Enumeration Formula for $(2, n)$-Cubes in Discrete Planes

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Abstract

We compute the number of local configurations of size $2{\times}n$ on naive discrete planes using combinatorics on words, 2-dimensional Rote sequences and Berstel-Pocchiola diagrams.

1. Introduction

At present, the enumeration of discrete segments as well as of Sturmian words are well known, either in terms of discrete geometry [1] or in the ones of word combinatorics [2, 3]. In dimension 3, a natural extension of the discrete segments is the notion of (m, n)-cubes [4, 5, 6], namely the local configurations of naive arithmetical discrete planes [7, 8] of which one orthogonal projection is a rectangle of size $m{\times}n$ in a coordinate plane of $\mathbb{Z}^3$.

The problem of enumerating the local configurations of discrete planes appears in various works, such as [6, 7, 8, 9] in terms of discrete geometry or in [10] in terms of word combinatorics. In [7, 8] and [10], the authors investigate the *specific complexity* of discrete planes, namely the number of local configurations of given size [6] or shape [7, 8, 10] occurring in a given discrete plane: let m and n be two positive integers, the number of (m, n)-cubes occurring in a discrete plane is at most mn [6]. Moreover, if the coordinates of the normal vector of the given discrete plane P are rationally independent, then the number of (m, n)-cubes occurring in P is exactly $m\,n$, whatever the positive integers m and n [10].

In [9], the authors deal with a slightly different question, that is the *global complexity* of naive discrete planes. More precisely, given two positive integers m and n, they investigate the following question: among the local configurations of voxels in one-to-one correspondence with a rectangle of size $m{\times}n$ in a coordinate plane of $\mathbb{Z}^3$ using an orthogonal projection, how many of them occur in a discrete plane? They obtain a lower and an upper bound for the number of (m, n)-cubes, for any m and n.

In the present paper, we focus on the exact enumeration of $(2, n)$-cubes. Our approach consists in representing discrete planes as two-dimensional sequences generated by a double rotation over the torus $\mathbb{R}/\mathbb{Z}$, partitioned by the intervals $[0, 1[$ and $[1, 2[$ [10]. In the following, such sequences are called *two-dimensional Rote sequences* since they provide a natural extension of sequences investigated by G. Rote in [11]. By this representation, the (m, n)-cubes are in a natural correspondence with the rectangular words of size $m{\times}n$ occurring in Rote sequences. Thanks to some symmetries we are able to exhibit a recurrence relation on the number $r(n)$ of such words and we obtain:

$$r(n) = 4\left(1 + \sum_{i=1}^{n} i \sum_{j=1}^{i-1} \varphi(j)\right),$$

where φ is Euler's totient function. Finally, since for any rectangular word w over the alphabet $\{0, 1\}$, w occurs in at least one two-dimensional Rote sequence if and only if so does $\overline{w}$ ($\overline{\cdot} : 0 \leftrightarrow 1$) and since w and $\overline{w}$

code the same local configuration, it follows that the cardinal of the set $\mathcal{M}_{2,n}$ of $(2,n)$-cubes is

$$\#\mathcal{M}_{2,n} = \frac{r(n)}{2} = 2\left(1 + \sum_{i=1}^{n} i \sum_{j=1}^{i-1} \varphi(j)\right).$$

2. Basic notions and notation

Definition 1 (Arithmetical discrete planes) *Let* $\mathbf{v} \in \mathbb{R}^3$ *be a non-zero vector,* $\mu \in \mathbb{R}$ *and* $\omega \in \mathbb{R}$. *The* arithmetical discrete plane $\mathrm{P}(\mathbf{v},\mu,\omega)$ *of* normal vector $\mathbf{v}$, shift μ *and* thickness ω *is the subset of* $\mathbb{Z}^3$ *defined by*

$$\mathrm{P}(\mathbf{v},\mu,\omega) = \left\{\mathbf{x} \in \mathbb{Z}^3 \mid 0 \le \langle \mathbf{v}, \mathbf{x}\rangle + \mu < \omega\right\},$$

where $\langle\,.\,,\,.\,\rangle$ *denotes the usual scalar product. If* $\omega = \|\mathbf{v}\|_\infty = \max\{|v_1|,|v_2|,|v_3|\}$ *(resp.* $\omega = \|\mathbf{v}\|_1 = |v_1|+|v_2|+|v_3|$*), then the arithmetical discrete plane* $\mathrm{P}(\mathbf{v},\mu,\omega)$ *is said to be* naive *(resp.* standard*).*

Up to an isometry on the unit cube of $\mathbb{R}^3$ centered on $(0,0,0)$, we assume, without loss of generality, that the normal vector $\mathbf{v} = (v_1,v_2,v_3)$ satisfies $0 \le v_1, v_2 \le v_3$ and $v_3 \neq 0$. Furthermore, by normalization, we may assume $v_3 = 1$.

From now on, we restrict our investigation to naive arithmetical discrete planes with normal vector $\mathbf{v} \in [0,1]^2 \times \{1\}$. For this reason, and for clarity issue, we refer to $\mathrm{P}(\mathbf{v},\mu,1)$ as $\mathrm{P}(\mathbf{v},\mu)$. Under these hypotheses, we have

$$\mathrm{P}(\mathbf{v},\mu) = \left\{(x,y,-\lfloor v_1\,x + v_2\,y + \mu\rfloor) \mid (x,y) \in \mathbb{Z}^2\right\}$$

and it becomes natural to code a discrete plane $\mathrm{P}(\mathbf{v},\mu)$ by the *height* of its points, that is:

$$\begin{array}{rcrcl} h_{\mathbf{v},\mu} & : & \mathbb{Z}^2 & \longrightarrow & \mathbb{Z} \\ & & (x,y) & \longmapsto & -\lfloor v_1\,x + v_2\,y + \mu\rfloor \end{array} \tag{1}$$

The two-dimensional sequence $(h_{\mathbf{v},\mu}(\mathbf{x}))_{\mathbf{x}\in\mathbb{Z}^2}$ is called the *height-coding* of $\mathrm{P}(\mathbf{v},\mu)$ (see Fig. 1).

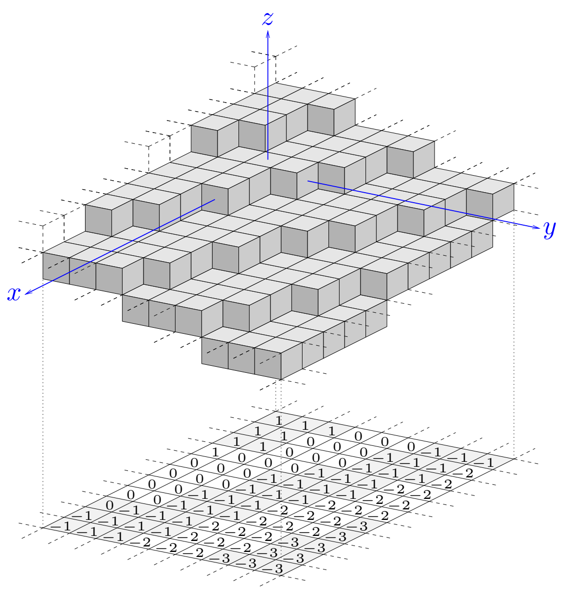


Figure 1: Height-coding of a naive discrete plane.

Let us now introduce the notion of local configuration.

Definition 2 (Local configurations) *Given $m, n \in \mathbb{N}^\star$, a* local configuration C *of size $m\times n$ is a subset of a naive discrete plane P such that its orthogonal projection $\Pi_3(\mathrm{C}) = \left\{(x, y) \in \mathbb{Z}^2 \mid \exists z \in \mathbb{Z},\, (x, y, z) \in \mathrm{C}\right\}$ is a rectangle of size $m\times n$ of $\mathbb{Z}^2$ that is, such that there exists $(x_0, y_0) \in \mathbb{Z}^2$ satisfying:*

$$\begin{aligned}\Pi_3(\mathrm{C}) &= \left\{(x, y) \in \mathbb{Z}^2 \mid x_0 \le x \le x_0 + m - 1 \text{ and } y_0 \le y \le y_0 + n - 1\right\}, \\ &= \{x_0, x_0 + 1, \ldots, x_0 + m - 1\}\times\{y_0, y_0 + 1, \ldots, y_0 + n - 1\}.\end{aligned}$$

The pair (x_0, y_0) is called an occurrence *of C in P. Examples of local configurations are given in Fig. 2*

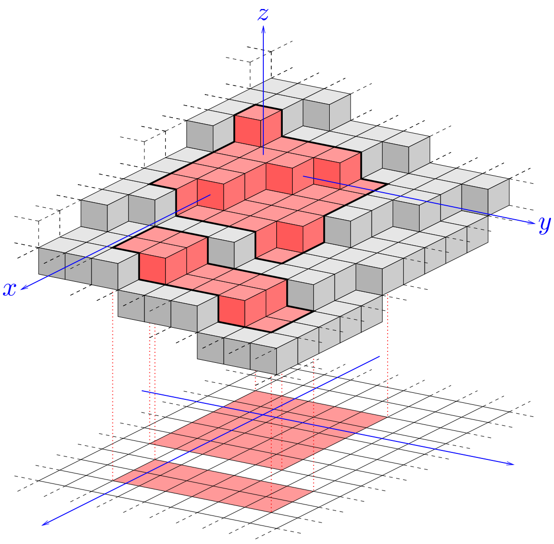


Figure 2: Two local configurations in a discrete plane: one of size 5×5 at occurrence $(-1, -1)$ and one of size 2×6 at occurrence $(5, 0)$.

