A NEW PROOF OF THE FABER-KRAHN INEQUALITY AND THE SYMMETRY OF OPTIMAL DOMAINS FOR HIGHER EIGENVALUES

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Abstract. We give a new proof of the Faber-Krahn inequality for the first eigenvalue of the Dirichlet-Laplacian. This proof is of a purely variational nature, proceeding along the following steps: proof of the existence of a domain which minimizes the first eigenvalue among all domains of prescribed volume, proof of (partial) regularity of the optimal domain and usage of a reflection argument in order to prove radiality. As a consequence, no rearrangement arguments are used and, although not the simplest of proofs of this statement, it has the advantage of its adaptability to study the symmetry properties of higher eigenvalues and also to other isoperimetric inequalities, as for example those involving Robin boundary conditions.

1. Introduction

It was conjectured by Rayleigh in 1877 that among all fixed membranes with a given area, the ball would minimize the first eigenvalue [15]. This assertion was proved by Faber and Krahn in the nineteen twenties using a rearrangement technique, and since then several proofs have appeared in the literature. We refer to the survey article in preparation by Ashbaugh and Laugesen for a complete presentation of this and related questions.

The purpose of this note is to give a new proof of the Rayleigh–Faber–Krahn inequality. It may be argued that, in spite of the fact that this is a basic result in the theory, no more proofs of this statement are needed, particularly if they are not simpler than other existing proofs. However, it is our belief that because the proof we propose here is somehow of a different nature, has applications to other problems, and follows a natural sequence of intuitive (but mathematically not easy) steps, it deserves some attention. These steps are the following: existence of an optimal shape, proof of its (partial) regularity, and use of its optimality in order to conclude that the optimizer must be radially symmetric.

One has in mind the incomplete proof of the isoperimetic inequality by Steiner in 1836 (see [16] and the survey article [3]). Steiner proved that if a domain is not the ball, then there must exist another domain with the same area but lower perimeter. Of course, the missing step is precisely the proof of the existence of a sufficiently smooth set which minimizes the perimeter among all domains of fixed area! Existence is not enough, since in order to use his argument, Steiner implicitly needed some smoothness. This proof became complete

Date: August 29, 2012.

D.B. was supported by the the ANR GAOS research programme, ANR-09-BLAN-0037. P.F. was partially supported by FCT's projects PTDC/MAT/101007/2008 and PEst-OE/MAT/UI0208/2011.

only in 1957, when De Giorgi proved that the Steiner argument could be carried in the class where existence holds, precisely the sets which have a finite perimeter defined using functions of bounded variation (see [8]). We also refer the reader to the open problem on the isoperimetric inequality for the buckling load of a clamped plate, where the same couple of questions remain unsolved. Willms and Weinberger (see [9, Theorem 11.3.7]) noticed that if a smooth, simply connected set would minimise the buckling load among all domains with fixed area (in two dimensions of the space), then necessarily the minimizer is the ball.

In the proof we give of the Faber–Krahn inequality, we use only variational arguments developed in the free boundary theory. To our knowledge, it is the first isoperimetric problem involving a partial differential operator for which the sequence of arguments *existence - regularity - radiality* is completely proved. Furthermore, this approach has the advantage to be adaptable to other isoperimetric inequalities where rearrangement or mass transport techniques fail. We have in mind isoperimetric inequalities involving Robin boundary conditions, as for example the minimization of the first Robin eigenfunction or the maximization of the Robin torsional rigidity among domains of fixed volume [6].

Let $\Omega \subseteq \mathbb{R}^N$ be an open set of finite measure, but otherwise with no assumptions either on smoothness or on boundedness. Thanks to the compact embedding $H^1_0(\Omega) \hookrightarrow L^2(\Omega)$, the spectrum of Dirichlet-Laplacian on Ω is discrete and consists on a sequence of eigenvalues which can be ordered (counting multiplicities) as

$$0 < \lambda_1(\Omega) \leq \lambda_2(\Omega) \leq .. \leq \lambda_k(\Omega) \leq .. \to +\infty.$$

Here, $H^1_0(\Omega)$ is the classical Sobolev space consisting on the completion of $C^\infty_0(\Omega)$ for the L^2-norm of the gradients.

