2}{3} \le a_4 \le \sqrt{3}-1$, $\mu_4(R_{a_4,2-a_4})$ must be given by either $(3,0)$ or $(1,1)$.

Finally, for $\sqrt{3}-1 \le a_3 \le 1$, $\mu_3(R_{a_3,2-a_3}) = \frac{\pi^2}{(2-a_3)^2} + \frac{\pi^2}{a_3^2}$ is given by the pair $(1,1)$. So for $\sqrt{3}-1 \le a_4 \le 1$, $\mu_4(R_{a_4,2-a_4})$ must be given by $(2,0)$, as $(1,0),(0,1),(1,1)$ have already been used for this range of a by μ_1, μ_2, μ_3 respectively.

Hence, we obtain that

$$\mu_4(R_{a_4,2-a_4}) = \begin{cases} \frac{16\pi^2}{(2-a_4)^2}, & 0 < a_4 \le \frac{2}{5} \\ \frac{\pi^2}{a_4^2}, & \frac{2}{5} \le a_4 \le \frac{1}{2} \\ \frac{9\pi^2}{(2-a_4)^2}, & \frac{1}{2} \le a_4 \le \frac{2}{7}(\sqrt{8}-1) \\ \frac{\pi^2}{(2-a_4)^2} + \frac{\pi^2}{a_4^2}, & \frac{2}{7}(\sqrt{8}-1) \le a_4 \le \sqrt{3}-1 \\ \frac{4\pi^2}{(2-a_4)^2} & \sqrt{3}-1 \le a_4 \le 1. \end{cases} \tag{3.38}$$

Thus $\mu_4^* = \frac{25\pi^2}{4}$ with $a_4^* = \frac{2}{5}$ and corresponding modes $(4,0),(0,1)$. ∎

3.2 Analysis of the minimization problem (3.4).

Our main result is the following.

Theorem 3.2 (i) *If $k = 1$ then variational problem* (3.4) *does not have a minimizer, and the infimum equals $\frac{\pi^2}{4}$.*

(ii) *If $k \ge 2$ then variational problem* (3.4) *does have a minimizer.*

(iii) *If $k \ge 2$ and $R_{a_k^*,b_k^*}$, $a_k^* \in (0,1], b_k^* = 2 - a_k^*$ are minimizers then*

$$\lim_{k\to\infty} a_k^* = 1, \tag{3.39}$$

i.e. any sequence of optimal rectangles for Problem (3.4) *converges to the unit square, as $k \to \infty$.*

Proof. If $k = 1$ then $(R_{\frac{1}{n},2-\frac{1}{n}})_n$ is minimizing and collapses to a segment of length 2. This proves the assertion under (i).

To prove (ii) we fix $k \ge 2$, and consider a minimizing sequence for problem (3.4), $(R_{a_n,2-a_n})_n$ with $a_n \in (0,1]$. By taking a subsequence if necessary, $(a_n)_n$ converges and, without loss of generality, we may assume that it is a monotone sequence. Then $(a_n)_n$ cannot converge to 0. If $a_n \to 0$, then for n large enough such that $0 < a_n \le \frac{2}{k+1}$, we have that

$$\mu_k(R_{a_n,2-a_n}) = \frac{\pi^2 k^2}{(2-a_n)^2} \to \frac{\pi^2 k^2}{4}. \tag{3.40}$$

However, by minimality and by Pólya's Inequality, we have that

$$\mu_k(R_{a_n,2-a_n}) \le \nu_k \le 4\pi k. \tag{3.41}$$

Clearly, this inequality leads to a contradiction as soon as $\frac{\pi^2 k^2}{4} > 4\pi k$. That is the case for $k \ge 6$. So, for $k \ge 6$, $a_n \to a_k^* > 0$, which gives an optimal rectangle, $R_{a_k^*,2-a_k^*}$.

Similarly to Section (3.1) we obtain the values of $\mu_k^*(R_{a_k^*,2-a_k^*})$ for $k = 2,3,4,5$ by direct computation. In the table below, we list these values as well as the corresponding values of a_k^* and the minimizing modes.

k	μ_k^*	a_k^*	minimizing modes
2	π^2	1	(1,0),(0,1)
3	$2\pi^2$	1	(1,1)
4	$\frac{2}{3}\pi^2(2+\sqrt{3})$	$\sqrt{3}-1$	(2,0),(1,1)
5	$4\pi^2$	1	(2,0),(0,2)

We note that a degenerating sequence of rectangles $R_{a_2^{(n)},2-a_2^{(n)}}$ with $a_2^{(n)} \to 0$, gives $\mu_2(a_2^{(n)}) \to \pi^2$. In addition, we remark that μ_3^* has only one minimizing mode $(1,1)$. By considering the derivative of the function $\frac{\pi^2}{(2-a)^2} + \frac{\pi^2}{a^2}$ with respect to a, we see that the point $a = 1$ is a minimum point. This is due to the fact that for the mode $(1,1)$ it is possible to obtain a vanishing derivative.