The set of local configurations is obviously invariant by a translation by a vector $\mathbf{u} \in \mathbb{Z}^3$. We define (m, n)-cubes as the equivalence classes of local configurations under the action of such translations.

Definition 3 ($(\boldsymbol{m}, \boldsymbol{n})$-cubes) *Two local configurations C and C' of size $m\times n$ are* equivalent *if $\mathrm{C}' = T_{\mathbf{u}}(\mathrm{C})$ for some vector $\mathbf{u} \in \mathbb{Z}^3$, where $T_{\mathbf{u}} : x \longmapsto x + \mathbf{u}$ is the translation of vector $\mathbf{u}$ over $\mathbb{Z}^3$. An equivalence class under this relation is called a (m, n)-cube. Given $m, n \in \mathbb{N}^\star$, the set of (m, n)-cubes is denoted by $\mathcal{M}_{m,n}$.*

Given a local configuration C with occurrence $(x_0, y_0) \in \mathbb{Z}^2$ in $P(\mathbf{v}, \mu)$, consider the local configuration $C' = T_{-\mathbf{u}}(C)$, where $\mathbf{u} = (x_0, y_0, -\lfloor v_1\, x_0 + v_2\, y_0 + \mu\rfloor)$. Obviously C' occurs at $(0, 0)$ in $P(\mathbf{v}, \mu')$ where $\mu' = \mu + v_1\, x_0 + v_2\, y_0 - \lfloor \mu + v_1\, x_0 + v_2\, y_0\rfloor$. Moreover, the height of the point at occurrence $(0, 0)$ in C' is 0 or equivalently C' contains the point $(0, 0, 0)$.

The following lemma is straightforward.

Lemma 4 *Any (m, n)-cube C has a unique representative with occurrence $(0, 0)$ and containing the point $(0, 0, 0)$. This representative is called the* canonical representative *of C.*

Therefore, the number of (m, n)-cubes is exactly the number of local configurations of size $m\times n$ with occurrence $(0, 0)$ and containing the point $(0, 0, 0)$.

Example 5 The local configuration of size 5×5 at occurrence $(-1, -1)$ in Fig. 2 is a non-canonical representative of a $(5, 5)$-cube. Its canonical representative is given in Fig. 3.

We can now state the main result of the present paper:

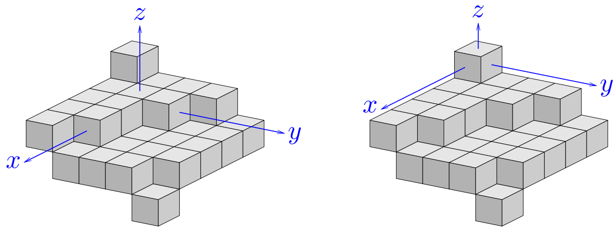


Figure 3: two representatives of a $(5, 5)$-cube: a non-canonical one (on the left) and the canonical one (on the right).

Theorem 6 *Let $n \in \mathbb{N}^\star$, the number of $(2, n)$-cubes is:*

$$\#\mathcal{M}_{2,n} = 2\left(1 + \sum_{i=1}^{n} i \sum_{j=1}^{i-1} \varphi(j)\right),$$

where $\varphi : \mathbb{N}^\star \longrightarrow \mathbb{N}^\star$ is Euler's totient function.

3. Coding (m, n)-cubes and arithmetical discrete planes with two-dimensional Rote sequences

As mentioned in Section 1, we have chosen to solve the problem of counting (m, n)-cubes by combinatorics on two-dimensional words. Let us first recall several basic notions and notation about rectangular words.

Let $\mathcal{A}$ be a finite alphabet and let $m, n \in \mathbb{N}^\star$. A *rectangular word of size* $m \times n$ over the alphabet $\mathcal{A}$ is a map $w : \{0, \ldots, m-1\} \times \{0, \ldots, n-1\} \longrightarrow \mathcal{A}$. We represent such a word as a matrix with m rows and n columns (see Fig. 4).

$$w = \begin{bmatrix} w_{0,0} & \cdots & w_{0,n-1} \\ \vdots & \ddots & \vdots \\ w_{m-1,0} & \cdots & w_{m-1,n-1} \end{bmatrix}.$$

Figure 4: A rectangular word of size $m \times n$.

Let $u : \mathbb{Z}^2 \longrightarrow \mathcal{A}$ be a two-dimensional sequence over the alphabet $\mathcal{A}$ and let w be a rectangular word of size $m \times n$ over the alphabet $\mathcal{A}$. We say that $(i_0, j_0) \in \mathbb{Z}^2$ is an *occurrence* of w in u if:

$$\forall (i, j) \in \{0, \ldots, m-1\} \times \{0, \ldots, n-1\},\ u_{i_0+i, j_0+j} = w_{i,j}.$$

Given a discrete plane $P(\mathbf{v}, \mu)$ with $\mathbf{v} \in [0, 1]^2 \times \{1\}$ and $\mu \in \mathbb{R}$, we have for any $(i, j) \in \mathbb{Z}^2$,

$$0 \leq \lfloor v_1\,(i+1) + v_2\,j + \mu \rfloor - \lfloor v_1\,i + v_2\,j + \mu \rfloor \leq 1,$$

and similarly

$$0 \leq \lfloor v_1\,i + v_2\,(j+1) + \mu \rfloor - \lfloor v_1\,i + v_2\,j + \mu \rfloor \leq 1.$$

Hence, if $h : \mathbb{Z}^2 \longrightarrow \mathbb{Z}$ is defined as in Eq. 1, then for all $(i, j) \in \mathbb{Z}^2$ we have $h_{i,j} - h_{i+1,j} = (h_{i,j} - h_{i+1,j}) \bmod 2 = (h_{i,j} \bmod 2 - h_{i+1,j} \bmod 2) \bmod 2$ and $h_{i,j} - h_{i,j+1} = (h_{i,j} \bmod 2 - h_{i,j+1} \bmod 2) \bmod 2$. Thus we may reconstruct any discrete plane (up to a vertical translation) from its coding $(h_{\mathbf{i}} \bmod 2)_{\mathbf{i} \in \mathbb{Z}^2}$ as a two-dimensional sequence over the two-letter alphabet $\{0, 1\}$. Such sequences are called *two-dimensional Rote sequences* (see Fig. 5). More precisely:

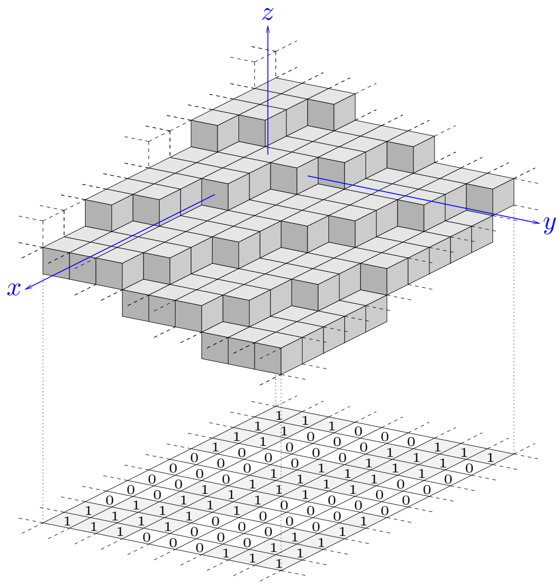


Figure 5: Coding of a naive discrete plane by a two-dimensional Rote sequence.

Definition 7 (Two-dimensional Rote sequences) *A* two-dimensional Rote sequence *(*Rote sequence *for short) is a two-dimensional sequence $u : \mathbb{Z}^2 \longrightarrow \{0, 1\}$ over the two-letter alphabet $\{0, 1\}$ such that there exists a triple $(\alpha, \beta, \mu) \in [0, 1]^2 \times \mathbb{R}$ satisfying:*

$$\forall (i, j) \in \mathbb{Z}^2,\ u_{i,j} = \lfloor \alpha\, i + \beta\, j + \mu \rfloor \bmod 2.$$

The triple (α, β, μ) is called a triple of parameters *of u.*

In other words, a two-dimensional sequence $u \in \{0, 1\}^{\mathbb{Z}^2}$ is a Rote sequence if and only if it codes the parity of the heights of the points of a naive discrete plane with normal vector $\mathbf{v} \in [0, 1]^2 \times \{1\}$.

Definition 8 (Two-dimensional Rote words) *A rectangular word w over the alphabet $\{0, 1\}$ is a* two-dimensional Rote word *(*Rote word *for short) if it occurs in at least one Rote sequence.*

A Rote word w is said to be normalized *if $w_{0,0} = 0$. The set of normalized Rote words of size $m \times n$ is denoted by $\mathrm{M}_{m,n}$.*

Thanks to the following lemma, we can restrict our attention to Rote words having an occurrence at $(0, 0)$ in a Rote sequence.