The Faber–Krahn inequality asserts that

$$\lambda_1(\Omega) \geq \lambda_1(B),$$

where B is the ball having the same measure as Ω. Equality holds if and only if Ω is a ball (up to a negligible set of points, which may be expressed in terms of capacity).

Roughly speaking the proof can be split into the following four steps:

Step 1. Prove that there exists a domain Ω^* which minimizes the first Dirichlet eigenvalue among all domains of fixed volume, i.e.

$$\exists \Omega^* \subseteq \mathbb{R}^N, |\Omega^*| = m, \ \forall \Omega \subseteq \mathbb{R}^N, |\Omega| = m \quad \lambda_1(\Omega^*) \leq \lambda_1(\Omega).$$

The existence result is carried in the family of quasi-open sets.

Step 2. Prove that the set Ω^* is (slightly) regular: more precisely prove that Ω^* is an open set.

Step 3. Use a reflection argument in order to deduce that Ω^* is radially symmetric, hence it is one annulus (since connectedness is an easy consequence of optimality).

Step 4. Prove that among all annuli of prescribed volume, the ball gives the lowest first eigenvalue.

It is clear that the proof of the Faber–Krahn inequality we propose in this note in not simple. The proof of Steps 1 and 2 requires quite refined techniques of calculus of variations, Sobolev spaces and free boundary problems. They have already been proved in different contexts and we shall not reproduce here the complete proofs; instead, we refer to the original papers. Step 3 is done using an idea that may be traced to Steiner's original manuscript [16, 3] and which has also been used in the context of the minimization of integral functionals in H^1-Sobolev spaces [12, 13]. We show in detail how this can be adapted to shape optimization in a functional context. Step 4 consists in a one dimensional analysis argument for which a precise computation can be carried.

In the last section of the paper we discuss briefly the symmetry of a minimizer of the k-th eigenvalue of the Dirichlet Laplacian among all quasi-open sets of prescribed measure, and show how our arguments may be used there. We point out that although one might expect minimisers of this type of spectral problems to always have some symmetry, say at least for the reflection with respect to one hyperplane, recent numerical evidence on this problem has raised the issue of whether or not this is actually true [2]. More precisely, the planar domain found (numerically) in that paper which minimises the thirteenth Dirichlet eigenvalue of the Laplace operator does not have any such symmetry and, furthermore, if a restriction is imposed enforcing that the optimiser does have some symmetry, namely invariance under reflection with respect to an axis, the resulting value of the optimal eigenvalue is worse than the unrestricted case. Although this case should at least receive some more numerical confirmation (a proof of such a statement, if true, is likely to be quite elusive), it does beg the question as to what is the minimum symmetry which we can guarantee these optimal domains will have. In this sense, the results presented here are a first step in this direction.

2. Setting the variational framework

Solving Step 1 requires to set up a very large framework for the existence question. As one cannot impose *a priori* constraints on the competing sets, in order to achieve existence the largest class of admissible shapes should be considered.

The most natural framework where the Dirichlet-Laplacian operator is well defined is the family of *quasi-open* sets. More precisely, $\Omega \subseteq \mathbb{R}^N$ is called quasi-open if for all $\varepsilon > 0$ there exists an open set U_ε such that the set $\Omega \cup U_\varepsilon$ is open, where

$$\mathrm{cap}(U_\varepsilon) := \inf\{\int_{\mathbb{R}^N} |\nabla u|^2 + |u|^2 dx : u \in H^1(\mathbb{R}^N), u \geq 1 \text{ a.e. } U_\varepsilon\} < \varepsilon.$$

Roughly speaking, quasi-open sets are precisely the level sets $\{\tilde{u} > 0\}$ of the "most continuous" representatives of Sobolev functions $u \in H^1(\mathbb{R}^N)$, i.e. the ones given by

$$\tilde{u}(x) = \lim_{r \to 0} \frac{\displaystyle\int_{B_r(x)} u(y)dy}{|B_r(x)|}. \tag{1}$$

The limit above exists for all points except a set of capacity zero.

