Discrete segments of $\mathbb{Z}^3$ constructed by synchronization of words *

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Abstract

We study a natural and naive composition algorithm what takes three input words written on two-letter alphabets and synchronizes then into a word on a three-letter alphabet. We show that in the case where the three input words are compatible Christoffel words, the algorithm provides a synchronization of the letters what allows the geometrical interpretation of the input words to be inherited by the output word forming a 3D discrete line segment. A second approach is considered while applying our composition algorithm to words defined by stripes meeting at a corner of a discrete planes. We show that, under certain conditions, the output of the algorithm correspond to the normal vector of the plane.

Keywords: Christoffel words, Billiard words, Discrete segments, Discrete planes.

1. Introduction

Discrete lines are fundamental objects in discrete mathematics. The finite version of discrete lines, called discrete segments, appeared for the first time in the work of Christoffel [Chr75] for the dimension 2 and was extensively studied by Borel Laubie [BL93], Reutenauer [BLRS09] and the community of discrete geometry (see [Rev91, KR04]). The infinite version of discrete lines in dimension 2 is called Sturmian words and is related to techniques in combinatorics on words [BS02], discrete dynamical systems [Fog07], square billiards [AR91, AMST94] and cutting words [HV07]. In dimension 3, the infinite version of discrete lines gives cubic billiards [AR91, AMST94] and leads to problem on multidimensional continued fractions [Rau82]. In discrete geometry, finite versions of discrete lines have been considered in dimension three, we refer to the work of Reveillès [FR96, FR06], Andres [And03], Toutant [Tou06] and Berthé and Labbé [BL11]. In fact, if we consider the construction of discrete segments related to continued fraction, the work of Berthé and Labbé [BL11] states many interesting parameters in order to study the properties of distinct continued fraction algorithms. A good candidate for discrete segments in dimension 3 is a finite version of cubic billiards since they inherit nice properties like, for instance, a short distance from each of the points of the discrete segment to the real line segment. Moreover it is exactly the composition of three square billiard words. Furthermore, a square billiard word has infinitely many palindromic prefixes while Borel and Reutenauer [BR05] have shown that it is usually not the case for cubic billiard words.

*Support by the ANR-2010-BLAN-0205 KIDICO

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In this paper we present a composition algorithm that takes three words written on two-letter alphabets as input. More precisely, the input words must be such that the first one is written over some alphabet $\{\mathbf{a}, \mathbf{b}\}$, the second on is written on an alphabet $\{\mathbf{a}, \mathbf{c}\}$ and the third one is written over the alphabet $\{\mathbf{b}, \mathbf{c}\}$. The algorithm combines the first letters of the three words in order output word over the alphabet $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$. The main goal of this paper is to understand the combinatorial and geometrical properties of the output given certain types of input words. Section 2 recalls the usual definitions of Christoffel and billiard words. Section 3 is dedicated to the composition algorithm. Section 4 shows the composition of three primitive (or non-primitive) Christoffel words with an arithmetic constraint that enables us to construct a Christoffel word in dimension three. Section 5 gives some experimental results and shows that, unlike in dimension two, in dimension three balancedness, palindromicity and complexity are in general distinct notions. The last section deals with discrete planes. Any corner of a discrete plane defines three stripes and by considering the normal vector of the consecutive faces of cube along a stripe, we obtain three words written on compatible alphabets. These words are used as input for the composition algorithm and we show that, on a well chosen corner, the output describes a 3D discrete line in the same direction than the normal vector.

2. Preliminaries

A *word* is a sequence, finite of not, of symbols called *letters*. An *alphabet* is a finite set of letters. A word $w = w_1 w_2 \cdots w_n$ where each w_i is a letter, is said to be written on the alphabet $\mathcal{A}$ if each $w_i \in \mathcal{A}$. In order to simply the presentation the i-th letter of w is also referred as $w[i]$. The *length* of the word w, denoted as $|w|$, is the total number of letters in w. The number of occurrences of a specific letter $\mathbf{a}$ in a word w is denoted by $|w|_{\mathbf{a}}$ so, $|w| = \sum_{\mathbf{a}\in\mathcal{A}} |w|_{\mathbf{a}}$. The word ε, called the *empty word*, is the unique word of length zero. The set $\mathcal{A}^*$ of all finite words on $\mathcal{A}$ forms a free monoid with neutral element ε when the concatenation operator $\cdot$ is considered. The notation w^k, with $k \in \mathbb{N}$, is the concatenation of k copy of the word w.

Let $p, f, s \in \mathcal{A}^*$ such that $w = pfs$, we say that p is a *prefix* of w, f is a factor of w and s is a suffix of w. Note that since any of the words p, f, s might be ε, prefixes and suffixes are also factors.

In the following, we will consider the three following properties of words:

- The word $\widetilde{w} = w_n w_{n-1} \cdots w_1$ is called the *mirror* of w, and a *palindrome* is a word w such that $w = \widetilde{w}$.
- An integer $p > 0$ is a *period* of w if for all $i \in \{1, 2, \ldots, n-p\}$, $w_i = w_{i+p}$.
- A finite word u is called a *periodic pattern* of w if $w = u^k$ for some $k \geq 1$.
- The word w is k*-balanced* if, for any two factors u, v of w and any letters $\mathbf{a} \in \mathcal{A}$,

$$|u| = |v| \implies \big||u|_{\mathbf{a}} - |v|_{\mathbf{a}}\big| \leq k.$$

Usually a 1-balanced word is called a *balanced word* and a pair u, v such that there exist a letter a such that $\big||u|_{\mathbf{a}} - |v|_{\mathbf{a}}\big| > 1$ is called *unbalanced*. See [Vui03] for a survey on balanced words and their generalizations.

A common geometrical interpretation for words on alphabet $\mathcal{A} = \{1, 2, \cdots, d\}$ is to associate the letter $i \in \mathcal{A}$ to the unit vector e_i, th. i-th vector of the canonical basis of $\mathbb{R}^d$, for $i = 1, \cdots, d$

so that words encode paths in a d-dimensional space. This coding is known as the Freeman code. The *abelianized* of a word w, denoted by $\overrightarrow{w}$, is the translation defined by this word :

$$\overrightarrow{w} = (|w|_1, |w|_2, \cdots, |w|_d).$$

Finally, given a word $w \in \mathcal{A}^*$ and a letter $\mathbf{a} \in \mathcal{A}$, the *projection* of w relatively to the letter $\mathbf{a}$, noted $w{\downarrow_{\mathbf{a}}}$, is the word obtained by removing every occurrences of $\mathbf{a}$ from w. Note that given a word $w \in \mathcal{A}^*$ such that $\overrightarrow{w} = (x_1, x_2, \ldots, x_d)$ then $\overrightarrow{w{\downarrow_{\mathbf{a}}}} = (x_1, x_2, \ldots, x_{\mathbf{a}-1}, 0, x_{\mathbf{a}+1}, \ldots, x_d)$.

2.1. Christoffel words

Christoffel words may be considered as the rational case of Sturmian word. They are called *Christoffel* by reference to [Chr75] while the geometrical point of view that we consider here is mostly due to [Ber90] and [BL93]. We refer the reader to [BLRS09] for a comprehensive self-contained survey on Christoffel words.

Let $\mathcal{A} = \{\mathbf{a}, \mathbf{b}\}$ be an ordered alphabet with $\mathbf{a} < \mathbf{b}$. A Christoffel word on $\mathcal{A}$ is defined by a pair of strictly positive and relatively prime integers p, q as follows.

1. Let S be the line segment that starts in $(0, 0)$ and ends in (p, q).
2. Consider the set of all discrete paths that start in $(0, 0)$, end in (p, q), contain only north (that is e_2) and east (that is e_1) steps and remain below S. Let P be the closest path to S.
3. The Freeman code of P is the *primitive Christoffel word* $C_{(\mathbf{a},\mathbf{b})}(p, q)$. By convention, letter $\mathbf{a}$ codes e_1 and $\mathbf{b}$ codes e_2 and q/p is called the *slope* of $C_{(\mathbf{a},\mathbf{b})}(p, q)$.

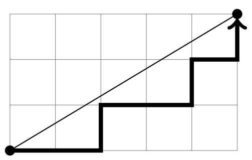

Figure 1: Construction of the Christoffel word $C_{(1,2)}(5, 3) = 1 \cdot 121121 \cdot 2$.

This definition is illustrated in Figure 1. Remark that by definition of $C_{(1,2)}(5, 3)$ this Christoffel word is on the alphabet $\{1, 2\}$ and has five 1's and three 2's, more generally $C_{(\mathbf{a},\mathbf{b})}(p, q)$ with relatively prime integers p, q means the Christoffel word on the alphabet $\{\mathbf{a}, \mathbf{b}\}$ with p $\mathbf{a}$'s and q $\mathbf{b}$'s . A *non-primitive Christoffel word* is a word $w = C^k$, where C is a primitive Christoffel word and $k \geq 2$. Note that, as defined here, these words are called *lower Christoffel words* by some authors. Moreover, the words $C_{(\mathbf{a},\mathbf{b})}(1, 0) = \mathbf{a}$ and $C_{(\mathbf{a},\mathbf{b})}(0, 1) = \mathbf{b}$ are usually called *trivial Christoffel words* but our definition excludes them. In order to lighten the presentation, when the alphabet is obvious, a Christoffel word of slope q/p may be simply denoted by $C(p, q)$.

The set of all Christoffel words over alphabet $\{\mathbf{a}, \mathbf{b}\}$ is denoted by $\mathcal{C}_{(\mathbf{a},\mathbf{b})}$. Clearly the construction of Christoffel words forces that a word $w \in \mathcal{C}_{(\mathbf{a},\mathbf{b})}$ starts with the letter $\mathbf{a}$ and ends with the letter $\mathbf{b}$.

Definition 2.1. A word $u \in \mathcal{A}^*$ is called a *central word* if there exists a primitive Christoffel word $w \in \mathcal{C}_{(\mathbf{a},\mathbf{b})}$ such that $w = \mathbf{a}\,u\,\mathbf{b}$.

The following theorem summarizes some of the well known properties of Christoffel words.

Theorem 2.2 ([dLM94, BdL97]). *Let $u, w \in \mathcal{A}^*$ such that $w = \boldsymbol{a}\, u\, \boldsymbol{b}$. Then the following conditions are equivalent:*

1. *w is a primitive Christoffel word.*
2. *u has periods p and q with $\gcd(p,q) = 1$ and $|u| = p + q - 2$.*
3. *The words $\boldsymbol{a}u\boldsymbol{a}$, $\boldsymbol{a}u\boldsymbol{b}$, $\boldsymbol{b}u\boldsymbol{a}$, $\boldsymbol{b}u\boldsymbol{b}$ are 1-balanced.*
4. *Either $u \in \{a\}^* \cup \{b\}^*$, or u is a palindrome and there exist two palindromes x, y such that $u = x\, \boldsymbol{ab}\, y$.*

2.2. Square billiards words

We give some information about the world of billiards [Bar95, Tab95]. Indeed, Sturmian words are characterized by coding of a square billiard word with irrational slope. Let us consider a square billiard table and a trajectory of a point along an irrational slope. Each time that the point hits the border of the billiard table, it bounces according to the reflection laws (the angles of the trajectory with the normal before and after the reflection are equal). Note that with this usual definition the trajectory is not defined in corners. Now, if we code by **a** (resp. by **b**) when the point hits the vertical (resp. horizontal) sides then the infinite word given by the coding of the trajectory is a Sturmian word. We only consider trajectories that start from $(0,0)$ so a billiard word is completely defined by a second point of $\mathbb{R}^2$. The set of all billiard words on the alphabet $\mathcal{A} = \{\mathbf{a}, \mathbf{b}\}$ is denoted by $\mathcal{B}_{(\mathbf{a},\mathbf{b})}$ and among the words of $\mathcal{B}_{(\mathbf{a},\mathbf{b})}$, the one defined by the line that passes through the origin and the point (x, y) is denoted $B_{(\mathbf{a},\mathbf{b})}(x, y)$. The *slope* of the billiard word $B_{(\mathbf{a},\mathbf{b})}(x, y)$ is the ratio y/x.