Lemma 9 *Let w be a Rote word. There exists a Rote sequence in which $(0, 0)$ is an occurrence of w.*

Proof. If $(i, j) \in \mathbb{Z}^2$ is an occurrence of w in the Rote sequence with parameters (α, β, μ), then $(0, 0)$ is an occurrence of w in the Rote sequence with parameters $(\alpha, \beta, \alpha\, i + \beta\, j + \mu)$. □

By definition of Rote sequences and Rote words, the map associating each (m, n)-cube with the rectangular word defined by the parity of the heights of the points of its canonical representative (see Fig. 6) is a surjection from the set of (m, n)-cubes to the set of normalized Rote words. A straightforward computation shows that the latter map is also an injection. Therefore, for any $m, n \in \mathbb{N}^\star$, the sets $\mathcal{M}_{m,n}$ and $\mathrm{M}_{m,n}$ are in one to one correspondence.

Definition 10 (Parameters of a Rote word) *Let w be a Rote word of size $m \times n$ and let $(\alpha, \beta, \mu) \in [0, 1]^2 \times \mathbb{R}$ satisfying, for all $(i, j) \in \{0, \ldots, m - 1\} \times \{0, \ldots, n - 1\}$, $w_{i,j} = \lfloor \alpha\, i + \beta\, j + \mu \rfloor \bmod 2$. Then, the triple (α, β, μ) is called a* triple of parameters *of w.*

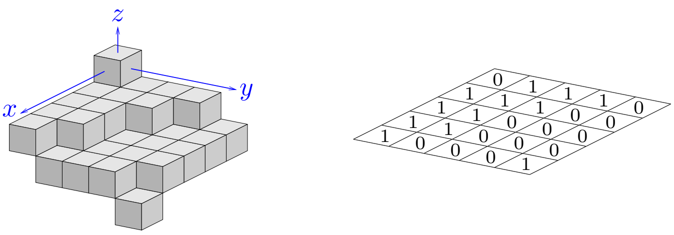


Figure 6: Canonical representative of an (m, n)-cube and its coding as a normalized Rote word.

In other words, a triple of parameters of a Rote word w is a triple of parameters of a Rote sequence u in which $(0, 0)$ is an occurrence of w. Of course, any given Rote word admits many triples of parameters. Note that by definition, it admits at least one.

Let us now show how we can restrict the set of parameters (α, β, μ) of Rote sequences without loss of generality on the Rote words we consider. The following lemma shows how to consider only non-extremal (0 or 1) parameters for α and β and how to limit the possible values of μ.

Lemma 11 *If w is a normalized Rote word, then there exists $(\alpha, \beta, \mu) \in]0, 1[^2 \times [0, 1[$ such that $(0, 0)$ is an occurrence of w in the Rote sequence with parameters (α, β, μ).*

Proof. Let w be a normalized Rote word of size $m \times n$ with parameters $(\alpha, \beta, \mu) \in [0, 1]^2 \times \mathbb{R}$. Obviously, taking $\mu' = \mu \bmod 2$ yields the same Rote word so that we may always assume $\mu \in [0, 2[$. Now since w is normalized, $\lfloor \alpha \times 0 + \beta \times 0 + \mu \rfloor \bmod 2 = \lfloor \mu \rfloor \bmod 2 = 0$ and we must have $\mu \in [0, 1[$.

We prove that for any triple of parameters $(\alpha, \beta, \mu) \in [0, 1] \times [0, 1[$ of w, we may find $\alpha' \in]0, 1[$ and $\mu' \in [0, 1[$ such that (α', β, μ') is also a triple of parameters of w. The same reasoning applied to β yields the result.

If $m = 1$ then any value of α yields the same Rote word so that we may take any $\alpha' \in]0, 1[$. In the rest of this proof, we assume $m \geq 2$.

Assume $\alpha = 0$ and let $M = \max_{j=0,\ldots,n-1}(\beta j + \mu - \lfloor \beta j + \mu \rfloor)$. We have $M \in [0, 1[$ and for any $\alpha' \in]0, (1 - M)/(m - 1)[$, we have $\alpha' \in]0, 1[$. Moreover, the Rote word with parameters (α', β, μ) is still w. Indeed, if $i = 0$ then for all j, we have $\lfloor \alpha' \times 0 + \beta j + \mu \rfloor = \lfloor \beta j + \mu \rfloor = \lfloor \alpha \times 0 + \beta j + \mu \rfloor$. For $i = 1, \ldots, m - 1$, we have, on one hand, $\alpha' i + \beta j + \mu \geq \beta j + \mu = \alpha i + \beta j + \mu$ and on the other hand:

$$\begin{aligned}
\lfloor \alpha i + \beta j + \mu \rfloor \quad \leq \quad \alpha' i + \beta j + \mu \quad &< \quad \frac{1 - M}{m - 1} i + \beta j + \mu \\
&\leq \quad 1 - M + \beta j + \mu \\
&\leq \quad 1 - M + \lfloor \beta j + \mu \rfloor + M \\
&= \quad \lfloor \beta j + \mu \rfloor + 1 \\
&= \quad \lfloor \alpha i + \beta j + \mu \rfloor + 1
\end{aligned}$$

Hence for all $(i, j) \in \{0, \ldots, m - 1\} \times \{0, \ldots, n - 1\}$, we have $\lfloor \alpha' i + \beta j + \mu \rfloor = \lfloor \alpha i + \beta j + \mu \rfloor$.

Now, assume $\alpha = 1$ and let μ' be such that $0 < \mu' - \mu < 1 - M$, where M is defined as above, and let $\alpha' = 1 - (\mu' - \mu)/(m - 1)$. Since $M \geq \beta \times 0 + \mu - \lfloor \beta \times 0 + \mu \rfloor = \mu$, we have $\mu' \in [0, 1[$. We also have $\alpha' \in]0, 1[$ and the Rote word with parameters (α', β, μ') is still w. Indeed, $\alpha' i + \beta j + \mu' = i + \beta j + \mu + (\mu' - \mu)(1 - i/(m - 1))$, and for all $i = 0, \ldots, m - 1$, $\alpha' i + \beta j + \mu' \geq \alpha i + \beta j + \mu \geq \lfloor \alpha i + \beta j + \mu \rfloor$. Moreover, if $i = m - 1$ then we have $\alpha' i + \beta j + \mu' = \alpha i + \beta j + \mu$ and for $0 \le i < m-1$, we have:

$$\begin{aligned}
\alpha' i + \beta j + \mu' &< i + \beta j + \mu + (1-M)\left(1 - \frac{i}{m-1}\right) \\
&\le i + \lfloor \beta j + \mu \rfloor + M + (1-M)\left(1 - \frac{i}{m-1}\right) \\
&= \lfloor \alpha i + \beta j + \mu \rfloor + 1 - (1-M)\frac{i}{m-1} \\
&\le \lfloor \alpha i + \beta j + \mu \rfloor + 1
\end{aligned}$$

Hence for all $(i,j) \in \{0,\ldots,m-1\} \times \{0,\ldots,n-1\}$, we have $\lfloor \alpha' i + \beta j + \mu' \rfloor = \lfloor \alpha i + \beta j + \mu \rfloor$.

Therefore, we may always find $\alpha' \in]0,1[$. Applying the same reasoning to β gives a triple of parameters $(\alpha', \beta', \mu'') \in]0,1[^2 \times [0,1[$ which still defines the same Rote word w. □

Corollary 12 *Let $m,n \in \mathbb{N}^\star$. Then*

$$\#\mathrm{M}_{m,n} = \#\left\{ \left[\lfloor \alpha i + \beta j + \mu \rfloor \bmod 2 \right]_{\substack{0 \le i < m \\ 0 \le j < n}},\ (\alpha,\beta,\mu) \in]0,1[^2 \times [0,1[\right\}.$$

4. Technical properties of Rote sequences

In the present section, we provide technical properties of Rote sequences and words which will allow us to prove the main result of the present paper, namely Th. 6. First we state a technical lemma about the words which may occur in a Rote sequence.

Lemma 13 *If u is a Rote sequence then the words $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ do not occur in u.*

Proof. If any of these words occurred in a Rote sequence, from Lemma 9, it would occur at occurrence $(0,0)$ in some Rote sequence with parameters $(\alpha,\beta,\mu) \in [0,1]^2 \times \mathbb{R}$. A straightforward computation shows that in each case, it is impossible to find suitable parameters (α,β,μ). □

To get an intuition of the previous lemma, consider for instance the word $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ as the coding of a local configuration. When moving right then down from the upper left corner, the parity does not change so that the height remains the same. If instead we move down then right, the parity changes twice so that the height decreases by 2 (see Fig. 7). Hence, this word cannot occur in the coding of a discrete plane.

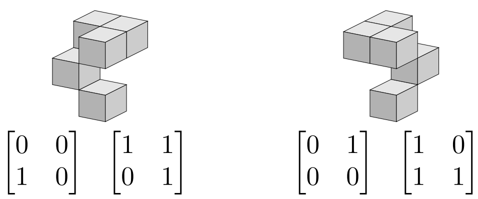


Figure 7: Impossible configuration in discrete planes.