If Ω is a quasi-open set, then the Sobolev space $H^1_0(\Omega)$ associated to the quasi-open set Ω is defined as a subspace of $H^1(\mathbb{R}^N)$, by

$$H^1_0(\Omega) = \bigcap_{\varepsilon>0} H^1_0(\Omega \cup U_\varepsilon).$$

If Ω is a quasi-open set of finite measure, the spectrum of the Dirichlet-Laplacian on Ω is defined in the same way as for open sets, being the inverse of the spectrum of the compact, positive, self-adjoint resolvent operator $R_\Omega : L^2(\Omega) \to L^2(\Omega)$, $R_\Omega f = u$, where $u \in H^1_0(\Omega)$ satisfies

$$\forall \varphi \in H^1_0(\Omega) \quad \int_\Omega \nabla u \nabla \varphi dx = \int_\Omega f \varphi dx. \tag{2}$$

In particular

$$\lambda_1(\Omega) := \min_{u \in H^1_0(\Omega)} \frac{\displaystyle\int_\Omega |\nabla u|^2 dx}{\displaystyle\int_\Omega |u|^2 dx}.$$

The minimizer function solves the equation

$$\begin{cases} -\Delta u = & \lambda_1(\Omega) u \ \text{ in } \ \Omega \\ \quad u = & 0 \ \ \partial\Omega \end{cases}$$

in the weak sense (2). Clearly, if Ω is open we find the classical definition of the first eigenvalue.

The problem we solve is the following: given $m > 0$ prove that the unique solution of

$$\min\{\lambda_1(\Omega) : \Omega \subseteq \mathbb{R}^N \text{ quasi-open}, |\Omega| = m\} \tag{3}$$

is the ball. Uniqueness is understood up to a set of zero capacity, which is precisely the size of a set for which one cannot distinguish the "precise" values of a Sobolev function.

3. Proof of the Faber-Krahn inequality

Proposition 3.1 (Step 1.)**.** *Problem* (3) *has a solution, i.e. there exists a quasi-open set* Ω^* *such that* $|\Omega^*| = m$ *such that for every quasi-open set* $\Omega \subseteq \mathbb{R}^N$, $|\Omega| = m$ *we have*

$$\lambda_1(\Omega^*) \le \lambda_1(\Omega).$$

Proof. (Idea) First of all, one can notice that

$$\alpha(m) := \inf\{\lambda_1(\Omega) : \Omega \subseteq \mathbb{R}^N \text{ quasi-open}, |\Omega| = m\} > 0$$

as a consequence of the Sobolev inequality in $\mathbb{R}^N$. In particular, by homothety arguments

$$\text{if } m < m' \text{ then } \alpha(m) > \alpha(m'). \tag{4}$$

The idea is very simple and is based on the concentration-compactness principle of P.L. Lions [11]. Assume that $(\Omega_n)_n$ is a minimizing sequence and let us denote $(u_n)_n$ a sequence of L^2-normalized associated first eigenfunctions. Then the sequence $(u_n)_n \subseteq H^1(\mathbb{R}^N)$ is bounded. Since $H^1(\mathbb{R}^N)$ is not compactly embedded in $L^2(\mathbb{R}^N)$, for a subsequence (still denoted using the same index) one of the three possibilities below occurs:

i) compactness: $\exists y_n \in \mathbb{R}^N$ such that $u_n(\cdot + y_n) \longrightarrow u$ strongly in $L^2(\mathbb{R}_n)$ and weakly in $H^1(\mathbb{R}^N)$.