It is well know that the construction of a billiard words of slope α is equivalent to drawing the straight line of slope α on a square lattice and considering its intersections with the horizontal and vertical lines of the square grid (see Figure 2).

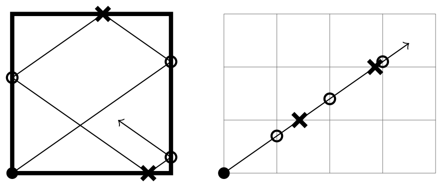

Figure 2: A billiard trajectory is conveniently seen as a straight line crossing a square grid.

In the case of a rational slope q/p, with $\gcd(p, q) = 1$, starting from $(0, 0)$ the straight line crosses all integer coordinates points of the from (kp, kq) for $k \in \mathbb{Z}$. Usually, such case is excluded since there is integer points present a problem with the classical definition of infinite billiard words. We avoid this problem by defining a *finite billiard word* as the word obtained by coding the trajectory between $(0, 0)$ to (p, q), where it is well defined as a cutting sequence. As explained, for instance, in [BR05], there is an equivalence between finite (resp. infinite) billiard words and Christoffel (resp. Sturmian) words. We focus on the finite case, as illustrated in Figure 3. In the following, we will constantly switch from one point of view to the other.

Proposition 2.3. *Given a pair of relatively prime numbers p, q with $p, q \geq 1$, the central word of the Christoffel word $C(p, q)$ is the finite billiard word $B(p, q)$.*

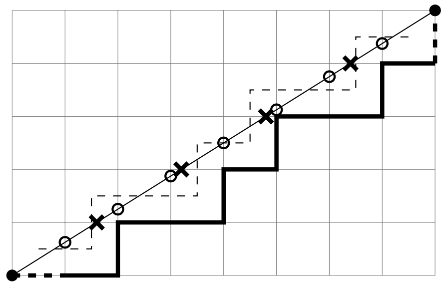

Figure 3: The billiard and the Christoffel word defined by vector $(8,5)$. The Christoffel word is $C_{(1,2)}(8,5) = 1 \cdot 12112121121 \cdot 2$ and the finite billiard $B_{(1,2)}(8,5) = 12112121121$. The positions where the real line segment from $(0,0)$ to $(8,5)$ crosses an horizontal line are identified by symbol $\circ$ and are coded by the letter 1 while positions where a vertical line is crossed are identified by $\times$ and are coded with letter 2. The dashed line is the part of the Christoffel path that corresponds to the central word starting from $(1/2,1/2)$. The correspondence with the billiard word is obvious since at each step the dashed line crosses one and only one one line of the square grid which is also crossed by the real line segment.

One of the implications of Proposition 2.3 is that square billiard words are palindromes. Geometrically, the trajectory from $(0,0)$ to (p,q) defines a cutting sequence that is symmetric around $(p/2,q/2)$. Also, by exchanging the role of p and q and doing the same for $\mathbf{a}$ and $\mathbf{b}$, we have that given p, q with $\gcd(p,q)=1$,

$$B_{(\mathbf{a},\mathbf{b})}(p,q) = B_{(\mathbf{b},\mathbf{a})}(q,p).$$

This equality translate to Christoffel words as follow: if u is the central word of $C_{(\mathbf{a},\mathbf{b})}(p,q)$, with $\gcd(p,q)=1$, then

$$C_{(\mathbf{b},\mathbf{a})}(q,p) = \mathbf{b}\, u\, \mathbf{a}.$$

By the above proposition, the billiard word $B(p,q)$ and the corresponding Christoffel word $C(p,q)$ are such that $C(p,q) = \mathbf{a} \cdot B(p,q) \cdot \mathbf{b}$. The word $B(p,q)$ is exactly two letters shorter then $C(p,q)$ because we do not consider the starting point and the ending point of the segment $(0,0)$ and (p,q), since in these points the segment meets a horizontal and a vertical line of the square lattice at the same time. Those *integer crossings* may not be coded in a billiard word. In order to define our version of discrete line segments we introduce *special letters* of the form $\overleftrightarrow{\mathbf{ab}}$, which simply encodes the fact that both letters $\mathbf{a}$ and $\mathbf{b}$ should occur at this position. Given a word w that contains special letters, we introduce the notion of *choice* that is to replace each special letter $\overleftrightarrow{\mathbf{ab}}$ by $\mathbf{ab}$ or $\mathbf{ba}$ in order to obtain a word that is written on normal letters. For this reason, we have that $\overleftrightarrow{\mathbf{ab}} = \overleftrightarrow{\mathbf{ba}}$. Note that given a special letter this choice may vary from an occurrence to another. Of course, by doing so, the result may not correspond to a cutting sequence as in the case of a billiard word. Finally, we use the convention that a special letter $\overleftrightarrow{\mathbf{ab}}$ abelianizes to $e_{\mathbf{a}} + e_{\mathbf{b}}$.

In order to emphasize the duality between Christoffel and billiard words, we consider the first and last letter of a Christoffel word and code them by a single special letter since they correspond to the coding of an integer point in the billiard word. Thus, from now on, a Christoffel word on $\{\mathbf{a},\mathbf{b}\}$, primitive or not, is written:

$$C(kp,kq) = (C(p,q))^k = \overrightarrow{\mathbf{a}} \cdot B(p,q) \cdot \underbrace{\overleftrightarrow{\mathbf{ba}} \cdot B(p,q) \cdots \overleftrightarrow{\mathbf{ba}} \cdot B(p,q)}_{k-1 \text{ times}} \cdot \overleftarrow{\mathbf{b}}\,.$$

A natural generalization of square billiards is cubic billiards. We play billiard in a cube starting from a corner and throw the ball (which again, is just a point) along a given direction. A cubic billiard word is obtain be noting the letter i each time the ball bounces on a side normal to the vector e_i. A direction (x, y, z) is said *irrational* if for any $k, l, m \in \mathbb{Z}$ we have $kx + ly + mz = 0$ if and only if $k = l = m = 0$. Moreover, the direction (x, y, z) is said to be *totally irrational* if it is irrational and (x^{-1}, y^{-1}, z^{-1}) is also irrational. We recall that the complexity function $p_w(n)$ of a word w indicates the number of factor of length n in w :

$$p_w(n) = \#\{u \mid u \text{ is a factor of } w \text{ and } |u| = n\}.$$

Theorem 2.4. *Let w be a 3D billiard word defined by an irrational direction (x, y, z).*

- *If (x, y, z) is totally irrational then $p_w(n) = n^2 + n + 1$ ([AMST94, Bar95, Bed03]).*
- *w is 2-balanced ([Vui03]).*

3. The composition algorithm

Each cubic billiard word is a mix of three Sturmian words and also a cutting sequence in the unit cubic grid [HV07]. Indeed, cubic billiard words are words on a three-letter alphabet $\{1, 2, 3\}$. Now, given b a cubic billiard word, the projection of b relatively to the letter 1 (resp. 2 or 3), is a Sturmian word (geometrically, the trajectory of the cubic billiard is observed from a direction normal to one of the sides. The result is a square billiard trajectory).

For example, this is the beginning of a billiard sequence:

$$B = 231232132312321322312312321323123213223123 \cdots$$

And here are the three associated Sturmian words:

$$B{\downarrow_3} = 2122121221221212212122212212 \cdots$$
$$B{\downarrow_2} = 3131331313313131331313313 \cdots$$
$$B{\downarrow_1} = 2323232323232223232323232322323 \cdots$$

Conversely, given three Sturmian words S_1, S_2, S_3 respectively written on alphabets $\{1, 2\}$, $\{1, 3\}$ and $\{2, 3\}$ there is an easy pseudo algorithm to construct iteratively the associated finite or infinite cubic billiard word B. At each step, if two of the three words begin by the same letter, then this first letter is removed from both words and added at the end of the cubic billiard word (which is initilized as the emtpy word). Otherwise, if the three words start with three different letters, the process stops and a finite word is obtained.

This construction is described by Algorithm 1 which we call the *composition algorithm*. Since, by definition, an algorithm should always terminate, it is written to take finite words as input. As shown in Figure 4, any triplet of finite words may be used as input. The only requirement is that the first word is written on alphabet $\{1, 2\}$, the second on $\{1, 3\}$ and the third on $\{2, 3\}$. No geometrical interpretation is required for these words. The output of this algorithm is called a *composition word*. Note that at line 1, three new symbols are added at the end of each word in order to guaranty that X, Y and Z are always non-empty words throughout the execution.

One could use infinite words as input by simply omitting line 1. In such case, there is no guaranty that the process stops and an infinite composition word may be constructed.

The following lemma formalize the fact that the composition algorithm does rebuild a word from its three projections.

Algorithm 1: Composition

Input: $w_{12} \in \{1,2\}^*,\ w_{13} \in \{1,3\}^*,\ w_{23} \in \{2,3\}^*$.

Output: $\chi \in \{1,2,3\}^*$

$X \leftarrow w_{12} \cdot \$$; $Y \leftarrow w_{13} \cdot \#$; $Z \leftarrow w_{23} \cdot \%$;
$\chi = \varepsilon$;
repeat
    if $X[1] == Y[1]$ **then**
        $\chi \leftarrow \chi \cdot 1$;
        remove the first letter of X and Y ;
    if $X[1] == Z[1]$ **then**
        $\chi \leftarrow \chi \cdot 2$;
        remove the first letter of X and Z ;
    if $Y[1] == Z[1]$ **then**
        $\chi \leftarrow \chi \cdot 3$;
        remove the first letter of Y and Z ;
until $X[1] \neq Y[1]$ **and** $X[1] \neq Z[1]$ **and** $Y[1] \neq Z[1]$;
return χ;

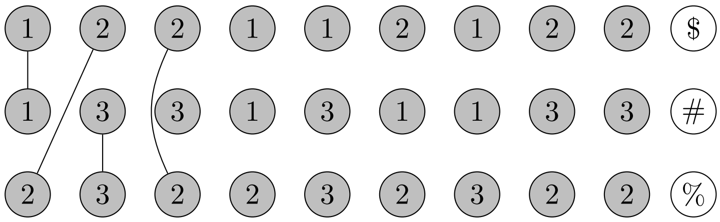

Figure 4: Illustration of the composition algorithm on the words $w_{12} = 122112122, w_{31} = 133131133, w_{23} = 232232322$. The algorithm reads the pairs of letters $1, 2, 3, 2$ then $X[1] = 1$, $Y[1] = 3$ and $Z[1] = 2$ so the algorithm stops and returns the composition word $\chi = 1232$.