Definition 14 *Let $m,n \in \mathbb{N}^\star$ and w a rectangular word of size $m \times n$ over the alphabet $\{0,1\}$. We denote by $\widetilde{w}$ the rectangular word of size $m \times n$ over the alphabet $\{0,1\}$ defined by:*

$$\widetilde{w} = \begin{bmatrix} w_{0,0} & \overline{w}_{0,1} & \cdots & \overline{w}_{0,n-1} \\ \overline{w}_{1,0} & \overline{w}_{1,1} & \cdots & \overline{w}_{1,n-1} \\ \vdots & \ddots & & \vdots \\ \overline{w}_{m-1,0} & \overline{w}_{m-1,1} & \cdots & \overline{w}_{m-1,n-1} \end{bmatrix}$$

where $\overline{w}$ is obtained from w by replacing 0s with 1s and 1s with 0s.

Geometrically, given a Rote word w coding a local configuration C, the map $\widetilde{\cdot}$ consists in raising the cube of C at occurrence $(0,0)$ by one unit (see Fig. 8). Equivalently, it consists in lowering each cube of C but the one at occurrence $(0,0)$ by one unit.

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix} \overset{\sim}{\longleftrightarrow} \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Figure 8: The action of the map $\widetilde{\cdot}$ on a Rote word of size $3{\times}5$ and its geometrical interpretation.

An important and useful property is:

Lemma 15 *If w is a normalized Rote word with parameters $(\alpha, \beta, 0)$, where $(\alpha, \beta) \in]0,1[^2$, then $\widetilde{w}$ is also a normalized Rote word.*

Proof. Let w be of size $m{\times}n$. If $m = n = 1$ then the result is obvious since $\widetilde{w} = w$. In the rest of the proof, we assume $(m,n) \neq (1,1)$. We may suppose α, β and 1 to be $\mathbb{Q}$-linearly independent. Indeed, let us set

$$\eta = \min_{\substack{0 \le i < m \\ 0 \le j < n \\ (i,j) \neq (0,0)}} \frac{1 + \lfloor \alpha\, i + \beta\, j \rfloor - (\alpha\, i + \beta\, j)}{i+j}.$$

Obviously $\eta > 0$ and one easily checks that for all $(\alpha', \beta') \in]\alpha, \alpha + \eta[\times]\beta, \beta + \eta[$ and all $(i,j) \in \{0, \ldots, m-1\} \times \{0, \ldots, n-1\}$, we have $\lfloor \alpha'\, i + \beta'\, j \rfloor = \lfloor \alpha\, i + \beta\, j \rfloor$. Therefore, by a density argument, we may always find $(\alpha', \beta') \in]\alpha, \alpha + \eta[\times]\beta, \beta + \eta[\cap]0,1[^2$ such that α', β' and 1 are $\mathbb{Q}$-linearly independent and the Rote word with parameters $(\alpha', \beta', 0)$ is w.

Now, let us set

$$\mu = 1 - \min_{\substack{0 \le i < m \\ 0 \le j < n \\ (i,j) \neq (0,0)}} (\alpha\, i + \beta\, j - \lfloor \alpha\, i + \beta\, j \rfloor)$$

and let w' be the Rote word of size $m{\times}n$ with parameters (α, β, μ). For all $(i,j) \in \{0, \ldots, m-1\}{\times}\{0, \ldots, n-1\}$, we have $w'_{i,j} = \lfloor \alpha\, i + \beta\, j + \mu \rfloor \bmod 2$. Let us show that $w' = \widetilde{w}$. Since α, β and 1 are $\mathbb{Q}$-linearly independent, $\alpha\, i + \beta\, j$ is never an integer so that $0 < \mu < 1$. For $(i,j) = (0,0)$, we have $\lfloor \alpha \times 0 + \beta \times 0 + \mu \rfloor = \lfloor \mu \rfloor = 0$ so that $w'_{0,0} = 0 = w_{0,0} = \widetilde{w}_{0,0}$. For $(i,j) \in \{0, \ldots, m-1\}{\times}\{0, \ldots, n-1\} \backslash \{(0,0)\}$, by definition of μ, we have $1 - \mu \le \alpha\, i + \beta\, j - \lfloor \alpha\, i + \beta\, j \rfloor$ or equivalently $\alpha\, i + \beta\, j + \mu \ge \lfloor \alpha\, i + \beta\, j \rfloor + 1$ which implies

$$\lfloor \alpha\, i + \beta\, j + \mu \rfloor \ge \lfloor \alpha\, i + \beta\, j \rfloor + 1.$$

Since $\mu < 1$, we also have $\lfloor \alpha\, i + \beta\, j + \mu \rfloor \le \lfloor \alpha\, i + \beta\, j \rfloor + 1$. Consequently, for all $(i,j) \in \{0, \ldots, m-1\}{\times}\{0, \ldots, n-1\} \backslash \{(0,0)\}$, we have

$$\lfloor \alpha\, i + \beta\, j + \mu \rfloor = \lfloor \alpha\, i + \beta\, j \rfloor + 1,$$

which implies $w'_{i,j} = \overline{w_{i,j}} = \widetilde{w}_{i,j}$. Hence for any $(i,j) \in \{0, \ldots, m-1\}{\times}\{0, \ldots, n-1\}$, we have $w'_{i,j} = \widetilde{w}_{i,j}$ so that $w' = \widetilde{w}$ and $\widetilde{w}$ is a Rote word. □

Let us now introduce some more notation.

Notation 16 Let $m, n \in \mathbb{N}^\star$. We set:

i) Let $\mathrm{M}^0_{m,n} = \left\{ \left[\lfloor \alpha\, i + \beta\, j \rfloor \bmod 2 \right]_{\substack{0\le i\le m-1 \\ 0\le j\le n-1}} \mid (\alpha,\beta) \in \left]0,1\right[^2 \right\}$ be the set of normalized Rote words of size $m{\times}n$ admitting a triple of parameters of the type $(\alpha,\beta,0)$ with $(\alpha,\beta)\in \left]0,1\right[^2$.

ii) Let $\mathrm{M}^1_{m,n} = \left\{ \widetilde{w} \mid w \in \mathrm{M}^0_{m,n} \right\}$. Note that Lemma 15 implies $\mathrm{M}^1_{m,n} \subseteq \mathrm{M}_{m,n}$. In other words, all elements of $\mathrm{M}^1_{m,n}$ are Rote words.

iii) Let $\mathrm{M}^{0,1}_{m,n} = \mathrm{M}^0_{m,n} \cup \mathrm{M}^1_{m,n}$ and $\mathrm{M}^\star_{m,n} = \mathrm{M}_{m,n} \setminus \mathrm{M}^{0,1}_{m,n}$.

Loosely speaking, a Rote word in $\mathrm{M}^0_{m,n}$ codes a configuration with occurrence $(0,0)$ in a discrete plane with shift $\mu = 0$, that is the discretization of a real plane going through the origin. However, note that Rote words with parameters $(1,\beta,0)$ or $(\alpha,1,0)$ do not belong to $\mathrm{M}^0_{m,n}$ as soon as $m>1$ and $n>1$. From Lemma 11 we may find for such words parameters $(\alpha,\beta,\mu)\in \left]0,1\right[^2 \times \left[0,1\right[$ but actually it is not possible to have $\mu = 0$.

Lemma 17 *Let $m,n\in\mathbb{N}^\star$. If $(m,n)\neq(1,1)$ then $\mathrm{M}^1_{m,n}\cap\mathrm{M}^0_{m,n}=\emptyset$.*

Proof. By symmetry, we may assume $m>1$ without loss of generality. Every $w\in\mathrm{M}^0_{m,n}$ has parameters $(\alpha,\beta,0)$ with $(\alpha,\beta)\in\left]0,1\right[^2$ so that $w_{1,0}=\lfloor\alpha\rfloor=0$. Thus $w'_{1,0}=1$ for every $w'\in\mathrm{M}^1_{m,n}$. Hence the result. □

Lemma 18 *Let $m,n\in\mathbb{N}^\star$ and w be a rectangular word of size $m{\times}n$ such that $w_{0,0}=0$. Then*

$$\left(w\in\mathrm{M}_{m,n}\wedge\widetilde{w}\in\mathrm{M}_{m,n}\right) \iff w\in\mathrm{M}^{0,1}_{m,n}.$$

Proof. If $(m,n)=(1,1)$ then the result is obvious because $\mathrm{M}_{1,1}=\mathrm{M}^0_{1,1}=\mathrm{M}^1_{1,1}=\{[0]\}$. In the rest of the proof, we assume $(m,n)\neq(1,1)$. By symmetry, we may assume $m>1$ without loss of generality.