ii) dichotomy: there exists $\alpha \in (0,1)$ and two sequences $\{u_n^1\}$ and $\{u_n^2\}$ such that

$$\|u_n - u_n^1 - u_n^2\|_{L^2(\mathbb{R}^N)} \to 0\,,$$
$$\int_{\mathbb{R}^N} |u_n^1|^2 dx \to \alpha \quad \int_{\mathbb{R}^N} |u_n^2|^2 dx \to 1-\alpha\,,$$
$$\mathrm{dist}(\mathrm{supp}\, u_n^1, \mathrm{supp}\, u_n^2) \to +\infty\,,$$

and

$$\liminf_{n\to\infty} \int_{\mathbb{R}^N} |\nabla u_n|^2 - |\nabla u_n^1|^2 - |\nabla u_n^2|^2 dx \geq 0\,. \tag{5}$$

iii) vanishing: for every $0 < R < \infty$

$$\lim_{n\to+\infty} \sup_{y\in\mathbb{R}^N} \int_{B(y,R)} u_n^2 dx = 0\,.$$

Situations ii) and iii) cannot occur to a minimizing sequence (see for instance [5]). Indeed, situation ii) leads to searching the minimizer in a class of domains of measure $m-\alpha$ for some $\alpha > 0$ in contradiction to (4). Situation iii) is excluded by an argument of Lieb [10] which proves that if iii) occurs then $\lambda_1(\Omega_n) \to +\infty$, in contradiction with the choice of a minimizing sequence. Only situation i) can occur and this leads to the existence of an optimal domain which is a quasi-open set. Indeed, we consider the set $\Omega := \{u > 0\}$. Then, if we choose the representative defined by (1), the set Ω is quasi-open and has a measure less than or equal to m. This latter assertion is a consequence of the strong L^2 convergence of u_n which has as a consequence that (at least for a subsequence) $1_\Omega(x) \leq \liminf_{n\to+\infty} 1_{\Omega_n}(x)$ a.e. $x \in \mathbb{R}^N$.

Moreover, we have

$$\lambda_1(\Omega) \leq \frac{\displaystyle\int_\Omega |\nabla u|^2 dx}{\displaystyle\int_\Omega |u|^2 dx} \leq \liminf_{n\to\infty} \frac{\displaystyle\int_\Omega |\nabla u_n|^2 dx}{\displaystyle\int_\Omega |u_n|^2 dx} = \liminf_{n\to\infty} \lambda_1(\Omega_n).$$

If necessary, taking a suitable dilation $\Omega^* = t\Omega$, for some $t \geq 1$ such that $|\Omega^*| = m$ and using the rescaling properties of λ_1, we conclude that Ω^* is a minimizing domain. □

Proposition 3.2 (Step 2.). *The optimal set Ω^* is open and connected.*

Proof. (Idea) Assuming one knows that Ω^* is open, the proof of the connectedness is immediate. Indeed, assume $\Omega^* = \Omega_1 \cup \Omega_2$ with Ω_1, Ω_2 open, disjoint and non-empty. Then $\lambda_1(\Omega^*)$ is either equal to $\lambda_1(\Omega_1)$ or to $\lambda_1(\Omega_2)$, hence one could find a set, say Ω_1, with the first Dirichlet eigenvalue equal to $\lambda_1(\Omega^*)$ but with measure strictly less than m. This is in contradiction to (4).

The proof of the openness has a technical issue and is using a local perturbation argument developed by Alt and Caffarelli in the context of free boundary problems [1, Lemma 3.2]. We refer the reader to [17, Theorem 3.2] and [7, Proposition 1.1] for complete proofs of Proposition 3.2.

The main idea is the following: assume $u \geq 0$ is an L^2-normalized eigenfunction on the optimal set Ω^*. Relying on the minimality of Ω^* and on the Rayleigh definition of the eigenvalue, one can prove that there exists r_0, C_0 such that for $r < r_0$ if

$$\frac{1}{r}\fint_{B_r(x_0)} u dx > C_0 \ \text{ then } \ u > 0 \text{ in } B_r(x_0). \tag{6}$$

Roughly speaking, this assertion relies on the following kind of estimate. We assume for simplicity that $r = 1$. Take $x_0 \in \mathbb{R}^N$ and $r > 0$. One introduces the perturbation of u given by

$$\tilde{u}(x) = u(x) \text{ in } \Omega \setminus B_1(x_0),$$
$$\Delta \tilde{u}(x) = 0 \text{ in } B_1(x_0).$$