Lemma 3.1. *Let $W \in \{1,2,3\}^*$ and let $w_{12} = W{\downarrow_3}$, $w_{13} = W{\downarrow_2}$, $w_{23} = W{\downarrow_1}$. The composition word χ obtained from w_{12}, w_{31}, w_{23} is equal to W.*

Proof. By recurrence on $n = |W|$. If $n = 0$ then the result is trivial. Now, suppose $n \geq 1$ and the result holds for all words of length $n - 1$. Let W' be such that $W = W'\mathbf{a}$ for some $\mathbf{a} \in \{1, 2, 3\}$. W.l.o.g, suppose that $\mathbf{a} = 1$, then by construction, there exist three words α, β, γ such that $w_{12} = \alpha 1$, $w_{13} = \beta 1$, $w_{23} = \gamma$ where α, β, γ are the three projections of W'.

Since α, β, γ are prefixes of, respectively, w_{12}, w_{13}, w_{23}, the execution of the composition algorithm with w_{12}, w_{13} and w_{23} will first read the words α, β and γ and thus, after $n - 1$ iterations of the loop, we have:

$$X = 1\$, \qquad Y = 1\#, \qquad Z = \%, \qquad \chi = W'.$$

So, at the n-th iteration 1 is added at the end of χ and the word $\chi = W$ is returned. $\square$

4. Discrete segments of $\mathbb{Z}^3$

We specialize the composition algorithm for the synchronization of three Christoffel words in order to obtain 3D discrete line segments. For this, we fix the starting point to $(0,0,0)$ and consider an end point $(a,b,c) \in \mathbb{N}^3$, with $a,b,c > 0$. We use the composition algorithm on the three central words u_{12}, u_{31} and u_{23} defined as follows :

$$\begin{aligned} C_{(1,2)}(a,b) &= 1 \cdot u_{12} \cdot 2, \\ C_{(3,1)}(c,a) &= 3 \cdot u_{31} \cdot 1, \\ C_{(2,3)}(b,c) &= 2 \cdot u_{23} \cdot 3. \end{aligned}$$

First, we consider the case where a, b, and c are *totally relatively primes*, that is $\gcd(a,b) = \gcd(a,c) = \gcd(b,c) = 1$. We use the composition algorithm on the three central words u_{12}, u_{31} and u_{23}. The *composition word* for vector (a,b,c), denoted by χ, is defined as the composition word obtained for three compatible Christoffel words. Previously, special letters of the form $\overleftrightarrow{12}$ were introduced to encode the fact that an integer coordinates point has been reached. By analogy, in dimension three, we introduced the special letter $\overleftrightarrow{123}$ in order to code that an integer crossing of $\mathbb{Z}^3$ has been reached. This letter encodes the fact that all three letters should occur at this position. We call the word $B(a,b,c) = \overleftrightarrow{123}\chi$ the *discrete segment* of vector (a,b,c). By analogy to the 3D billiard word, the special letter $\overleftrightarrow{123}$ encodes the starting point $(0,0,0)$. Note that, by convention, the special letter $\overleftrightarrow{123}$ abelianizes to $(1,1,1)$.

4.1. Composition of primitive words

First, we consider the case a, b and c are totally relatively primes, that is $\gcd(a,b) = \gcd(b,c) = \gcd(a,c) = 1$. This implies that the three words used as input for the composition algorithm do not contain special letters.

For example, we compute the discrete segment for vector $(7,5,4)$ by considering

$$\begin{aligned} C_{(1,2)}(7,5) &= \overrightarrow{1}\,1212112121\,\overleftarrow{2}, \\ C_{(3,1)}(4,7) &= \overrightarrow{3}\,131131131\,\overleftarrow{1}, \\ C_{(2,3)}(5,4) &= \overrightarrow{2}\,2323232\,\overleftarrow{3}. \end{aligned}$$

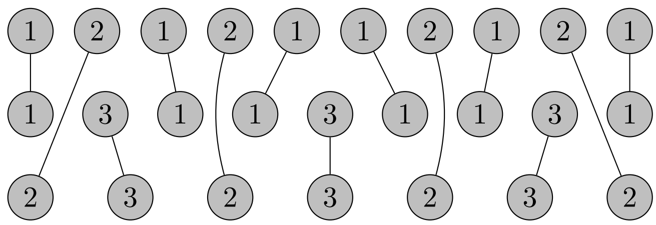

Figure 5: Illustration of Algorithm 1 on the central words defined by vector $(7,5,4)$. Words w_{12}, w_{31}, w_{23} are set as the central words of the Christoffel words of slope $\frac{5}{7}$, $\frac{4}{7}$ and $\frac{4}{5}$. The composition word for vector $(7,5,4)$ is $\chi = 1231213121321$ and the discrete segment is $B(7,5,4) = \overleftrightarrow{123}\chi$.

The three input central words are thus, $u_{12} = 1212112121$, $u_{31} = 131131131$ and $u_{23} = 2323232$. At the beginning of the algorithm, the composition word is empty, $\chi = \varepsilon$, and it is incrementally constructed by taking letters which are equals in the words u_{12}, u_{31} and u_{23}. At the first iteration, we have $u_{12}[1] = 1$, $u_{31}[1] = 1$, $u_{23}[1] = 2$, thus we append the letter 1 to χ and then $\chi = 1$. Then we consider the next letters in u_{12} and u_{31} but not in u_{23} since its letter we not used yet. At the second iteration, we have $u_{12}[2] = 2$, $u_{31}[2] = 3$ and $u_{23}[1] = 2$, thus the letter 2 is append to χ and so on until the three input words are completely read and $\chi = 1231213121321$ (see Figure 5). The central word obtained is palindromic and the discrete segment $B(7, 5, 4) = \overleftrightarrow{1231231213121321}$ abelianizes to $(7, 5, 4)$. Note that the only way that a composition word for a vector (a, b, c) abelianizes to $(a - 1, b - 1, c - 1)$ is that the composition is successful on every letters of the input words.

Proposition 4.1. *The composition word χ for vector (a, b, c), where $\gcd(a, b) = \gcd(b, c) = \gcd(a, c) = 1$ constructed by Algorithm 1 is palindromic and its abelianized is $\overrightarrow{\chi} = (a-1, b-1, c-1)$.*

Proof. Let B be the cubic billiard word obtained as the cutting sequence of the segment from $(0, 0, 0)$ to (a, b, c). Since a, b and c are totally relatively primes, the open segment never crosses a unit cube's edge or vertex and thus the word B is well defined on alphabet $\{1, 2, 3\}$. Also, for symmetry reasons of the cutting sequence according to the point $(a/2, b/2, c/2)$ we have that B is a palindrome.

Each projection of B is a square billiard word:

$$
\begin{aligned}
B{\downarrow_3} &= B_{(1,2)}(a, b),\\
B{\downarrow_2} &= B_{(1,3)}(a, c),\\
B{\downarrow_1} &= B_{(2,3)}(b, c).
\end{aligned}
$$

The composition word χ is computed with the composition algorithm using the central words of the Christoffel words $C_{(1,2)}(a, b)$, $C_{(3,1)}(c, a)$ and $C_{(2,3)}(b, c)$ as input. Using the fact that $B_{(1,3)}(a, c) = B_{(3,1)}(c, a)$, Proposition 2.3 and Lemma 3.1, we conclude that the composition word χ is B.

□

4.2. Billiard words as jump words

We now introduce a formalism which is a monodimensional vision of billiard words. This allows us to deal with the rational relations between a, b and c.

An equivalent definition of hypercubic billiards is to unfold the trajectory and to consider jump words on a hypercubic grid. We define a *jump word* J on alphabet $\{1, 2, \ldots, d\}$, associated with a hypercubic billiard in dimension d of angle vector $\alpha = (\alpha_1, \alpha_2, \cdots \alpha_d)$ with rationally independent coordinates as follows. Let $P_j \subset \mathbb{R} \times \{1, 2, \ldots, d\}$ be the set

$$P_J = \bigcup_{i=1}^{d} \{\, (n\alpha_i, i) \mid n \in \mathbb{N}\}.$$

Sort the elements of P_J in increasing order according to the first coordinate. Then J is the concatenation of the second coordinates (see Figure 6). Note that for $d = 2$ and α_1, α_2 rationally independent, we have Sturmian words and for $d = 3$ and $\alpha_1, \alpha_2, \alpha_3$ rationally independent we have cubic billiard words [Bar95].

We use this construction for $d = 3$, with each $\alpha_i \in \mathbb{Q}$ in order to synchronize Christoffel words, primitive or not. There is a problem if at least two letters have the same position, that is $n\alpha_i = m\alpha_j$ for $i \neq j$ and $n, m \neq 0$. This is called a *collision* and this occurs when there is a rational relation

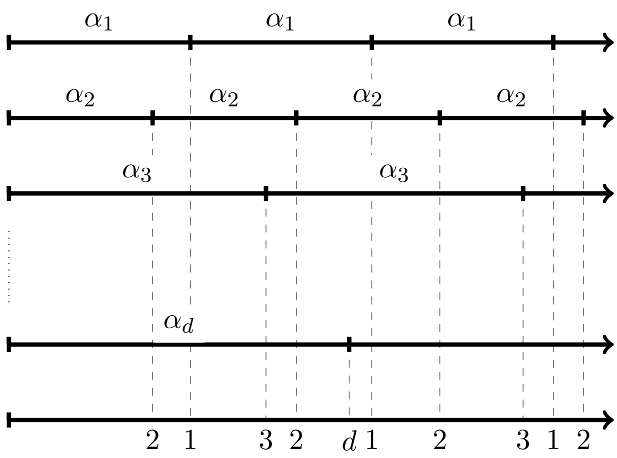

Figure 6: Illustration of the construction of a n-dimensional hypercubic billiard word using a jump word. The natural order on the $n\alpha_i$ defines the billiard word.

between a, b and c. This implies that at least one of the Christoffel words is not primitive. We have a choice for coding this collision by 12 or 21 and thus, we use the special letter $\overleftrightarrow{12}$.