By definition of $\mathrm{M}^{0,1}_{m,n}$, the assertion $w\in\mathrm{M}^{0,1}_{m,n} \implies (w\in\mathrm{M}_{m,n}\wedge\widetilde{w}\in\mathrm{M}_{m,n})$ is an immediate consequence of Lemma 15. Now, assume $w\in\mathrm{M}_{m,n}$ and $\widetilde{w}\in\mathrm{M}_{m,n}$ and let $(\alpha,\beta,\mu)\in\left]0,1\right[^2\times\left[0,1\right[$ and $(\alpha',\beta',\mu')\in\left]0,1\right[^2\times\left[0,1\right[$ be respectively triples of parameters of w and $\widetilde{w}$. By construction of $\tilde{w}$, there exist odd numbers $\delta_{i,j}$ such that

$$\forall (i,j)\in\{0,\dots,m-1\}\times\{0,\dots,n-1\}\setminus\{(0,0)\},\ \ \lfloor\alpha'\, i+\beta'\, j+\mu'\rfloor+\delta_{i,j}=\lfloor\alpha\, i+\beta\, j+\mu\rfloor.$$

For all $(i,j)\in\{0,\dots,m-2\}\times\{0,\dots,n-1\}\setminus\{(0,0)\}$, we have

$$\delta_{i+1,j}-\delta_{i,j}=\underbrace{\left(\lfloor\alpha\,(i+1)+\beta\, j+\mu\rfloor-\lfloor\alpha\, i+\beta\, j+\mu\rfloor\right)}_{\in\{0,1\}\text{ because }\alpha\in[0,1]}-\underbrace{\left(\lfloor\alpha'\,(i+1)+\beta'\, j+\mu'\rfloor-\lfloor\alpha'\, i+\beta'\, j+\mu'\rfloor\right)}_{\in\{0,1\}\text{ because }\alpha'\in[0,1]}.$$

Hence $\delta_{i+1,j}-\delta_{i,j}\in\{-1,0,1\}$ and actually $\delta_{i+1,j}-\delta_{i,j}=0$ because it is even. Similarly, we get $\delta_{i,j+1}-\delta_{i,j}=0$ for all $(i,j)\in\{0,\dots,m-1\}\times\{0,\dots,n-2\}\setminus\{(0,0)\}$ so that all $\delta_{i,j}$'s are equal for $(i,j)\in\{0,\dots,m-1\}\times\{0,\dots,n-1\}\setminus\{(0,0)\}$. Therefore, there exists an odd number δ such that

$$\forall (i,j)\in\{0,\dots,m-1\}\times\{0,\dots,n-1\}\setminus\{(0,0)\},\ \ \lfloor\alpha'\, i+\beta'\, j+\mu'\rfloor+\delta=\lfloor\alpha\, i+\beta\, j+\mu\rfloor. \tag{2}$$

Since we assumed $m>1$, we have $\delta=\lfloor\alpha+\mu\rfloor-\lfloor\alpha'+\mu'\rfloor\in\{-1,0,1\}$ so that $\delta\in\{-1,1\}$. Let $\theta=(\delta+1)/2\in\{0,1\}$. If $\delta=-1$ then $\mu'+\delta<0=\theta\le\mu$ and if $\delta=1$ then $\mu<1=\theta\le\mu'+\delta$ so that in both cases, there exists $\lambda\in[0,1]$ such that $\lambda\mu+(1-\lambda)(\mu'+\delta)=\theta$. Let $(\alpha'',\beta'')=\lambda(\alpha,\beta)+(1-\lambda)(\alpha',\beta')$. Since (α'',β'') is a convex combination of (α,β) and (α',β'), we have $(\alpha'',\beta'')\in\left]0,1\right[^2$. Let w'' be the Rote word of size $m{\times}n$ with parameters $(\alpha'',\beta'',0)$. Clearly, $w''\in\mathrm{M}^0_{m,n}$. For all $(i,j)\in\{0,\dots,m-1\}\times\{0,\dots,n-1\}\setminus\{(0,0)\}$, we have

$$\begin{aligned}\alpha''\, i+\beta''\, j &= (\lambda\,\alpha+(1-\lambda)\,\alpha')\, i+(\lambda\,\beta+(1-\lambda)\,\beta')\, j+\lambda\mu+(1-\lambda)(\mu'+\delta)-\theta\\ &= \lambda\,(\alpha\, i+\beta\, j+\mu)+(1-\lambda)\,(\alpha'\, i+\beta'\, j+\mu'+\delta)-\theta,\end{aligned}$$

and from Eq. 2,

$$\lfloor \alpha'' i + \beta'' j \rfloor = \lfloor \alpha i + \beta j + \mu \rfloor - \theta = \lfloor \alpha' i + \beta' j + \mu' \rfloor + \delta - \theta.$$

If $\delta = -1$ then $\theta = 0$ and $\lfloor \alpha'' i + \beta'' j \rfloor = \lfloor \alpha i + \beta j + \mu \rfloor$ so that $w'' = w$. If $\delta = 1$ then $\theta = 1$ and $\lfloor \alpha'' i + \beta'' j \rfloor = \lfloor \alpha' i + \beta' j + \mu' \rfloor$ so that $w'' = \widetilde{w}$. Finally, either $w'' = w$ or $w'' = \widetilde{w}$, that is, either $w \in \mathrm{M}^0_{m,n}$ or $\widetilde{w} \in \mathrm{M}^0_{m,n}$. It follows that $w \in \mathrm{M}^{0,1}_{m,n}$. □

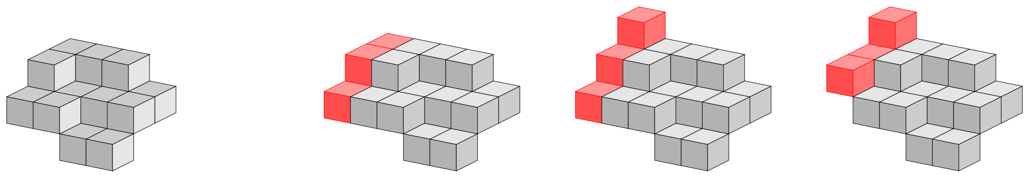
5. Proof of the main theorem

In this section, we focus more specifically on $\mathrm{M}_{2,n}$ and we will establish a recurrence on $\#\mathrm{M}_{2,n}$. For this, we study how a Rote word in $\mathrm{M}_{m,n}$ *extends* to Rote words in $\mathrm{M}_{m,n+1}$.

Definition 19 (Extensions of a Rote word) *Let w be a normalized Rote word of size $m \times n$ with $m, n \in \mathbb{N}^\star$. A* (left) extension *of w is a normalized Rote word w' of size $m \times (n+1)$ satisfying:*

$$w' = \begin{bmatrix} 0 & w_{0,0} & \cdots & w_{0,n-1} \\ w'_{1,0} & w_{1,1} & \cdots & w_{0,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ w'_{m-1,0} & w_{m-1,0} & \cdots & w_{m-1,n-1} \end{bmatrix} \quad \text{or} \quad w' = \begin{bmatrix} 0 & \overline{w}_{0,0} & \cdots & \overline{w}_{0,n-1} \\ w'_{1,0} & \overline{w}_{1,1} & \cdots & \overline{w}_{0,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ w'_{m-1,0} & \overline{w}_{m-1,0} & \cdots & \overline{w}_{m-1,n-1} \end{bmatrix}$$

We denote by $\mathrm{Ext}(w)$ the set of extensions of w. Fig. 9 gives a geometrical interpretation of the extensions of a Rote word.



$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Figure 9: The three extensions in $\mathrm{M}_{3,5}$ of a Rote word in $\mathrm{M}_{3,4}$ and their geometrical interpretations.

Lemma 20 *Let $w \in \mathrm{M}_{2,n}$ with $n \in \mathbb{N}^\star$. Then w admits at most three extensions, i.e. $\#\mathrm{Ext}(w) \leq 3$.*

Proof. If $n > 1$ and $\left[\begin{smallmatrix} 0 & w'_{0,1} & \cdots & w'_{0,n} \\ w'_{1,0} & w'_{1,1} & \cdots & w'_{1,n} \end{smallmatrix}\right]$ is an extension of $\left[\begin{smallmatrix} 0 & w_{0,1} & \cdots & w_{0,n-1} \\ w_{1,0} & w_{1,1} & \cdots & w_{1,n-1} \end{smallmatrix}\right]$ then $\left[\begin{smallmatrix} 0 & w'_{0,1} \\ w'_{1,0} & w'_{1,1} \end{smallmatrix}\right]$ is obviously an extension of $\left[\begin{smallmatrix} 0 \\ w_{1,0} \end{smallmatrix}\right] \in \mathrm{M}_{2,1}$. Now, $\mathrm{M}_{2,1}$ contains two elements, $\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$. For each of them, one of its four potential extensions is not a Rote word according to Lemma 13. Thus it admits at most three extensions. Hence the result. □

Lemma 21 *Let $w \in \mathrm{M}_{2,n}$ with $n \in \mathbb{N}^\star$. Then w admits an extension in $\mathrm{M}^0_{2,n+1}$ if and only if it admits an extension in $\mathrm{M}^1_{2,n+1}$*

Proof. Let w' be an extension of w in $\mathrm{M}^0_{2,n+1}$, then $\widetilde{w'} \in \mathrm{M}^1_{2,n+1}$ and we check that $\widetilde{w'}$ is an extension of w too. □

From Lemmas 15, 17, 18, 20 and 21, we get:

Lemma 22 *Let w be a normalized Rote word of size $2\times n$, with $n \in \mathbb{N}^\star$. Exactly four cases occur:*

i) *w admits exactly one extension $w' \in \mathrm{M}^\star_{2,n+1}$.*
ii) *w admits exactly two extensions w' and w'' satisfying $\{w', w''\} \subseteq \mathrm{M}^\star_{2,n+1}$.*
iii) *w admits exactly two extensions w' and w'' respectively in $\mathrm{M}^0_{2,n+1}$ and $\mathrm{M}^1_{2,n+1}$.*
iv) *w admits exactly three extensions respectively in $\mathrm{M}^0_{2,n+1}$, $\mathrm{M}^1_{2,n+1}$ and $\mathrm{M}^\star_{2,n+1}$.*