For a suitable Lagrange multiplier $C > 0$ we can write

$$\lambda_1(\Omega^*) + C|\Omega^*| \leq \lambda_1(\{\tilde{u} > 0\}) + C|\{\tilde{u} > 0\}|.$$

Assuming $u(x_0) > 0$, we get

$$\frac{\displaystyle\int_{\Omega^*} |\nabla u|^2 dx}{\displaystyle\int_{\Omega^*} u^2 dx} + C|\Omega^*| \leq \frac{\displaystyle\int_{\Omega^* \cup B_1(x_0)} |\nabla \tilde{u}|^2 dx}{\displaystyle\int_{\Omega^* \cup B_1(x_0)} \tilde{u}^2 dx} + C|\Omega^* \cup B_1(x_0)|.$$

Exploiting the coincidence between u and $\tilde{u}$ on $\Omega^* \setminus B_1(x_0)$, we get

$$\int_{B_1(x_0)} |\nabla(u - \tilde{u})|^2 dx \leq C_0' |\Omega^{*c} \cap B_1(x_0)|.$$

On the other hand, the following inequality holds true for every positive function

$$\int_{B_1(x_0)} 1_{\{u=0\}} dx \left(\fint_{\partial B_1(x_0)} u \right)^2 \leq C(N) \int_{B_1(x_0)} |\nabla(\tilde{u} - u)|^2 dx.$$

Consequently, if $\fint_{B_1(x_0)} u$ is large enough, then $|\Omega^{*c} \cap B_1(x_0)| = 0$. Since $-\Delta u \leq \lambda_1 u$ in the sense of distributions in $\mathcal{D}'(\mathbb{R}^N)$, one can deduce that if for some x we have that $u(x) > 0$ then for small enough radii $\fint_{B_r(x)} u > \frac{u(x)}{2}$ and so from the preceding argument u is strictly positive in a ball around x.

□

In fact, it can be proven that moreover the set Ω^* is bounded, has finite perimeter and has a smooth boundary. The precise smoothness depends on the dimension, but we do not insist on this point since only *openness* is enough for our purposes.

Proposition 3.3 (Step 3.). *The optimal set Ω^* has radial symmetry.*

Proof. As mentioned in the Introduction, the idea to prove radial symmetry for minimizers of integral functionals in $H^1(\mathbb{R}^N)$ has been used before (see, for instance, [12, 13]), while its usage in a geometrical setting appears in Steiner's arguments [16, 3]. Roughly speaking, if Ω^* is a minimizer then one can cut it by a hyperplane $\mathcal{H}$ in two pieces of equal measure. We can assume that $\mathcal{H}$ is given by the equation $x_1 = 0$. Then both the left and right parts together with their respective reflections are admissible (they have the correct measure) and are also minimizers. Indeed, let u be a non-zero first eigenfunction on Ω^*. We put the indices l, r to the corresponding quantities on the left, right parts of Ω^*, respectively. We denote $\Omega_l = \Omega \cap \{x \in \mathbb{R}^N : x_1 \leq 0\}$.

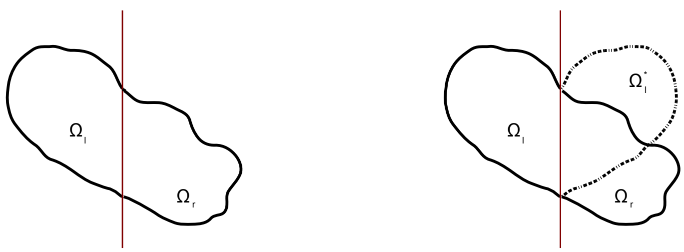

FIGURE 1. A region and one of its symmetrised counterparts with respect to a hyperplane

Then,

$$\lambda_1(\Omega^*) = \frac{\int_{\Omega_l^*} |\nabla u_l|^2 dx + \int_{\Omega_r^*} |\nabla u_r|^2 dx}{\int_{\Omega_l^*} u_l^2 dx + \int_{\Omega_r^*} u_r^2 dx}.$$