To prove assertion (iii) of the theorem we note that by minimality and Pólya's Inequality,

$$\mu_k(R_{a_k^*,2-a_k^*}) \le \nu_k \le 4\pi k. \tag{3.42}$$

Recall that if R_{a_1,b_1}, R_{a_2,b_2} are two rectangles such that $a_1 \le a_2$ and $b_1 \le b_2$, then for every $k \ge 0$ $\mu_k(R_{a_1,b_1}) \ge \mu_k(R_{a_2,b_2})$. The latter is a direct consequence of the expression of the eigenvalues on rectangles. Assume for some subsequence (still denoted with the same index k) that $a_k^* \to \alpha$. Then, for every $\delta > 0$, there exists K_δ such that for $k \ge K_\delta$ we have that

$$R_{a_k^*,2-a_k^*} \subset R_{\alpha+\delta,2-\alpha+\delta}. \tag{3.43}$$

We have that

$$\mu_k(R_{\alpha+\delta,2-\alpha+\delta}) \le \mu_k(R_{a_k^*,2-a_k^*}) \le \nu_k \le 4\pi k. \tag{3.44}$$

Using the Weyl asymptotic on $R_{\alpha+\delta,2-\alpha+\delta}$, and letting $k \to \infty$, we obtain

$$\frac{4\pi}{(\alpha+\delta)(2-\alpha+\delta)} \le 4\pi. \tag{3.45}$$

By subsequently letting $\delta \to 0$, we obtain that $\alpha(2-\alpha) \ge 1$, which leads to $\alpha = 1$. Hence, $\lim_{k\to\infty} a_k^* = 1$, and this limit is independent of the subsequence (a_k^*).

∎

Remark If $m \ge 3$ then problem (3.4) with fixed k does not have a solution, since a sequence of cuboids with one very long edge has vanishing k-th eigenvalue.

Remark In order to analyze problem (3.3), we first observe that for every $k \ge 1$ and every $m \ge 2$ the problem

$$\max\{\mu_k(R) : R \text{ cuboid}, R \subseteq \mathbb{R}^m,\ |R| = 1\}, \tag{3.46}$$

has a solution. Indeed, if a maximizing sequence is degenerating, then one of the edges of the cuboid is vanishing and so another one is blowing up. This second phenomenon produces vanishing eigenvalues, so it is excluded.

Now, concerning problem (3.3) in $\mathbb{R}^m$, $m \ge 3$, we claim that there exists a solution. Indeed, a maximizing sequence of cuboids cannot have two (or more) vanishing edges, since this implies that another edge is blowing up, so the k-th eigenvalue is vanishing. There are only two possibilities: either there is convergence to a non-degenerate cuboid, or (only) one edge is vanishing. In the latter case, for a sufficiently short edge, the eigenvalues of the cuboid will be given by the eigenvalues of the $(m-1)$-dimensional complement cuboid which satisfies a volume constraint. That is, if $\left(R_{a_1^{(n)},\cdots,a_m^{(n)}}\right)_n$ is a maximizing sequence of cuboids such that for all $i \in \{1,\cdots,m\}$, $a_i^{(n)} \to a_i$ and, without loss of generality, $a_1^{(n)} \to 0$, then the perimeter constraint becomes $a_2 a_3 \cdots a_m = 4$. Thus, the eigenvalues of $R_{a_1,\cdots,a_m}$ are the eigenvalues of the $(m-1)$-dimensional cuboid with edges of length $a_2, a_3, \cdots, a_m$ subject to a volume constraint. At this point, making the vanishing edge longer would increase the eigenvalues.

For every k, let R_k^* be a maximizing cuboid. Then, for $k \to +\infty$ the sequence $(R_k^*)_k$ has to collapse. By considering the Weyl asymptotic on R_k^*, μ_k^* would behave like $k^{\frac{2}{m}}$. However, if one chooses a particular sequence which collapses towards a fixed $(m-1)$-dimensional cuboid, then μ_k^* would behave like $k^{\frac{2}{m-1}}$.

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