Proof. The first three cases are direct consequences of Lemmas 15, 18, 20 and 21. For the fourth case, it suffices to prove that a Rote word cannot admit exactly three extensions all in $\mathrm{M}^\star_{2,n+1}$. The three extensions of w are necessarily among

$$w_1 = \begin{bmatrix} 0 & \\ 0 & w \end{bmatrix}, w_2 = \begin{bmatrix} 0 & \\ 1 & w \end{bmatrix}, w_3 = \begin{bmatrix} 0 & \\ 0 & \overline{w} \end{bmatrix}, \text{ and } w_4 = \begin{bmatrix} 0 & \\ 1 & \overline{w} \end{bmatrix}.$$

Hence either $\{w_1, w_4\} \subseteq \mathrm{Ext}(w)$ or $\{w_2, w_3\} \subseteq \mathrm{Ext}(w)$. Since $w_1 = \widetilde{w_4}$ and $w_2 = \widetilde{w_3}$, Lemma 18 implies that either $\{w_1, w_4\} \subseteq \mathrm{M}^{0,1}_{2,n+1}$ or $\{w_2, w_3\} \subseteq \mathrm{M}^{0,1}_{2,n+1}$ and the result follows. □

Thus, for every $n \geq 1$, the set $\mathrm{M}_{2,n}$ may be partitioned into the following disjoint sets:

i) $\mathrm{W}^1_n = \{w \in \mathrm{M}_{2,n} \mid \#\mathrm{Ext}(w) = 1 \text{ and } \mathrm{Ext}(w) \subseteq \mathrm{M}^\star_{2,n+1}\}$.
ii) $\mathrm{W}^2_n = \{w \in \mathrm{M}_{2,n} \mid \#\mathrm{Ext}(w) = 2 \text{ and } \mathrm{Ext}(w) \subseteq \mathrm{M}^\star_{2,n+1}\}$.
iii) $\mathrm{W}^3_n = \left\{w \in \mathrm{M}_{2,n} \mid \#\mathrm{Ext}(w) = 2 \text{ and } \mathrm{Ext}(w) \subseteq \mathrm{M}^{0,1}_{2,n+1}\right\}$.
iv) $\mathrm{W}^4_n = \{w \in \mathrm{M}_{2,n} \mid \#\mathrm{Ext}(w) = 3 \text{ and } \forall k \in \{0, 1, \star\}, \# \left(\mathrm{Ext}(w) \cap \mathrm{M}^k_{2,n+1}\right) = 1\}$.

Since any element of $\mathrm{M}_{2,n+1}$ is necessarily an extension of an element of $\mathrm{M}_{2,n}$ and since two distinct elements of $\mathrm{M}_{2,n}$ have distinct extensions, we get the following relations for all $n \geq 1$:

$$\begin{aligned} \#\mathrm{M}_{2,n} &= \#\mathrm{W}^1_n + \#\mathrm{W}^2_n + \#\mathrm{W}^3_n + \#\mathrm{W}^4_n, \\ \#\mathrm{M}_{2,n+1} &= \#\mathrm{W}^1_n + 2\,\#\mathrm{W}^2_n + 2\,\#\mathrm{W}^3_n + 3\,\#\mathrm{W}^4_n, \\ \#\mathrm{M}^0_{2,n+1} &= \#\mathrm{W}^3_n + \#\mathrm{W}^4_n. \end{aligned}$$

From these relations, we deduce:

$$\forall n \geq 1,\ \#\mathrm{M}_{2,n+1} = \mathrm{M}_{2,n} + 2\,\#\mathrm{M}^0_{2,n+1} + (\#\mathrm{W}^2_n - \#\mathrm{W}^3_n). \tag{3}$$

In the following, we will show that W^2_n and W^3_n have the same cardinality which will allow us to simplify the relation above. In other words, there are as many Rote words with exactly two extensions in $\mathrm{M}^\star_{2,n+1}$ as Rote words with exactly two extensions outside of $\mathrm{M}^\star_{2,n+1}$. To prove this property, we need to define a new operator Θ on normalized Rote words of size $2\times n$ as follows:

$$\begin{array}{rccc} \Theta & : & \displaystyle\bigcup_{n\in\mathbb{N}^\star} \mathrm{M}_{2,n} & \longrightarrow & \displaystyle\bigcup_{n\in\mathbb{N}^\star} \mathcal{W}_{2,n} \\ & & w & \longmapsto & \left\{ \begin{array}{ll} \begin{bmatrix} 0 & w_{1,1} & \dots & w_{1,n-1} \\ 1 & \overline{w}_{0,1} & \dots & \overline{w}_{0,n-1} \end{bmatrix} & \text{if } w_{1,0} = 0, \\ \begin{bmatrix} 0 & \overline{w}_{1,1} & \dots & \overline{w}_{1,n-1} \\ 0 & w_{0,1} & \dots & w_{0,n-1} \end{bmatrix} & \text{if } w_{1,0} = 1. \end{array} \right. \end{array}$$

where $\mathcal{W}_{2,n}$ is the set of words of size $2\times n$ over the alphabet $\{0, 1\}$.

The action of Θ on a Rote word w is simply to exchange the two lines of w and then to complement one of them. The line which is complemented is chosen in such a way that the result is normalized. The geometrical interpretation is given in Fig. 10. The following lemma states two important properties of Θ.

$$\begin{bmatrix}0&0&1&0&0\\0&1&1&0&1\end{bmatrix} \overset{\Theta}{\longleftrightarrow} \begin{bmatrix}0&1&1&0&1\\1&1&0&1&1\end{bmatrix}$$

Figure 10: The action of Θ on a Rote word of size $2{\times}5$ and its geometrical interpretation.

Lemma 23 *Let w be a normalized Rote word of size $2{\times}n$ with $n \in \mathbb{N}^\star$, and $(\alpha, \beta, \mu) \in]0,1[^2 \times [0,1[$ be parameters of w. Then*

1. *$\Theta(w)$ is a normalized Rote word with parameters $(1-\alpha, \beta, \alpha+\mu-\lfloor\alpha+\mu\rfloor) \in]0,1[^2 \times [0,1[$.*
2. *$\mathrm{Ext}(\Theta(w)) = \Theta(\mathrm{Ext}(w)) = \{\Theta(w') \mid w' \in \mathrm{Ext}(w)\}$.*

The proof of these two properties is straightforward.

We are now ready to prove:

Proposition 24 *For all $n \in \mathbb{N}^\star$, $\#\mathrm{W}_n^2 = \#\mathrm{W}_n^3$.*

Proof. Lemma 23 implies that for all $w \in \mathrm{M}_{2,n}$, w and $\Theta(w)$ have the same number of extensions. Thus $\Theta(\mathrm{W}_n^2 \cup \mathrm{W}_n^3) \subseteq \mathrm{W}_n^2 \cup \mathrm{W}_n^3$. Since Θ is an involution we have actually the equality

$$\Theta(\mathrm{W}_n^2 \cup \mathrm{W}_n^3) = \mathrm{W}_n^2 \cup \mathrm{W}_n^3.$$

We will show that $\Theta(\mathrm{W}_n^2) = \mathrm{W}_n^3$.

Let $w = \begin{bmatrix}L_0\\L_1\end{bmatrix} \in \mathrm{W}_n^3$ with $L_0, L_1 \in \{0,1\}^n$ and $L_{0,0} = 0$. Lemma 21 implies that $\mathrm{Ext}(w) = \{w', \widetilde{w'}\}$ for some normalized Rote word w'. The unique possibility is $\mathrm{Ext}(w) = \left\{\begin{bmatrix}0&L_0\\0&L_1\end{bmatrix}, \begin{bmatrix}0&\overline{L_0}\\1&\overline{L_1}\end{bmatrix}\right\}$ so that

$$\mathrm{Ext}(\Theta(w)) = \Theta(\mathrm{Ext}(w)) = \left\{\begin{bmatrix}0&\overline{L_1}\\1&\overline{L_0}\end{bmatrix}, \begin{bmatrix}0&\overline{L_1}\\0&\overline{L_0}\end{bmatrix}\right\}$$

Thus $\Theta(w) \notin \mathrm{W}_n^3$ since $\widetilde{\begin{bmatrix}0&\overline{L_1}\\1&\overline{L_0}\end{bmatrix}} \neq \begin{bmatrix}0&\overline{L_1}\\0&\overline{L_0}\end{bmatrix}$. Therefore $\Theta(w) \in \mathrm{W}_n^2$ which proves that $\Theta(\mathrm{W}_n^3) \subseteq \mathrm{W}_n^2$ or equivalently $\mathrm{W}_n^3 \subseteq \Theta(\mathrm{W}_n^2)$ because Θ is an involution.

Now we prove by contradiction that $\Theta(\mathrm{W}_n^2) \subseteq \mathrm{W}_n^3$. Let $w \in \mathrm{W}_n^2$ be such that $\Theta(w) \notin \mathrm{W}_n^3$. Then we have $\Theta(w) \in \mathrm{W}_n^2$ or equivalently $w \in \Theta(\mathrm{W}_n^2)$ so that $w \in \mathrm{W}_n^2 \cap \Theta(\mathrm{W}_n^2)$.