In order to understand the relation between the preceding construction and billiard words, we restate the properties in terms of jump words. In the previous example we compute a discrete segment from $(0,0,0)$ to $(7,5,4)$. Thus in terms of jump words we have three possible jumps $\alpha_1 = \frac{16}{7}$, $\alpha_2 = \frac{16}{5}$ and $\alpha_3 = \frac{16}{4}$. Thus we construct the positions, in $\mathbb{R}$, of letter 1 by considering the values $n\alpha_1$ for $n \in \mathbb{N}$. These positions are : $0,\ \frac{16}{7},\ \frac{2*16}{7},\ \frac{3*16}{7},\ \frac{4*16}{7},\ \frac{5*16}{7},\ \frac{6*16}{7},\ \frac{7*16}{7},\cdots$. Similarly, the positions of letter 2 are given by $0,\ \frac{16}{5},\ \frac{2*16}{5},\ \frac{3*16}{5},\ \frac{4*16}{5},\ \frac{5*16}{5},\cdots$. And finally, the positions of letter 3 are given by $0,\ \frac{16}{4},\ \frac{2*16}{4},\ \frac{3*16}{4},\ \frac{4*16}{4},\cdots$. Notice that the positions of the letters, in $\mathbb{R}$, are the same up to period 16. Now we sort the positions of the letters in the period. The three first letters are in position 0 and then quotients are in the following usual order:

$$\begin{array}{cccccccccccccccc}
0, & \frac{16}{7}, & \frac{16}{5}, & \frac{16}{4}, & \frac{2*16}{7}, & \frac{2*16}{5}, & \frac{3*16}{7}, & \frac{2*16}{4}, & \frac{4*16}{7}, & \frac{3*16}{5}, & \frac{5*16}{7}, & \frac{3*16}{4}, & \frac{4*16}{5}, & \frac{6*16}{7}, & 16. \\
\overleftrightarrow{123} & 1 & 2 & 3 & 1 & 2 & 1 & 3 & 1 & 2 & 1 & 3 & 2 & 1 & \overleftrightarrow{123}.
\end{array}$$

Remark that the positions in the period, excluding the extremities 0 and 16, are two-by-two distinct and are invariant by the central symmetry of center $\frac{2*16}{4} = 8$, which means that the above set of fractions is invariant under the function $x \mapsto (2*8) - x$. This is a geometrical way to show the palindromicity of the composition word $\chi = 1231213121321$.

4.3. Composition of non-primitive words

Now, lets consider the case where pairwise gcds are greater than one. We compute the discrete segment from $(0,0,0)$ to $(6,4,3)$. In terms of jump words, the jumps are $\alpha_1 = \frac{13}{6}$, $\alpha_2 = \frac{13}{4}$ and $\alpha_3 = \frac{13}{3}$.

$$\begin{array}{cccccccc}
0, & \frac{13}{6}, & \frac{13}{4}, & \frac{2*13}{6} = \frac{13}{3}, & \frac{3*13}{6} = \frac{2*13}{4}, & \frac{4*13}{6} = \frac{2*13}{3}, & \frac{3*13}{4}, & \frac{5*13}{6}, & 13. \\
\overleftrightarrow{123} & 1 & 2 & \overleftrightarrow{13} & \overleftrightarrow{12} & \overleftrightarrow{13} & 2 & 1 & \overleftrightarrow{123}.
\end{array}$$

As mentioned, a special letter occurs when $n\alpha_i = m\alpha_j$. In order to use the composition algorithm on words containing special letters, we introduce a new rule for composition : if one of the three words begins with $\overleftrightarrow{\mathbf{ab}}$, a second one with $\mathbf{a}$ and a third one with $\mathbf{b}$, the first letter is removed from all three words and the special letter $\overleftrightarrow{\mathbf{ab}}$ is added at the end of the composition word. Figure 7 illustrate the execution of the composition algorithm on the following words:

$$\begin{aligned} C_{(1,2)}(6,4) = &\ \left(C_{(1,2)}(3,2)\right)^2 = \overrightarrow{1}121\overleftrightarrow{21}121\overleftarrow{2}\,, \\ C_{(3,1)}(3,6) = &\ \left(C_{(3,1)}(1,2)\right)^3 = \overrightarrow{3}1\overleftrightarrow{13}1\overleftrightarrow{13}1\overleftarrow{1}\,, \\ C_{(2,3)}(4,3) = &\ \left(C_{(2,3)}(4,3)\right)^1 = \overrightarrow{2}23232\overleftarrow{3}\,. \end{aligned}$$

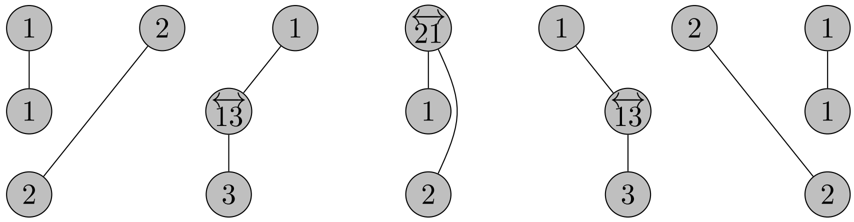

Figure 7: Illustration of the composition algorithm for vector $(6,4,3)$. Input words are $w_{12} = 121\overleftrightarrow{21}121$, $w_{31} = 1\overleftrightarrow{13}1\overleftrightarrow{13}1$ and $w_{23} = 23232$. The composition word $\chi = 12\overleftrightarrow{13}\,\overleftrightarrow{12}\,\overleftrightarrow{13}21$.

In order to show that the composition words for vector $(a,b,c) \in \mathbb{N}^3$ with $a,b,c > 0$ and $\gcd(a,b,c) = 1$ is successful, we first show that special letters synchronizes with the appropriate letters in the other words.

Lemma 4.2. *Let $(a,b,c) \in \mathbb{N}^3$ with $a,b,c > 0$ and $\gcd(a,b,c) = 1$. Throughout the execution of the composition algorithm with input words u_{12}, u_{31} and u_{23} defined as :*

$$\begin{aligned} C_{(1,2)}(a,b) = &\ \overrightarrow{1} \cdot u_{12} \cdot \overleftarrow{2} \\ C_{(3,1)}(c,a) = &\ \overrightarrow{3} \cdot u_{31} \cdot \overleftarrow{1} \\ C_{(2,3)}(b,c) = &\ \overrightarrow{2} \cdot u_{23} \cdot \overleftarrow{3} \end{aligned}$$

if at one point one of the three words X, Y, Z starts with letter $\overleftrightarrow{\boldsymbol{ab}}$ then, the two other ones either both starts with the same letter or one starts by $\boldsymbol{a}$ and the second one by $\boldsymbol{b}$.

Proof. First we show that at a given iteration of the composition algorithm, only one of the three words X, Y, Z may start by a special letter. By contradiction, suppose it it not the case. This means that, two special letters are associated to the same fraction in the jump word. For instance we have a point labeled by 1 and 2 (for the word u_{12}) and by 1 and 3 (for the word u_{31}). We show that this is also labeled by 2 and 3 which means a special letter for the word u_{23}. This implies that not only are all three words are non-primitive, but the three special letters occur in the same position which is an integer point between $(0,0,0)$ and (a,b,c) (these points excluded).

More precisely, we find k and ℓ such that $\frac{k*(a+b+c)}{a} = \frac{\ell*(a+b+c)}{b}$ which implies a collision for the letters 1 and 2 and we find ℓ and m such that $\frac{\ell*(a+b+c)}{b} = \frac{m*(a+b+c)}{c}$ which implies a collision for the letters 1 and 3. Then we also have the equality $\frac{k*(a+b+c)}{a} = \frac{m*(a+b+c)}{c}$ which implies a collision for the letter 2 and 3. The whole equality $\frac{k*(a+b+c)}{a} = \frac{\ell*(a+b+c)}{b} = \frac{m*(a+b+c)}{c}$ implies another integer point between $(0,0,0)$ and (a,b,c). Indeed, we have the following equalities $\frac{k}{a} = \frac{l}{b} = \frac{m}{c}$ and furthermore as $\gcd(a,b) > 1$ we have $\frac{k}{a} = \frac{l}{b} = \frac{m}{c} = \frac{p}{q}$ where $\frac{p}{q}$ is in reduce form (that is $a \neq q$). Thus $\frac{k}{a} = \frac{k'p}{a'q}$, $\frac{l}{b} = \frac{l'p}{b'q}$ and $\frac{m}{c} = \frac{m'p}{c'q}$. It follows that (a',b',c') with $a',b',c' \in \mathbb{N} - \{0\}$ appears in the segment between $(0,0,0)$ and (a,b,c) in contradiction with the hypothesis $\gcd(a,b,c) = 1$.

Finally, it remains to show that the letters correspond. The three words u_{12}, u_{31} and u_{23} are in correspondence with the structure $\cup_{i=1}^{3}\{n\alpha_i | n \in \mathbb{N} - \{0\}\}$ where $\alpha_1 = (a+b+c)/a$, $\alpha_2 = (a+b+c)/b$, $\alpha_3 = (a+b+c)/c$, with the restriction to the $a+b+c-3$ first letters of this infinite periodic word.

Again by contradiction, suppose that the words X starts by $\overleftrightarrow{12}$, Y start by 1 and Z start by 3 (the only other problematic case would be Y starts by 3 and Z start by 2 which is similar). Then it means that in the jump word either we have a special letter $\overleftrightarrow{123}$ which is impossible by the hypothesis $\gcd(a,b,c) = 1$ or we have the letter 3 before letter $\overleftrightarrow{12}$ and this letter has not been synchronized. Thus we find a contradiction because word Y may not start with letter 1 (indeed in order to synchronize this case Y must starts by the letter 3). □

We now state the main result of the section.

Theorem 4.3. *The composition word χ for vector (a,b,c), where $\gcd(a,b,c) = 1$, constructed by Algorithm 1 with input words u_{12}, u_{31} and u_{23} defined as :*

$$\begin{array}{rl} C_{(1,2)}(a,b) = & \overrightarrow{1} \cdot u_{12} \cdot \overleftarrow{2} \\ C_{(3,1)}(c,a) = & \overrightarrow{3} \cdot u_{31} \cdot \overleftarrow{1} \\ C_{(2,3)}(b,c) = & \overrightarrow{2} \cdot u_{23} \cdot \overleftarrow{3} \end{array}$$

is such that $\overrightarrow{\chi} = (a-1, b-1, c-1)$.

Proof. Lemma 4.2 ensures that the synchronization of special letters of the form $\overleftrightarrow{\mathbf{ab}}$ is correct. As for the lemma, the word u_{12} is in correspondence with $\cup_{i=1,2}\{n\alpha_i \,|\, n \in \mathbb{N} - \{0\}\}$, the second one u_{31} with $\cup_{i=1,3}\{n\alpha_i \,|\, n \in \mathbb{N} - \{0\}\}$ and the third one u_{23} with $\cup_{i=2,3}\{n\alpha_i \,|\, n \in \mathbb{N} - \{0\}\}$, where $\alpha_1 = (a+b+c)/a$, $\alpha_2 = (a+b+c)/b$, $\alpha_3 = (a+b+c)/c$. The synchronization of the letters 1, 2 and 3 also comes from the fact that the relative order of the letters in each of the three 2D jump word. More precisely, the word χ is in correspondence with the structure $\cup_{i=1}^{3}\{n\alpha_i | n \in \mathbb{N} - \{0\}\}$ and with the restriction to the $a+b+c-3$ first letters of this infinite periodic word. Indeed, since the input word w_{12} has length $a+b-2$, w_{13} has length $a+c-2$ and w_{23} has length $b+c-2$, the composition word may not be longer then $a+b+c-3$.

All letters of the composition word are associated with a distinct positions except for the coding of two dimensional integer points implying special letters. The bound on the length of χ ensures that all letters of χ are in correspondence with positions before $a+b+c$ and thus, before the first integer point is crossed be the cutting sequence. Thus using the method of synchronization either there is no collision or if there is a special letter of the form $\overleftrightarrow{\mathbf{ab}}$ and both cases leads to a synchronized word.