Using, for positive numbers $a, b, c, d > 0$, the algebraic inequality

$$\frac{a+b}{c+d} \geq \min\left(\frac{a}{c}, \frac{b}{d}\right),$$

we get (assuming for instance the left part is minimal)

$$\lambda_1(\Omega^*) \geq \frac{\int_{\Omega_l^*} |\nabla u_l|^2 dx}{\int_{\Omega_l^*} u_l^2 dx}.$$

Defining the reflection transformation $\mathcal{R} : \mathbb{R}^N \to \mathbb{R}^N$ $\mathcal{R}(x) = (-x_1, x_2, .., x_N)$, we introduce the reflected domain

$$\Omega^l := \Omega_l^* \cup \mathcal{R}\Omega_l^*,$$

together with the reflected test function

$$\overline{u}(x) = u_l(x) \text{ if } x_1 \leq 0 \text{ and } \overline{u}(x) = u_l(\mathcal{R}x) \text{ if } x_1 \geq 0.$$

Then $|\Omega^l| = m$, $u^l \in H_0^1(\Omega^l)$ (this is immediate using the density of C_0^∞-function in H^1) and we get

$$\lambda_1(\Omega^*) \geq \frac{\int_{\Omega_l^*} |\nabla u_l|^2 dx}{\int_{\Omega_l^*} u_l^2 dx} = \frac{\int_{\Omega^l} |\nabla |\overline{u}|^2 dx}{\int_{\Omega^l} |\overline{u}|^2 dx} \geq \lambda(\Omega^l).$$

Relying on the minimality of Ω^* and on the inequalities above, we conclude that Ω^l is also a minimizer and that $\overline{u}$ is an eigenfunction on Ω^l. Finally, this means that u_l has two analytic extensions: one in the open set Ω_r which is u_r and another one in $\mathcal{R}\Omega_l$ which is $\overline{u}$. Using the maximum principle, there cannot be a point of the complement of Ω_r where u_r is vanishing which is interior for $\mathcal{R}\Omega_l$, and vice-versa. Finally, this implies that $\Omega_r = \mathcal{R}\Omega_l$ and so Ω^* is symmetric with respect to $\mathcal{H}$. Since such a hyperplane can be found in every direction, we conclude that Ω^* has to be radially symmetric. □

Proposition 3.4 (Step 4.)**.** *The optimal set Ω^* is the ball.*

Proof. Since the optimal set is connected, using the proposition above, we know it is an annulus. Precisely, for some $t \geq 0$ this annulus can be written

$$\Omega^* = K(0, t, r(t)) := \{x \in \mathbb{R}^N : t < |x| < r(t)\},$$

where $\omega_{N-1}(r^N(t) - t^N) = m$. In order to prove that the solution of the Faber-Krahn inequality is the ball, it is enough to study the mapping $t \mapsto \lambda_1(K(0, t, r(t)))$ and to prove it is increasing. Using the the shape derivative formula for λ_1 (and denoting u an L^2-normalized eigenfunction on $K(0, t, r(t))$, we get

$$\frac{d}{dt}\lambda_1(K(0, t, r(t)))) = \int_{\partial B(0)_t} \left(\frac{\partial u}{\partial n}\right)^2 d\mathcal{H}^{N-1} - \frac{t^{N-1}}{r^{N-1}(t)} \int_{\partial B(0)_{r(t)}} \left(\frac{\partial u}{\partial n}\right)^2 d\mathcal{H}^{N-1}.$$

Since u is radially symmetric, proving that $\frac{d}{dt}\lambda_1(K(0, t, r(t)))) > 0$ is equivalent to proving that $|u'(t)|^2 > |u'(r(t))|^2$ (in this notation u depends only on the radius).