- Let us first assume that w has the form $\begin{bmatrix}0&L_0\\0&L_1\end{bmatrix}$. From Lemma 13, $\begin{bmatrix}0&0&L_0\\1&0&L_1\end{bmatrix}$ is not a Rote word so that the only possible extensions of w are:

$$w_1 = \begin{bmatrix}0&0&L_0\\0&0&L_1\end{bmatrix}, \quad \widetilde{w_1} = \begin{bmatrix}0&1&\overline{L_0}\\1&1&\overline{L_1}\end{bmatrix}, \quad w_2 = \begin{bmatrix}0&1&\overline{L_0}\\0&1&\overline{L_1}\end{bmatrix}.$$

  Since $w \in \mathrm{W}_n^2$, we cannot have $\mathrm{Ext}(w) = \{w_1, \widetilde{w_1}\}$. Neither can we have $\mathrm{Ext}(w) = \widetilde{\{\widetilde{w_1}, w_2\}}$ because we would have $\mathrm{Ext}(\Theta(w)) = \{\Theta(\widetilde{w_1}), \Theta(w_2)\} = \left\{\begin{bmatrix}0&0&\overline{L_1}\\0&1&\overline{L_0}\end{bmatrix}, \begin{bmatrix}0&1&\overline{L_1}\\1&0&\overline{L_0}\end{bmatrix}\right\}$. But since $\widetilde{\begin{bmatrix}0&0&\overline{L_1}\\0&1&\overline{L_0}\end{bmatrix}} = \begin{bmatrix}0&1&\overline{L_1}\\1&0&\overline{L_0}\end{bmatrix}$, it would imply $\Theta(w) \in \mathrm{W}_n^3$ which contradicts the hypothesis. The only remaining possibility is

$\mathrm{Ext}(w) = \{w_1, w_2\}$. We will show that in this case, $\widetilde{w_1}$ is also an extension of w, which again contradicts the hypothesis. It is sufficient to show that $\widetilde{w_1}$ is a Rote word.

Let (α, β, μ), $(\alpha', \beta', \mu') \in]0,1[^2 \times [0,1[$ be respective parameters of w_1 and w_2. Using an argument similar to the one used in the proof of Lemma 18, there exists an odd integer δ such that

$$\forall (i,j) \in \{0,1\}\times\{1,\ldots,n\},\ \lfloor \alpha' i + \beta' j + \mu' \rfloor = \lfloor \alpha i + \beta j + \mu \rfloor + \delta$$

and we have $\lfloor \beta + \mu \rfloor = \lfloor \mu \rfloor = 0$ and $\lfloor \beta' + \mu' \rfloor = \lfloor \mu' \rfloor + 1 = 1$ so that $\delta = 1$. We also have $\lfloor \alpha + \mu \rfloor = \lfloor \alpha' + \mu' \rfloor = 0$ and since $0 \le \alpha + \mu < 1$ and $0 \le \alpha' + \mu' < 1$, we have $1 + \alpha + \mu - \alpha' - \mu' > 0$. Let $\lambda = \frac{1-\alpha'-\mu'}{1+\alpha+\mu-\alpha'-\mu'} \in]0,1]$ and $(\alpha'', \beta'', \mu'') = \lambda(\alpha, \beta, \mu+1) + (1-\lambda)(\alpha', \beta', \mu')$. Since (α'', β'') is a convex combination of (α, β) and (α', β'), we have $(\alpha'', \beta'') \in]0,1[^2$. Moreover $\alpha'' + \mu'' = 1$ so that $\mu'' \in [0,1[$. Let w'' be the Rote word of size $2\times(n+1)$ with parameters $(\alpha'', \beta'', \mu'')$. Since $\alpha''+\mu'' = 1$, we have $w''_{1,0} = 1$ and for all $(i,j) \in \{0,1\}\times\{1,\ldots,n\}$, we have

$$w''_{i,j} = \lfloor \alpha'' i + \beta'' j + \mu'' \rfloor = \lfloor \alpha i + \beta j + \mu \rfloor + 1 = \lfloor \alpha' i + \beta' j + \mu' \rfloor .$$

Finally, we get $w'' = \widetilde{w_1}$ so that $\widetilde{w_1}$ is also an extension of w. This contradicts the hypothesis. Therefore $\Theta(w) \in W_n^3$.

- If w has the form $\begin{bmatrix} 0 & L_0 \\ 1 & L_1 \end{bmatrix}$, the proof proceeds similarly. The possible extensions of w are:

$$w_3 = \begin{bmatrix} 0 & 0 & L_0 \\ 1 & 1 & L_1 \end{bmatrix} \quad w_4 = \begin{bmatrix} 0 & 1 & \overline{L_0} \\ 1 & 0 & \overline{L_1} \end{bmatrix} \quad \widetilde{w_4} = \begin{bmatrix} 0 & 0 & L_0 \\ 0 & 1 & L_1 \end{bmatrix}$$

and the only possibility is $\mathrm{Ext}(w) = \{w_3, w_4\}$. We consider again (α, β, μ) and (α', β', μ'), the parameters of w_3 and w_4, and we show that $\widetilde{w_4}$ is the Rote word with parameters $(\alpha'', \beta'', \mu'') = \lambda(\alpha, \beta, \mu) + (1-\lambda)(\alpha', \beta', \mu' - 1)$ where $\lambda = \frac{1-\mu'}{1+\mu-\mu'}$. Finally, we get $\mathrm{W}_n^2 \cap \Theta(\mathrm{W}_n^2) = \emptyset$ so that $\Theta(\mathrm{W}_n^2) \subset \mathrm{W}_n^3$ hence $\Theta(\mathrm{W}_n^2) = \mathrm{W}_n^3$.

□

Thanks to this equality, we may remove the last term in the right-hand side of Eq. 3 which becomes

$$\forall n \ge 1,\ \#\mathrm{M}_{2,n+1} = \mathrm{M}_{2,n} + 2\,\#\mathrm{M}^0_{2,n+1}. \tag{4}$$

In the following, we will exhibit a closed formula for $\#\mathrm{M}^0_{2,n}$ which will terminate the proof of Th. 6.

Proposition 25 *For all* $n \in \mathbb{N}^\star$, $\#\mathrm{M}^0_{2,n} = n\left(1 + \sum_{i=2}^{n-1} \varphi(i)\right)$.

Proof. The result is obvious if $n = 1$. In the following, we assume $n \ge 2$. A given rote word $w \in \mathrm{M}^0_{2,n}$ is the coding of a unique (m,n)-cube C. Let $(z_{i,j})_{(i,j)\in\{0,1\}\times\{0,\ldots,n-1\}}$ be the height coding of the canonical representative of C. For any $(\alpha, \beta) \in]0,1[^2$, $(\alpha, \beta, 0)$ is a triple of parameters of w if and only if

$$\forall (i,j) \in \{0,1\}\times\{0,\ldots,n-1\},\ z_{i,j} = -\lfloor \alpha i + \beta j \rfloor,$$

or equivalently

$$\forall (i,j) \in \{0,1\}\times\{0,\ldots,n-1\},\ -z_{i,j} \le \alpha i + \beta j < -z_{i,j} + 1.$$

This means that the point of coordinates (α, β) lies in the stripe between the two parallel lines of respective equations $i\,x + j\,y = -z_{i,j}$ and $i\,x + j\,y = -z_{i,j} + 1$ (see Fig. 11). The intersection of all these stripes when

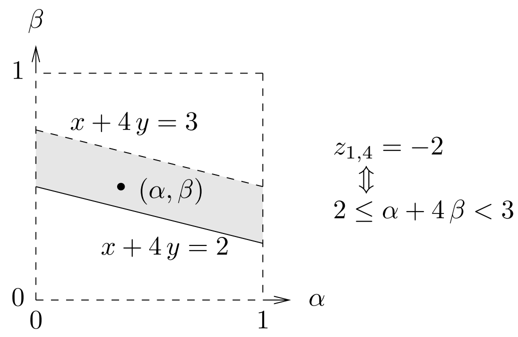


Figure 11: The shaded stripe represents the possible values for α and β when $z_{1,4} = -2$.

i and j vary define the set of all parameters of w of the form $(\alpha, \beta, 0)$ with $(\alpha, \beta) \in]0, 1[$. Note that by definition of $\mathrm{M}^0_{M,n}$, this set is non empty.

Hence, the elements of $\mathrm{M}^0_{2,n}$ are in one-to-one correspondence with the faces of the graph generated by the intersections between the unit square $]0, 1[^2$ and the lines of equation $i\,x + j\,y = k$ with $i \in \{0, 1\}$, $j \in \{0, \dots, n-1\}$, $k \in \mathbb{Z}$ and $(i, j) \neq (0, 0)$. In fact, we have an intersection with the unit square only if and only if $k \in \{1, \dots, i + j - 1\}$ (see Fig. 12 and Fig. 13).