□

When building a discrete segment, if the composition contains special letter of the form $\overleftrightarrow{\mathbf{ab}}$, each special letter must be replaced by a corresponding pair of letters.

Regarding the palindromicity of central words, it is obvious that there exists a choice that leads to a palindromic composition words if and only if the number of special letters is even. In such case, it suffices to replace special letters of the form $\overleftrightarrow{\mathbf{ab}}$ by, say, **ab** on the left part of the word and by **ba** on the right side. On the other hand, an odd number of special letter forces that one of them is located in the center of the word so that neither the choice **ab** or **ba** may yield a palindrome. For instance, as shown in Figure 8, for vector $(6, 4, 3)$ no matter what are the choices for the three special letters, the resulting word may never be palindromic.

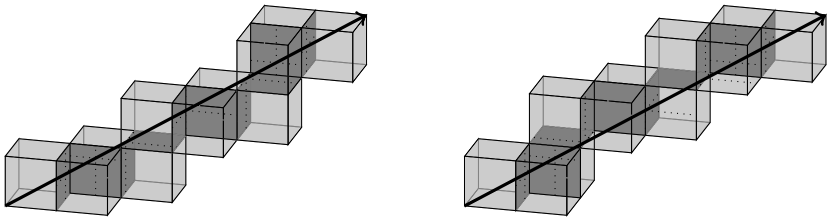

Figure 8: Two discrete segments for vector $(6, 4, 3)$. On the left, the special letters $\overleftrightarrow{13}$, $\overleftrightarrow{12}$ and $\overleftrightarrow{13}$ are replaced respectively by 13, 12 and 13, leading to the discrete segment $\overleftrightarrow{123}$1213121321, while on the right special letters $\overleftrightarrow{13}$, $\overleftrightarrow{12}$ and $\overleftrightarrow{13}$ are replaced by 31, 21 and 31 leading to $\overleftrightarrow{123}$1231213121.

Finally, we remark that the choices made for special letters allow to form object that do not always correspond to any of the classical definitions of discrete segments.

5. Experimentations

The goal of this section is to investigate the properties of the central discrete segments in terms of balancedness, palindromicity and distance to the real segment.

In the case where $\gcd(a, b, c) = 1$ but at least one of the $\gcd(a, b)$, $\gcd(a, c)$, $\gcd(b, c)$ is greater than one, as seen previously, there is a non-trivial choice for each special letter. We now focus on experimental observation in order to observe the effect of those non-trivial choices on some combinatorial and geometric properties of the composition word.

Given a composition word w containing $k \geq 1$ special letters, a specific choice for those special letters is described by a list of k bits. We thus represent those choices by a vector $[c_1, c_2, \ldots, c_k]$ where $c_i = 0$ if the i-th special letter $\overleftrightarrow{\mathbf{ab}}$ is replaced by **ab** and $c_i = 1$ if $\overleftrightarrow{\mathbf{ab}}$ is replaced by **ba** (see Figure 9).

The property called *distance* is defined as follow. Given a point $p \in \mathbb{R}^3$ and a line $l \subset \mathbb{R}^3$, we note $d(p, l)$ the distance from the point p to the line l. Given a word $w \in \{1, 2, 3\}^*$, the distance from w to the real line l is the maximal distance from one of the points of the path w (w being seen as a Freeman code) to l. More precisely, the distance from w to l is $\max\{d(\overrightarrow{p}, l) \mid p \text{ is a prefix of } w\}$. Let W be the discrete segment for vector (a, b, c). If W does not contain any special letter, then the property called *distance of* W the distance from W to the line l that passes through the origin and (a, b, c). On the other hand, if W does contain special letters, then there each choice for the special letters defines a word on $\{1, 2, 3\}$. Among these choices those that minimizes the distance of W to the line l are called *minimal choices* and the *minimal distance of* W is the distance obtained by its minimal choices.

- Vector $(5, 5, 3)$

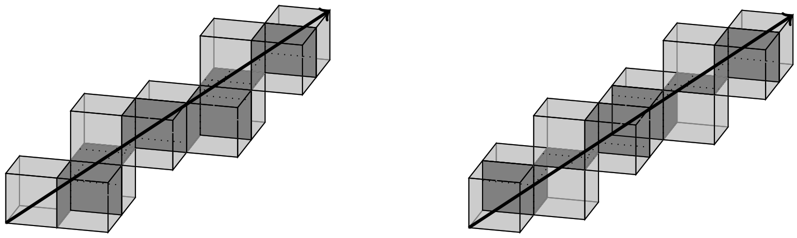

Figure 9: Vector $(5,5,3)$ produces the composition word $\overleftrightarrow{12}3\overleftrightarrow{12}\,\overleftrightarrow{12}3\overleftrightarrow{12}$. On the left, the composition choice, $[0,0,0,0] \to 1231212312$ and, on the right, the choice $[1,0,1,0] \to 2131221312$.

$$\begin{aligned}
(C_{(1,2)}(1,1))^5 &= \overrightarrow{1}\,\overleftrightarrow{12}\,\overleftrightarrow{12}\,\overleftrightarrow{12}\,\overleftrightarrow{12}\,\overleftarrow{2}\,,\\
(C_{(3,1)}(3,5))^1 &= \overrightarrow{3}\,131131\,\overleftarrow{1}\,,\\
(C_{(2,3)}(5,3))^1 &= \overrightarrow{2}\,232232\,\overleftarrow{3}\,,
\end{aligned}$$

$$\begin{aligned}
w_{12} &= \overleftrightarrow{12}\,\overleftrightarrow{12}\,\overleftrightarrow{12}\,\overleftrightarrow{12},\\
w_{31} &= 131131,\\
u_{23} &= 232232,\\
\chi &= \overleftrightarrow{12}3\overleftrightarrow{12}\,\overleftrightarrow{12}3\overleftrightarrow{12}.
\end{aligned}$$

Each of the four special letters $\overleftrightarrow{12}$ may be replaced by either 12 or 21 in order to obtain a word on $\{1,2,3\}$. This sums up to 16 possibles choices. Among those choices, only two allow to achieve minimum balance and five yield to palindromes :

- Minimum balance : 1.

  $[0,0,0,0] \to 12 \cdot 3 \cdot 12 \cdot 12 \cdot 3 \cdot 12.$
  $[1,1,1,1] \to 21 \cdot 3 \cdot 21 \cdot 21 \cdot 3 \cdot 21.$

- Palindromes

  $[0,0,1,1] \to 12 \cdot 3 \cdot 12 \cdot 21 \cdot 3 \cdot 21.$
  $[0,1,0,1] \to 12 \cdot 3 \cdot 21 \cdot 12 \cdot 3 \cdot 21.$
  $[1,0,1,0] \to 21 \cdot 3 \cdot 12 \cdot 21 \cdot 3 \cdot 12.$
  $[1,1,0,0] \to 21 \cdot 3 \cdot 21 \cdot 12 \cdot 3 \cdot 12.$

- Distance. In this particular case it appears that all 16 choices are minimal choices that all yield to the distance $\sqrt{42/59} \approx 0.844$.

This shows that even though specific a choice allows to obtain a balanced word or a palindrome, unlike in dimension two both properties may not happen at the same time.

- Vector $(3,4,8)$

$$\begin{aligned}
(C_{(1,2)}(3,4))^1 &= \overrightarrow{1}\,21212\,\overleftarrow{2}\,,\\
(C_{(3,1)}(8,3))^1 &= \overrightarrow{3}\,331333133\,\overleftarrow{1}\,,\\
(C_{(2,3)}(1,2))^4 &= \overrightarrow{2}\,3\,\overleftrightarrow{23}\,3\,\overleftrightarrow{23}\,3\,\overleftrightarrow{23}\,3\,\overleftarrow{3}\,,
\end{aligned}$$

$$\begin{aligned}
w_{12} &= 21212,\\
w_{31} &= 331333133,\\
w_{23} &= 3\,\overleftrightarrow{23}\,3\,\overleftrightarrow{23}\,3\,\overleftrightarrow{23}\,3,\\
\chi &= 3\,\overleftrightarrow{23}\,1\,3\,\overleftrightarrow{23}\,3\,1\,\overleftrightarrow{23}\,3.
\end{aligned}$$

  - Minimum balance : 1.

    $[0,0,1] \to 3 \cdot 23 \cdot 13 \cdot 23 \cdot 31 \cdot 32 \cdot 3.$

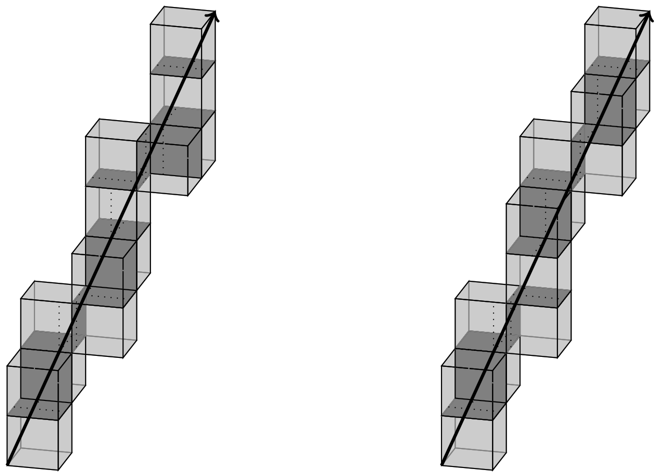

Figure 10: Vector $(3, 4, 8)$ produces the composition word $3\,\overleftrightarrow{23}\,13\,\overleftrightarrow{23}\,31\,\overleftrightarrow{23}\,3$. On the left, the composition choice, $[0, 0, 0] \rightarrow 323132331233$ and, on the right, the choice $[0, 1, 1] \rightarrow 323133231323$.

$$[0, 1, 1] \rightarrow 3 \cdot 23 \cdot 13 \cdot 32 \cdot 31 \cdot 32 \cdot 3.$$

- Palindromes : since χ has an odd number of special letters, no choice leads to a palindrome.
- Distance. The minimal distance is $\sqrt{69/89} \approx 0.881$ and is achieved only with the following choices :

$$[0, 0, 0] \rightarrow 3 \cdot 23 \cdot 13 \cdot 23 \cdot 31 \cdot 23 \cdot 3.$$

This example shows that in some cases, the smallest balance may not be reached by applying the same choice everywhere. Moreover, in this case, the choice that minimizes the distance does not minimize balance.