The equation satisfied by the radial function u is

$$-u''(s) - \frac{N-1}{s}u'(s) = \lambda_1 u(s), \text{ on } (t, r(t)),$$
$$u(t) = u(r(t)) = 0.$$

Denoting $v(s) = |u'(s)|^2$, we get that

$$v'(s) = 2u'(s)u''(s) = -\frac{2(N-1)}{s}|u'(s)|^2 - 2\lambda_1 u'(s)u(s),$$

and summing between t and $r(t)$ we get

$$v(r(t)) - v(t) = -\frac{2(N-1)}{s}\int_t^{r(t)} |u'(s)|^2 ds < 0.$$

The last inequality is obvious since u is not constant, hence the mapping $t \mapsto \lambda_1(K(0, t, r(t)))$ is strictly increasing on $(0, +\infty)$. Consequently the ball, corresponding to $t = 0$, is the global minimizer. □

4. Further remarks: higher order eigenvalues

For $k \geq 2$ one can also consider the isoperimetric problem

$$\min\{\lambda_k(\Omega) : \Omega \subseteq \mathbb{R}^N \text{ quasi-open}, |\Omega| = m\}. \tag{7}$$

When $k = 2$ the minimiser consists of two equal and disjoint balls of measure $m/2$, this being a direct consequence of the inequality for the first eigenvalue and one which was already considered by Krahn. For $k \geq 3$, and apart from numerical results (see, for instance, [2]) only few facts are known: a solution to problem (7) exists (let us call it Ω_k^*), is bounded and has finite perimeter (see [4] and [14]).

Relying on the reflection argument, one can get some more information on the symmetry of Ω_k^*, depending on the dimension of the space. Roughly speaking, the larger the space dimension, the more symmetric must the minimizer be.

Assume that $u_1, .., u_k$ are L^2-normalized eigenfunctions corresponding to $\lambda_1(\Omega_k^*), .., \lambda_k(\Omega_k^*)$ such that

$$\forall i, j = 1, .., k, i \neq j, \quad \int u_i u_j dx = 0, \int \nabla u_i \nabla u_j = 0.$$

Problem (7) can be re-written as

$$\min_{|\Omega|=m} \min_{S_k \subset H_0^1(\Omega)} \max_{u \in S_k} \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega |u|^2 dx},$$

where S_k denotes any subspace of dimension k.

Assume $\mathcal{H}$ is a hyperplane splitting Ω_k^* into Ω^l and Ω^r such that

$$|\Omega^l| = |\Omega^r|,$$
$$\forall i, j = 1, .., k, i \neq j, \int_{\Omega^l} u_i u_j dx = 0,$$
$$\forall i, j = 1, .., k, i \neq j, \int_{\Omega^l} \nabla u_i \nabla u_j dx = 0,$$
$$\forall i = 1, .., k-1, \quad \frac{\int_{\Omega^l} |\nabla u_i|^2 dx}{\int_{\Omega^l} |u_i|^2 dx} = \lambda_j(\Omega_k^*).$$

Notice that the number of constraints equals k^2.

Assuming that

$$\frac{\int_{\Omega^l} |\nabla u_k|^2 dx}{\int_{\Omega^l} |u_k|^2 dx} \leq \frac{\int_{\Omega^r} |\nabla u_k|^2 dx}{\int_{\Omega^r} |u_k|^2 dx},$$

and reflecting Ω^l together with the functions $u_1|_{\Omega^l}, .., u_k|_{\Omega^l}$, we get

$$\lambda_k(\Omega^l \cup \mathcal{R}\Omega^l) \leq \max\{\lambda_{k-1}(\Omega_k^*), \frac{\int_{\Omega^l} |\nabla u_k|^2 dx}{\int_{\Omega^l} |u_k|^2 dx}\} \leq \lambda_k(\Omega_k^*).$$

Consequently, the set $\Omega^l \cup \mathcal{R}\Omega^l$ is also a minimizer and is symmetric with respect to $\mathcal{H}$.

If Ω_k^* were open, then we could conclude that $1_{\Omega_k^*} = 1_{\Omega^l \cup \mathcal{R}\Omega^l}$. Following the same arguments as [13, Theorem 2], we would get that the set Ω_k^* would be symmetric with respect to an affine subspace of dimension $k^2 - 1$. As Ω_k^* is not known to be open, we can only assert the existence of a minimizer which has $N - (k^2 - 1)$ hyperplanes of symmetry. In either case, we see that it is only for k equal to one that *full symmetry* may be obtained in this way.

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