We observe that the vertices of the graph are all located either on the horizontal lines in the graph or on the lower or upper edge of the square. Indeed, two distinct lines of respective equations $i\,x + j\,y = k$ and $i'\,x + j'\,y = k'$ intersect if and only if $i\,j' - i'\,j \neq 0$. Their intersection point is

$$(x, y) = \left(\frac{k\,j' - k'\,j}{i\,j' - i'\,j}, \frac{i\,k' - i'\,k}{i\,j' - i'\,j} \right)$$

and since $i, i' \in \{0, 1\}$ and $j, j' \in \{0, \dots, n-1\}$, we have $|i\,j' - i'\,j| \leq n - 1$. Therefore, if $(x, y) \in [0, 1]^2$ then $y = r/d$ for some integers r, d such that $d > 0$ and $0 \leq r \leq d \leq n - 1$. If $r = 0$ then (x, y) is on the lower edge of the square, if $r = d$ then (x, y) is on the upper edge of the square and otherwise (x, y) is on the horizontal line of equation $r\,y = d$. Hence the number of faces of the subgraph between the horizontal line $y = r/d$ $(0 \leq r < d \leq n - 1)$ and the one immediately above it is exactly one plus the number of distinct lines which intersect $y = r/d$ at $x \in]0; 1]$. Since (i, j, k) and $(\lambda\,i, \lambda\,j, \lambda\,k)$ define the same line, we only have to consider triples (i, j, k) such that $\gcd(i, j, k) = 1$. Thus the number of faces of the subgraph is exactly

$$\begin{aligned} & 1 + \#\{(i, j, k) \mid 0 \leq i \leq 1 \wedge 1 \leq j \leq n - 1 \wedge (k - i)/j \leq r/d < k/j \wedge \gcd(i, j, k) = 1\} \\ = \; & 1 + \#\{(j, k) \mid 1 \leq j \leq n - 1 \wedge r\,j/d < k \leq r\,j/d + 1\} \\ = \; & n \end{aligned}$$

Consequently, the total number of faces of the graph is n times the number of such subgraphs which itself is one plus the number of horizontal lines distinct from the edges of the square. We have such a line for each irreducible fraction k/j where $1 \leq k < j \leq n - 1$. The number of these fractions is exactly $\sum_{i=2}^{n-1} \varphi(i)$ so that the number of subgraphs is $1 + \sum_{i=2}^{n-1} \varphi(i)$. □

Remark 26 Note that if we replace $1 + \sum_{i=2}^{n-1} \varphi(i)$ with $\sum_{i=1}^{n-1} \varphi(i)$ in Lemma 25, the formula is not valid anymore for $n = 1$.

The previous proof is strongly inspired by J. Berstel and M. Pocchiola's one [3] of the following theorem:

Theorem 27 ([1, 2, 3]) *Let $n \in \mathbb{N}^\star$. The number $s(n)$ of Rote words of size $1 \times n$ is*

$$s(n) = 2 \left(1 + \sum_{i=1}^{n-1} (n - i) \varphi(i) \right).$$

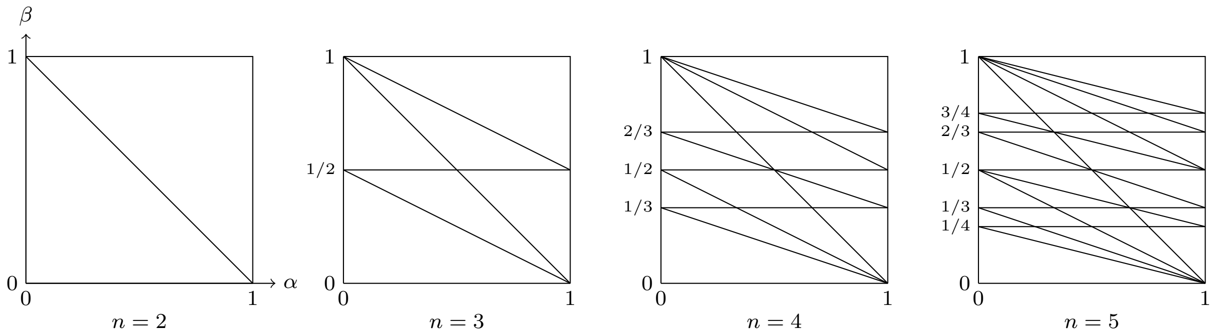


Figure 12: Graphs associated with normalized Rote words in $\mathrm{M}^0_{2,n}$, $n \in \{2, 3, 4, 5\}$.

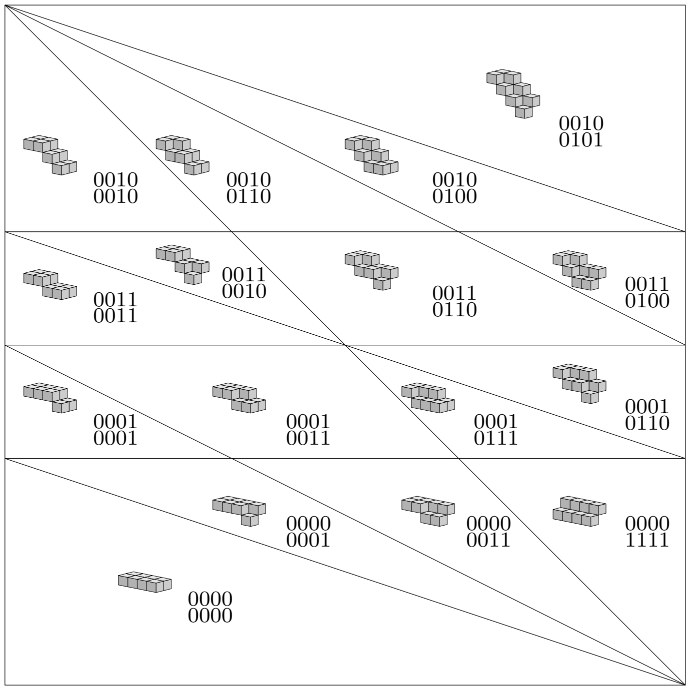


Figure 13: Graph and 3-dimensional geometric configurations associated with normalized Rote words in $\mathrm{M}^0_{2,4}$.

We are now ready to finalize the proof of Th. 6.

Proof (of Th. 6). Since $\mathcal{M}_{2,n}$ and $\mathrm{M}_{2,n}$ are in one to one correspondence, we have $\#\mathcal{M}_{2,n} = \#\mathrm{M}_{2,n}$. Using Eq. 4 and Prop. 25, the result follows by a straightforward recurrence. □

Example 28 The first values of $\#\mathcal{M}_{2,n}$ for $n \in \{1, \ldots, 10\}$ are 2, 6, 18, 50, 110, 230, 398, 686,1082,1642.

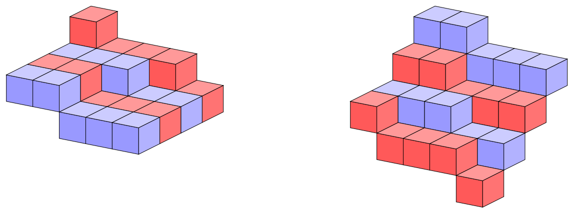
Corollary 29 *Let $n \in \mathbb{N}^\star$. The number $r(n)$ of Rote words of size $2\times n$ is*

$$r(n) = 2\, s(n) = 4 \left(1 + \sum_{i=1}^{n} i \sum_{j=1}^{i-1} \varphi(j) \right),$$

6. Conclusion and perspectives

We have computed the number of local configurations of size $2{\times}n$ in discrete planes or equivalently the number of rectangular words over two letters generated by a double rotation over the torus $\mathbb{R}/\mathbb{Z} \equiv [0,1[$ partitioned as $[0,1[\equiv [0,1[\cup[1,2[$ using combinatorics on words, 2-dimensional Rote sequences and Berstel-Pocchiola diagrams.

In a forthcoming work, it would be interesting to investigate the generalization of this result, that is the number of local configurations of size $m{\times}n$. Most of our definitions and lemmas are still valid in the general case. For instance, the definitions of $\tilde{w}$, $\mathrm{M}^0_{m,n}$ and $\mathrm{M}^1_{m,n}$ do not depend on m and Lemmas 17 and 18 hold in the general case. The operator Θ may also be easily extended to the general case as follows: if w is a normalized Rote word of size $m{\times}n$ with parameters $(\alpha,\beta,\mu) \in]0,1[^2 \times [0,1[$ then $\Theta(w)$ is the normalized Rote word of size $m{\times}n$ with parameters $(1-\alpha,\beta,(m-1)\alpha+\mu-\lfloor(m-1)\alpha+\mu\rfloor)$. The action of Θ on a normalized Rote word is to reverse the order of its rows and then to complement either the odd ones or the even ones, so as to get a normalized word. This is depicted in Fig. 14. However, in the general case, the



$$\begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix} \xleftrightarrow{\;\Theta\;} \begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

Figure 14: Generalization of the operator Θ.

partition of $\mathrm{M}_{m,n}$ given in Lemma 22 may be much more complex and a simple relation between $\#\mathrm{M}^0_{m,n}$ and $\#\mathrm{M}_{m,n}$ like Eq. 4 does not seem to exist. Moreover, even a general formula for $\#\mathrm{M}^0_{m,n}$ seems hard to establish.

One possible approach would be to extend Berstel and Pocchiola's scheme of proof. We could consider the graph generated by the intersections of the unit cube with the planes of equations $\alpha\, i+\beta\, j+\mu = k$ with $(i,j,k) \in \{0,\ldots,m-1\}{\times}\{0,\ldots,n-1\}{\times}\mathbb{Z}$. applying Euler's theorem to this graph might lead to a general formula.

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