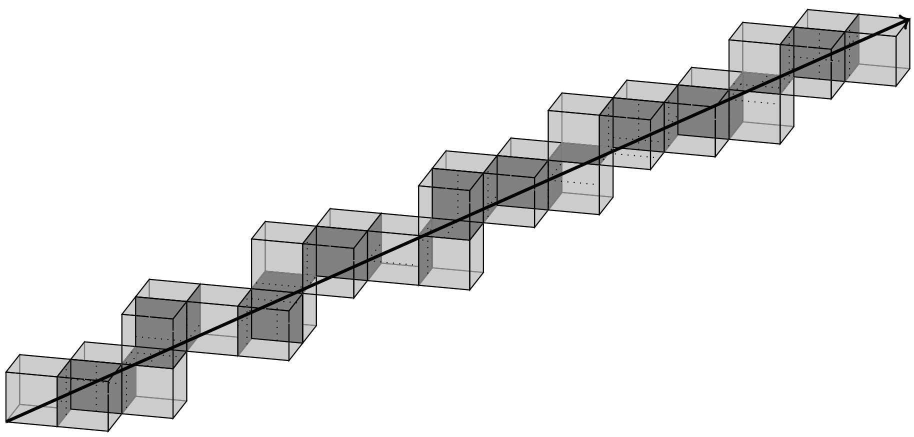

Figure 11: Vector $(15, 10, 6)$, with $[1, 1, 1, 1, 1, 1, 1] \rightarrow 1213 \cdot 21 \cdot 12 \cdot 31 \cdot 21 \cdot 1 \cdot 32 \cdot 1 \cdot 21 \cdot 31 \cdot 21 \cdot 21 \cdot 3121.$

- Vector $(15, 10, 6)$

$$
\begin{aligned}
(C_{(1,2)}(3,2))^5 &= \overrightarrow{1}\,121\,\overleftrightarrow{12}\,121\,\overleftrightarrow{12}\,121\,\overleftrightarrow{12}\,121\,\overleftrightarrow{12}\,121\,\overleftarrow{2}\,, \\
(C_{(3,1)}(2,5))^3 &= \overrightarrow{3}\,11311\,\overleftrightarrow{13}\,11311\,\overleftrightarrow{13}\,11311\,\overleftarrow{1}\,, \\
(C_{(2,3)}(5,3))^2 &= \overrightarrow{2}\,232232\,\overleftrightarrow{23}\,232232\,\overleftarrow{3}\,, \\
\\
w_{12} &= 121\,\overleftrightarrow{12}\,121\,\overleftrightarrow{12}\,121\,\overleftrightarrow{12}\,121\,\overleftrightarrow{12}\,121, \\
w_{31} &= 11311\,\overleftrightarrow{13}\,11311\,\overleftrightarrow{13}\,11311, \\
w_{23} &= 232232\,\overleftrightarrow{23}\,232232, \\
\chi &= 1213\,\overleftrightarrow{12}\;12\,\overleftrightarrow{13}\,\overleftrightarrow{12}\,1\,\overleftrightarrow{23}\,1\,\overleftrightarrow{12}\;\overleftrightarrow{13}\,21\,\overleftrightarrow{12}\,3121.
\end{aligned}
$$

  Vector $(15, 10, 6)$ is the smallest with the property that $\gcd(a, b, c) = 1$ but $\gcd(a, b)$, $\gcd(a, c)$ and $\gcd(b, c)$ are all greater than one. The composition word χ contains seven special letters leading to 128 possible choices.

  - Minimum balance : 2. This balance is achieved by all 128 possible choices which means that no choice leads to a balance of 1.
  - Palindromes : since χ has an odd number of special letters, no choice leads to a palindrome.
  - Distance. The two following choices yeild the minimal distance of $2/19\sqrt{65} \approx 0.849$.

    $[0, 1, 1, 1, 1, 1, 1] \to 1213 \cdot 12 \cdot 12 \cdot 31 \cdot 21 \cdot 1 \cdot 32 \cdot 1 \cdot 21 \cdot 31 \cdot 21 \cdot 21 \cdot 3121.$
    $[1, 1, 1, 1, 1, 1, 1] \to 1213 \cdot 21 \cdot 12 \cdot 31 \cdot 21 \cdot 1 \cdot 32 \cdot 1 \cdot 21 \cdot 31 \cdot 21 \cdot 21 \cdot 3121.$

6. Composition algorithm on discrete planes

Even though we have focused on the application of the composition algorithm on Christoffel words, we recall that this algorithm may be used on any triplet of words as long as their respective alphabets coincide. We now consider triplets of words defined from discrete planes. We use the arithmetical definition of discrete planes introduced in [Rev91].

Definition 6.1.

- A *voxel* is a unit cube aligned with the axes whose center has integer coordinates. If (x, y, z) is the center of the voxel V, by convention, we say that V is located in (x, y, z).
- Each face of a voxel is called a *surfel*.
- Given a normal vector $v = (a, b, c) \in \mathbb{R}^3$, the *arithmetic discrete plane* $P(a, b, c, \mu, \omega)$ is defined as the set of voxels located in (x, y, z) satisfying the double inequality

$$0 \leq ax + by + cz - \mu < \omega$$

  where the two parameters μ and ω are, respectively, the *intercept* and the *thickness*.

In order to define words on a discrete plane, we first introduce the notion of surface and stripes. Each surfel is adjacent to exactly two voxels. A surfel is called *visible* if, among its two adjacent voxels, one is in the discrete plane and the other is not. When the thickness ω is large enough, the visible surfels form two infinite and simply connected surfaces. For more details on discrete planes and their topology see [AAS97]. We only consider the case where the intercept $\mu = 0$, thickness $\omega = \infty$ and where the normal vector is positive, that is $a, b, c > 0$. Such a discrete plane is simply denoted $P(a, b, c)$. Note that by fixing the thickness to infinity, the voxels that are include in the plane are those located in (x, y, z) satisfying $ax + by + cz \geq 0$ and the visible surfels form only one surface.

Definition 6.2. A *corner* of a digital plane $P(a, b, c)$ is a voxel $(x, y, z) \in P(a, b, c)$ such that $(x-1, y, z), (x, y-1, z), (x, y, z-1) \notin P(a, b, c)$.

Since voxels are aligned with the axes, each surfel is normal to one of the vectors e_i. Using the convention that $e_i \mapsto i$, a surfel is associated with a letter in $\{1, 2, 3\}$ and a path on a surface made of surfel defines a word. A surfel normal to e_i is called of *type* i.

Definition 6.3. Let $A \in \mathbb{Z}$, a *x-stripe* of $P(a, b, c)$ is the set of visible surfels intersected by the real plane $\{(A, y, z) \mid y, z \in \mathbb{R}\}$.

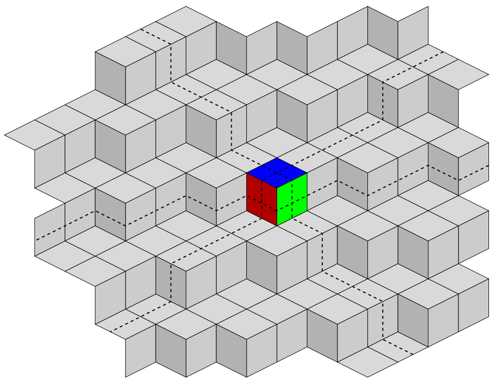

Figure 12: Three stripes defined by a corner on a discrete plane.

The y-stripes and z-stripes are defined in a similar way so that any corner of a discrete plane defines three stripes (see Figure 12). Each of these tripes define a biinfinite path of surfels on the discrete surface.

Let V be a corner located in (x, y, z) in a discrete plane $P(a, b, c)$. For $i = 1, 2, 3$, let S_i be a visible surfel of type i adjacent to V. Each S_i in intersected by exactly two stripes. For instance, S_1 is intersected by a y-stripe and a z-stripe, while S_2 is intersected by the same z-string and an x-stripe. Let $\mathcal{S}_x$ be the biinfinite sequence of surfel intersected by the x-stripe. The sequences $\mathcal{S}_y$ and S_z are defined similarly so that we have:

$$\begin{aligned} \mathcal{S}_z &= \mathcal{S}_{12} \cdot S_1 \cdot S_2 \cdot \mathcal{S}'_{12}, \\ \mathcal{S}_y &= \mathcal{S}_{31} \cdot S_3 \cdot S_1 \cdot \mathcal{S}'_{31}, \\ \mathcal{S}_x &= \mathcal{S}_{23} \cdot S_2 \cdot S_3 \cdot \mathcal{S}'_{23}, \end{aligned}$$

where S_{ij} is a left-infinite sequence of surfels of types i and j and $\mathcal{S}'_{ij}$ is a right-infinite sequence of surfels also of types i and j. From the six infinite sequences we define the six infinite words W_{ij} and W'_{ij} as follow :

$$W_{12} = \Phi(\widetilde{\mathcal{S}_{12}}), \quad W_{31} = \Phi(\widetilde{\mathcal{S}_{31}}), \quad W_{23} = \Phi(\widetilde{\mathcal{S}_{23}}), \quad W'_{12} = \Phi(\mathcal{S}'_{12}), \quad W'_{31} = \Phi(\mathcal{S}'_{31}), \quad W'_{23} = \Phi(\mathcal{S}'_{23}),$$

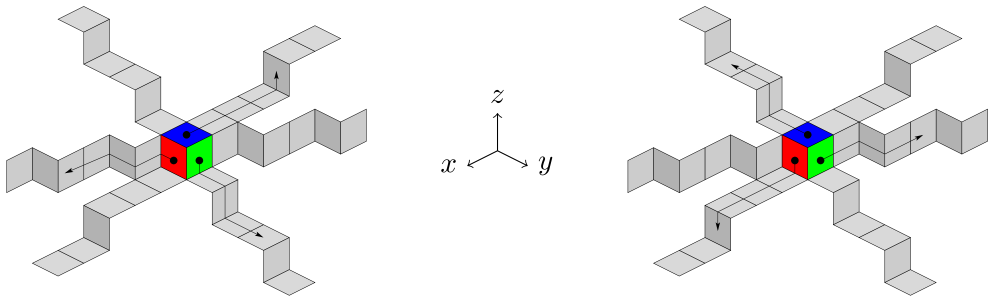

Figure 13: Six infinite words defined from a corner, on both cases exactly one word starts from each of the visible surfels of the corner. On the left, starting from S_1 (in red) the word $1 \cdot W_{12} = 1 \cdot 21121 \cdots$, starting from S_2 (in green) the word $2 \cdot W_{23} = 2 \cdot 32232 \cdots$, and from S_3 (in blue) the word $3 \cdot W_{31} = 3 \cdot 333133 \cdots$. On the right, starting from S_1 (in red) the word $1 \cdot W'_{31} = 1 \cdot 333133 \cdots$, starting from S_2 (in green) the word $2 \cdot W'_{12} = 2 \cdot 212212 \cdots$, and from S_3 (in blue) the word $3 \cdot W'_{23} = 3 \cdot 323323 \cdots$.

where $\widetilde{\mathcal{S}}$ is the reversal of the sequence $\mathcal{S}$ and $\Phi(\mathcal{S})$ is the word obtained by replacing each surfel of $\mathcal{S}$ by its type. Figure 13 illustrates this construction.

Being written on appropriate alphabets, theses infinite words may be used as input for the composition algorithm. Each corner of a discrete plane defines two *planar composition words* χ_0 and χ_1 as follow :

- χ_0 is the output of Algorithm 1 with the input W_{12}, W_{31}, W_{23}.
- χ_1 is the output of Algorithm 1 with the input W'_{12}, W'_{31}, W'_{23}.

Figure 14 illustrates the construction of planar composition words. In the case where these two words are finite, two *planar composition vectors* are defined as $\lambda_0 = \overrightarrow{\chi_0} + (1,1,1)$ and $\lambda_1 = \overrightarrow{\chi_1} + (1,1,1)$. Vector $(1,1,1)$ is added to the abelianized of the planar composition words since the letter given by the three surfels S_1, S_2, S_3 were not considered in the composition process.

6.1. Composition vectors from a rational plane

In this section we show that on well chosen corners of a discrete plane with a rational normal vector, the composition vectors are equal to the normal vector.

Definition 6.4. Given a normal vector $v = (a,b,c) \in \mathbb{R}^3$, the *height* of a point $u = (x,y,z) \in \mathbb{R}^3$ is the scalar product $\langle u, v\rangle = ax + by + cz$.

By analogy, we call the *height* of voxel the height of its center. Obviously, the voxels adjacent to visible surfels are those whose heights are in $[0, max(|a|,|b|,|c|)]$ and the corners are those whose heights are in $[0, min(|a|,|b|,|c|)]$. In the case where $v = (a,b,c) \in \mathbb{N}^3$ with $a,b,c > 0$, then the finite patterns of $P(a,b,c)$ are repeated according to a regular lattice. In fact, $P(a,b,c)$ is invariant by translation of any vector (x,y,z) such that $ax + by + cz = 0$. This implies that two corners having the same height produce the same composition vectors.

Theorem 6.5. *Given $(a,b,c) \in \mathbb{N}^3$ with $a,b,c > 0$ and $\gcd(a,b,c) = 1$, let V_0 be the voxel centered at $(0,0,0)$, then V_0 is a corner of $P(a,b,c)$ and the planar composition words obtained from V_0 are finite and the planar composition vectors are such that $\lambda_0 = \lambda_1 = (a,b,c)$.*

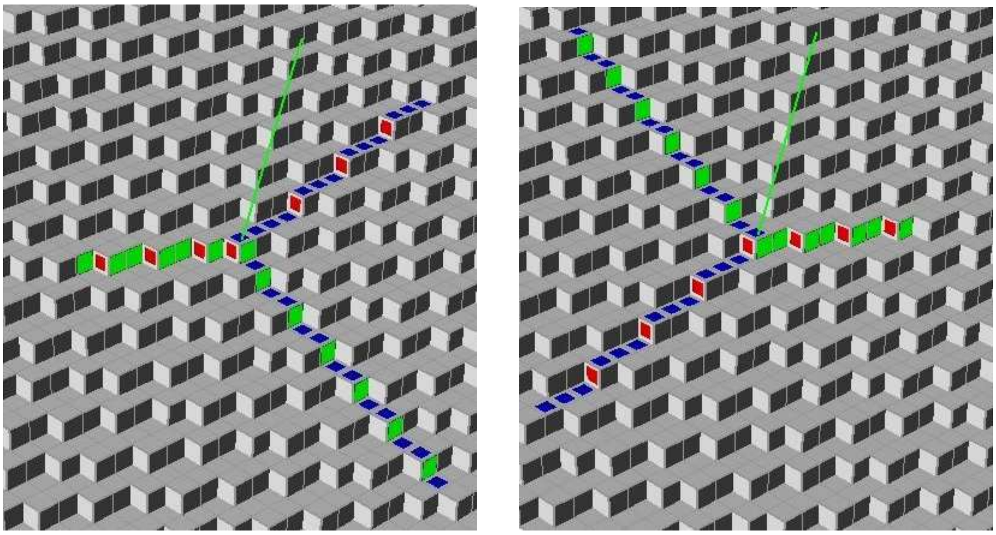

Figure 14: Two planar compositions from $(0,0,0)$ in $P(4,7,13)$. On the left, the three stripes determine the words : $1 \cdot W_{12} = 1 \cdot 2122122122\cdots$ (shown in red and green), $3 \cdot W_{31} = 3 \cdot 3331333133313331\cdots$ (shown in red and blue) and $2 \cdot W_{23} = 2 \cdot 3233233233233233233\cdots$. The planar composition word is obtained is $\chi_0 = 323312332313233213323$ and the planar composition vector correspond to the normal $\lambda_0 = (4,7,13)$. Colored surfels are those successfully used in the composition algorithm. On the right, same thing for χ_1.

Proof. The voxel V_0 is a corner of $P(a,b,c)$ since $V_0 \in P(a,b,c)$ while $V_0 - e_1$, $V_0 - e_2, V_0 - e_3$ are not. First, we show that starting from V_0, the words read on the three stripes are the compatible Christoffel $C_{(1,2)}(a,b)$, $C_{(3,1)}(c,a)$ and $C_{(2,3)}(b,c)$. Then Theorem 4.3 implies that both composition vectors are (a,b,c).

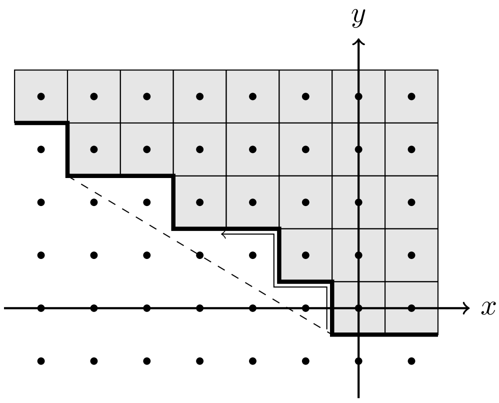

Figure 15: Side view of the z-stripe that passes at the origin of the discrete plane $D(3,5,7)$. Integer points of Z^3 are displayed by dots, voxels of the discrete plane are shown in gray and the thicker line shows the visible surfels. The arrow shows the surfel that are read starting from the visible surfel normal to the x axis adjacent to the origin. The word read is $1 \cdot W_{12} = 1 \cdot 2122122\cdots$. The dashed line illustrates the analogy with the definition of Christoffel words (see Figure 1).

By the construction the plane $P(a,b,c)$ is the set of voxels located in (x,y,z) with $ax+by+$

$cz \geq 0$. By fixing $z = 0$, the stripe forms a 4-connected path that starts in $(-1/2, -1/2, 0)$, passes through $(a - 1/2, b - 1/2, 0)$ and stays as close as possible to the real straight line joining these two points while staying above it (see Figure 15). Up to a translation and a symmetry, Christoffel words are constructed exactly the same way. Let $k_1 = \gcd(a, b)$, $k_2 = \gcd(a, c)$ and $k_3 = \gcd(b, c)$ and pose u_{12}, u_{31} and u_{23} as the central words satisfying $C_{(1,2)}(a/k_1, b/k_1) = 1 \cdot u_{12} \cdot 2$, $C_{(3,1)}(c/k_2, a/k_2) = 3 \cdot u_{31} \cdot 1$ and $C_{(2,3)}(b/k_3, c/k_3) = 2 \cdot u_{23} \cdot 3$. We have:

$$\begin{array}{lcl} (u_{12}21)^{k_1} \text{ is a prefix of } W_{12}, & & (u_{12}12)^{k_1} \text{ is a prefix of } W'_{12}, \\ (u_{31}13)^{k_2} \text{ is a prefix of } W_{31}, & \text{and} & (u_{31}31)^{k_2} \text{ is a prefix of } W'_{31}, \\ (u_{23}32)^{k_3} \text{ is a prefix of } W_{23}, & & (u_{23}23)^{k_3} \text{ is a prefix of } W'_{23}, \end{array}$$

The situation is similar to the hypothesis of Theorem 4.3 up to the difference that if $k_1 > 1$, then the central words in u_{12} are separated by the pair of letter 21 instead of the special letter $\overleftrightarrow{12}$. This is not a problem since Lemma 4.2 ensures that the corresponding letters will be read on the two other words. Same thing holds for u_{31} and u_{23}. So, by theorem Theorem 4.3 in both cases, the first $a+b+c-3$ iterations of the composition algorithm reads these prefixes and the compositions words $\chi_0 = \chi_1$ with $\overrightarrow{\chi_0} = \overrightarrow{\chi_1} = (a-1, b-1, c-1)$ are constructed. Then, at the $(a+b+c-2)$-th iteration, in the case of the computation of χ_0, we have $X[1] = 2$, $Y[1] = 1$ and $Z[1] = 3$ and in the case of the computation of χ_1 we have $X[1] = 1$, $Y[1] = 3$ and $Z[1] = 2$. Thus, in both cases, the algorithm stops and the finite words χ_0 and χ_1 are returned. $\square$

A natural question that arises from the previous theorem is to describe the composition vectors obtained from other corners than the origin. As shown in Table 1, these vectors seems to form successive approximations of the normal vector.

Corner's height	λ_0	λ_1
0	(4,7,13)	(4,7,13)
1	(1,2,3)	(3,5,10)
2	(1,2,3)	(2,3,6)
3	(1,2,3)	(1,1,2)

Corner's height	λ_0	λ_1
0	(40, 29, 16)	(40, 29, 16)
1	(40, 29, 16)	(40, 29, 16)
2	(25, 18, 10)	(15, 11, 6)
3	(25, 18, 10)	(15, 11, 6)
4	(10, 7, 4)	(15, 11, 6)
5	(10, 7, 4)	(15, 11, 6)
6	(10, 7, 4)	(5, 4, 2)
7	(7, 5, 3)	(3, 2, 1)
8	(3, 2, 1)	(2, 2, 1)
9	(3, 2, 1)	(2, 2, 1)
10	(3, 2, 1)	(2, 2, 1)
11	(3, 2, 1)	(2, 2, 1)
12	(3, 2, 1)	(2, 2, 1)
13	(1, 1, 1)	(2, 2, 1)
14	(1, 1, 1)	(2, 2, 1)
15	(1, 1, 1)	(2, 2, 1)

Table 1: Exemple of planar composition vectors computed on the different corners of a discrete planes. On the left, the normal vector is $(4, 7, 13)$ and $(40, 29, 16)$ on the right.

We suspect these composition vectors to be related with the successive approximations of the normal vector obtained from a multidimensional continued fraction algorithm. Indeed, developments in discrete geometry have successfully linked the structure of discrete lines with the continued fraction development of its slope [BS02, Pyt02]. While there exist no canonical expansion to continued fractions in higher dimension, numerous generalizations of Euclid's algorithm have been studied. Links between the structure of discrete planes and generalized continued fraction algorithms applied on the normal vector have been showed via the use of substitutions [ABEI01, Fer09]. In a different approach, works on the connectivity of thin discrete planes have showed that the generalized continued fraction algorithm called *fully subtractive* intervenes naturally in the study of discrete planes [DJT09, BJJP13]. We propose as an open question to explicit a link between the composition vectors and generalized continued fraction algorithms.

The following proposition states that on a rational plane, no corner may provide a better synchronization than the origin.

Proposition 6.6. *Given* $(a,b,c) \in \mathbb{N}^3$ *with* $a,b,c > 0$ *and* $\gcd(a,b,c) = 1$*. Let* χ *be one of the two planar composition word computed a corner of* $P(a,b,c)$*, then* $|\chi| \leq a+b+c-3$*.*

Proof. Since $a,b,c \in \mathbb{N}$, all x-stripes define the same periodic biinfinite word with period pattern $C_{(1,2)}(a,b)$. The same holds for the two other type of stripes.

W.l.o.g. suppose that the planar composition word computed is χ_0. Starting from a given corner, the three words read on the stripes will begin by the following prefixes :

- W_{12} begins with $U_{12} \cdot 21$ which is conjugate to $C_{(1,2)}(a,b)$.
- W_{31} begins with $U_{31} \cdot 13$ which is conjugate to $C_{(3,1)}(c,a)$.
- W_{23} begins with $U_{23} \cdot 32$ which is conjugate to $C_{(2,3)}(b,c)$.

Note that since stripes are read starting from a corner, each conjugate ends with two different letters.

Now, lets consider the execution of the composition algorithm with these three words as input. If the process stops before any of the words U_{12}, U_{31} or U_{23}, has been completely read, then the result holds. On the other hand, w.l.o.g. suppose that U_{12} is the first of the three words has been completely read. Then since there are exactly as many occurrences of the letter 1 in U_{12} then in U_{31}, what is left te read in U_{31} is only occurrences of the letter 3. The same holds for U_{23} which contains the same number of occurences of the letter 3 then U_{31}. Those letters will now synchronize and, when all three words U_{12}, U_{31} and U_{23} have been read, we have $X[1] = 2$, $Y[1] = 1$ and $Z[1] = 3$. The process stops and χ_0 is returned with $|\chi_0| = a+b+c-3$. □

6.2. Composition on irrational discrete planes and conjectures

We now consider the composition words obtained from discrete planes with irrational normal vector.

Theorem 6.7. *Given an irrational vector* (α,β,γ) *with* $\alpha,\beta,\gamma > 0$*, the voxel* V_0 *centered at* $(0,0,0)$ *is a corner of* $P(\alpha,\beta,\gamma)$ *and the composition words obtained from* V_0 *are infinite.*

Proof. The fact that V_0 is a corner of $P(\alpha,\beta,\gamma)$ is obtained as is Theorem 6.5. By construction, near the V_0, we find three standard Sturmian words S_1, S_2 and S_3. Each Sturmian words is given by $\cup_{i=1,2}\{n\alpha_i \,|\, n \in \mathbb{N} - \{0\}\}$ or $\cup_{i=1,3}\{n\alpha_i \,|\, n \in \mathbb{N} - \{0\}\}$ or $\cup_{i=2,3}\{n\alpha_i \,|\, n \in \mathbb{N} - \{0\}\}$ with $\alpha_1 = \frac{\alpha+\beta+\gamma}{\alpha}$, $\alpha_2 = \frac{\alpha+\beta+\gamma}{\beta}$, $\alpha_3 = \frac{\alpha+\beta+\gamma}{\gamma}$. As the normal is totally irrational there is no rational relation between α, β and γ thus no collision in the three jump words and no collision in the union of the three jump words that is in $\cup_{i=1}^{3}\{n\alpha_i \mid n \in \mathbb{N}-\{0\}\}$. In other terms, letters in the jump word $\cup_{i=1}^{3}\{n\alpha_i \mid n \in \mathbb{N}-\{0\}\}$ are two by two distinct. Otherwise, if there is a collision of two letters (for example for the letters 1 and 2) this means that $\exists \ell, m \in \mathbb{N}$ such that $\ell\frac{\alpha+\beta+\gamma}{\alpha} = m\frac{\alpha+\beta+\gamma}{\beta}$ and this is impossible because the normal is irrational. Thus our algorithm of synchronization pick up the letters one by one and never stops and thus construct an infinite billiard word by the synchronization of three standard Sturmian words with $\alpha_1 = \frac{\alpha+\beta+\gamma}{\alpha}$, $\alpha_2 = \frac{\alpha+\beta+\gamma}{\beta}$, $\alpha_3 = \frac{\alpha+\beta+\gamma}{\gamma}$. $\square$

We believe that such an infinite synchronization may only happen in this specific case, that is with an irrational normal vector and starting from the origin. From our observations of the rational and irrational case, we provide the following conjecture.

Conjecture 6.8. *Given a vector (a,b,c) with $a,b,c > 0$, let V and V' be two corners of $P(a,b,c)$. Let χ_0, χ_1 be the planar composition words computed at V and χ_0', χ_1' those computed at V', then*

- *If V is not centered at the origin then χ_0 and χ_1 are a finite words,*
- *height$(V) \leq$ height$(V') \implies (|\chi_0| \geq |\chi_0'|)$ and $(|\chi_1| \geq |\chi_1'|)$.*

7. Conclusion

In this paper, we investigate the first step in the study of synchronization of Christoffel words. Our ambitious goal was to design Christoffel words in dimension three by composition of three usual Christoffel words. This idea is driven by the search of a fundamental object to prove an analogous version of the theorem "Lyndon + Christoffel = Digitally Convex" [BLPR09] and thus obtain a more combinatorial notion of digital convexity in dimension three. We present some construction on discrete planes but wonder about the corresponding constructions for discrete surfaces given by implicit formula ? May an efficient normal estimator be derived from this technique ?

[AAS97] E. Andres, R. Acharya, and C. Sibata. Discrete analytical hyperplanes. *Graphical Models and Image Processing*, 59(5):302 – 309, 1997.

[ABEI01] P. Arnoux, V. Berthé, H. Ei, and S. Ito. Tilings, quasicrystals, discrete planes, generalized substitutions, and multidimensional continued fractions. In *Discrete models: combinatorics, computation, and geometry*, Discrete Math. Theor. Comput. Sci. Proc., AA, pages 59–78. Maison Inform. Math. Discrèt. (MIMD), Paris, 2001.

[AMST94] P. Arnoux, C. Mauduit, I. Shiokawa, and J.-I. Tamura. Complexity of sequences defined by billiard in the cube. *Bull. Soc. Math. France*, 122(1):1–12, 1994.

[And03] E. Andres. Discrete linear objects in dimension n: the standard model. *Graph. Models*, 65(1-3):92–111, January 2003.

[AR91] P. Arnoux and G. Rauzy. Représentation géométrique de suites de complexité $2n+1$. *Bull. Soc. Math. France*, 119(2):199–215, 1991.

[Bar95] Y. Baryshnikov. Complexity of trajectories in rectangular billiards. *Comm. Math. Phys.*, 174(1):43–56, 1995.

[BdL97] J. Berstel and A. de Luca. Sturmian words, Lyndon words and trees. *Theoret. Comput. Sci.*, 178(1-2):171–203, 1997.

[Bed03] N. Bédaride. Billiard complexity in rational polyhedra. *Regular and chaotic dynamics*, 8(1):97–104, 2003.

[Ber90] J. Berstel. Tracé de droites, fractions continues et morphismes itérés. In *Mots*, Lang. Raison. Calc., pages 298–309. Hermès, Paris, 1990.

[BL93] J.-P. Borel and F. Laubie. Quelques mots sur la droite projective réelle. *J. Théor. Nombres Bordeaux*, 5(1):23–51, 1993.

[BJJP13] V. Berthé, D. Jamet, T. Jolivet, X. Provençal. Critical Connectedness of Thin Arithmetical Discrete Planes. In *Discrete geometry for computer imagery*, volume 7749 of *Lecture Notes in Comput. Sci.*, pages 107–118. Springer, Heidelberg, 2013.

[BL11] V. Berthé and S. Labbé. An arithmetic and combinatorial approach to three-dimensional discrete lines. In *Discrete geometry for computer imagery*, volume 6607 of *Lecture Notes in Comput. Sci.*, pages 47–58. Springer, Heidelberg, 2011.

[BLPR09] S. Brlek, J.-O. Lachaud, X. Provençal, and C. Reutenauer. Lyndon + Christoffel = digitally convex. *Pattern Recognition*, 42(10):2239 – 2246, 2009.

[BLRS09] J. Berstel, A. Lauve, C. Reutenauer, and F. V. Saliola. *Combinatorics on words*, volume 27 of *CRM Monograph Series*. American Mathematical Society, Providence, RI, 2009. Christoffel words and repetitions in words.

[BR05] J.-P. Borel and C. Reutenauer. Palindromic factors of billiard words. *Theoret. Comput. Sci.*, 340(2):334–348, 2005.

[BS02] J. Berstel and P. Séébold. Sturmian words. In M. Lothaire, editor, *Algebraic Combinatorics on Words*, volume 90 of *Encyclopedia of Mathematics and Its Applications*, pages 45–110. Cambridge University Press, 2002.

[Chr75] E. B. Christoffel. Observatio arithmetica. *Annali di Mathematica*, 6:145–152, 1875.

[DJT09] E. Domenjoud, D. Jamet, and J.-L. Toutant. On the connecting thickness of arithmetical discrete planes. In *Discrete Geometry for Computer Imagery*, volume 5810 of *Lecture Notes in Computer Science*, pages 362–372. Springer Berlin Heidelberg, 2009.

[dLM94] A. de Luca and F. Mignosi. Some combinatorial properties of Sturmian words. *Theoret. Comput. Sci.*, 136(2):361–385, 1994.

[Fer09] T. Fernique. Generation and recognition of digital planes using multi-dimensional continued fractions. *Pattern Recognition*, 42(10):2229 – 2238, 2009.

[FR06] F. Feschet, J.-P. Reveillès. A Generic Approach for n-Dimensional Digital Lines, In *Discrete Geometry for Computer Imagery*, volume 4245 of *Lecture Notes in Computer Science*, pages 29–40. Springer Berlin Heidelberg, 2006.

[FR96] O. Figueiredo, J.-P. Reveillès. New Results abourd 3D digital lines. *Proc. SPIE*, 2836:98–108, 1996.

[Fog07] N. Pytheas Fogg. Substitutions par des motifs en dimension 1. *ITA*, 41(3):267–284, 2007.

[HV07] P. Hubert and L. Vuillon. Complexity of cutting words on regular tilings. *European J. Combin.*, 28(1):429–438, 2007.

[KR04] R. Klette and A. Rosenfeld. Digital straightness - a review. *Discrete Applied Mathematics*, 139:197–230, 2004. The 2001 International Workshop on Combinatorial Image Analysis.

[Pyt02] N. Pytheas Fogg. *Substitutions in Dynamics, Arithmetics and Combinatorics*, volume 1794 of *Lecture Notes in Mathematics*. Springer Verlag, 2002. Ed. by V. Berthé and S. Ferenczi and C. Mauduit and A. Siegel.

[Rau82] G. Rauzy. Nombres algébriques et substitutions. *Bull. Soc. Math. France*, 110(2):147–178, 1982.

[Rev91] J.-P. Reveillès. *Géométrie discrète, calcul en nombres entiers et algorithmique*. PhD thesis, Université Louis Pasteur, Strasbourg, December 1991.

[Tab95] S. Tabachnikov. Billiards. *Panoramas et Synthèses*, (1):vi+142, 1995.

[Tou06] J.-L. Toutant. Characterization of the closest discrete approximation of a line in the 3-dimensional space. In *International Symposium on Visual Computing, Springer Berlin / Heidelberg: ISCV 2006*, pages 618–627, 2006.

[Vui03] L. Vuillon. Balanced words. *Bull. Belg. Math. Soc. Simon Stevin*, 10(suppl.):787–805, 2003.