

Non lattice periodic tilings of \mathbb{R}^3 by single polycubes

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Abstract

In this paper, we study a class of polycubes that tile the space by translation in a non lattice periodic way. More precisely, we construct a family of tiles indexed by integers with the property that T_k is a tile having $k \geq 2$ has anisohedral number. That is k copies of T_k are assembled by translation in order to form a metatile. We prove that this metatile is lattice periodic while T_k is not a lattice periodic tile.

Keywords: Tilings of \mathbb{R}^3 , polycubes, tilings by translation, lattice periodic tilings, anisohedral number.

1 Introduction

Finding a single tile that tiles the plane (or the space) by translation in an aperiodic way is still a challenge. *Tile* is used in a general meaning (with rectifiable or not boundary) that is an object of the plane (or of the space) that makes a partition (without overlapping and without holes) of the plane (or of the space) by translated copies of an origin tile. In fact along this article we denote by *polyomino* [6] a simply connected union of unit squares and by *polycube* [8] a simply connected union of unit cubes. Indeed, many nice constructions are given in

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particular to exhibit a single 3D tile that tiles in a non periodic way (this kind of tile is called einstein by Danzer [3]). The first construction using a tile T and infinitely many copies of T with an irrational rotation of the original tile was given by Danzer [3]. A recent article by Socolar and Taylor uses an einstein in 2D with coloration and matching rules in order to find layers that are non periodic [10, 11]. In two dimension, it is still an open problem to find a single tile that tiles the plane by translation only in a non periodic way. More precisely by a theorem of Beauquier-Nivat [1, 13] if a tile T with rectifiable boundary tiles the plane then T tiles also the plane in a *lattice periodic* way, that is the position of each tile is given by an integer combination of two non collinear vectors. Thus, if we search a tile in the plane that tiles in a non lattice periodic way then it must have a non rectifiable boundary; in other terms such a candidate for a non periodic tiling of the plane with a single tile must be with a fractal boundary. Furthermore, to measure the degree of aperiodicity of a tiling, it is classical to define the anisohedral number [7]. By definition adapted to the case of tilings by a single tile, the *anisohedral number* is k if we use the distinct union of k copies of T to construct a metatile that tiles the plane by translation in a lattice periodic way. In fact, the tiling with T will be aperiodic if the anisohedral number of T is infinite for all tilings with T (see [9, 8]). In dimension 2, we have many constructions of a tiling such that the metatile is given by copies by translation and rotation of T [7, 12]. Nevertheless, in 3D it was an open problem to find anisohedral numbers only by translation of a single tile and in this article we give an example of polycube for each anisohedral number. A discrete model for tilings of the plane is given by polyominoes and there exists a nice characterization of polyominoes that tile the plane by translation by contour words of polyominoes [1]. In 3D, we use polycubes to investigate tilings of the space and we show that the dimension 3 is much more difficult than the dimension 2 (see also [5]). Indeed, in dimension 2 each rectifiable tile tiles the plane in a lattice periodic way while in dimension 3 we construct an infinite family of polycubes that tiles the space in a non lattice periodic way.

In this article, we construct a family of polycubes T_k with the property that $k \geq 2$ copies of T_k are assembled by translation in order to form a metatile and this metatile is lattice periodic and any assembly of strictly less than k copies of T_k by translation is not a lattice periodic. In addition to that T_2 is the smallest tile in number of cubes in the polycube with an anisohedral number equals to 2. Unfortunately the number of cubes increases with k and then when k tends to infinity, the number of cubes goes to infinity, thus it is still an open problem to find a single tile that tiles the space by translation on an aperiodic

way.

2 A 2-anisohedral tile

We begin our study by searching the smallest polycube that has anisohedral number equals to 2. We give the first candidate with 8 cubes. The picture Fig. 1 presents the representation of the polycube T_2 .

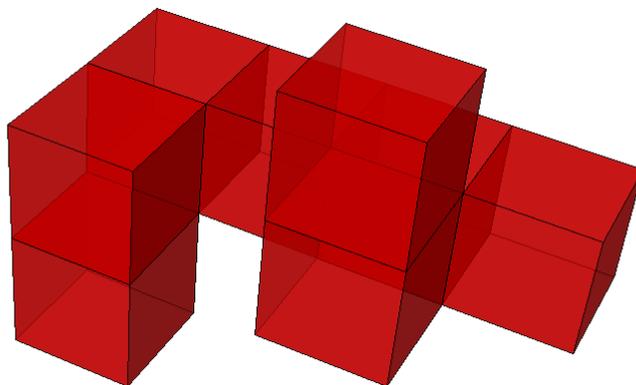


Figure 1: A representation in 3D of the polycube T_2 .

In order to fix the construction, each polycube is given by a list of positions of its unit cubes.

T_2 is constructed by the union of an horizontal bar of 4 unit cubes $B = \{(0, 3, 0), (0, 2, 0), (0, 1, 0), (0, 0, 0)\}$ with two extra vertical bars of 2 unit cubes $U = \{(1, 2, 1), (1, 2, 0)\}$ and $L = \{(1, 0, 0), (1, 0, -1)\}$ (see Fig. 1).

Theorem 2.1. *The polycube*

$$T_2 = \{(0, 3, 0), (0, 2, 0), (0, 1, 0), (0, 0, 0), (1, 2, 1), (1, 2, 0), (1, 0, 0), (1, 0, -1)\}$$

is the smallest tile in number of unit cubes with anisohedral number equals to 2.

Proof. We have to prove that T_2 tiles the space in a non lattice periodic way. In fact by construction of T_2 there is a gap between the two vertical bars U and L , namely a unit cube with position $(1, 1, 0)$. Thus we must fill the position $(1, 1, 0)$ by a copy of T_2 . There is only two ways of filling the gap by a copy of T_2 . We take either a copy of T_2 by

the translation $(0, 1, 1)$ or by the translation $(0, -1, -1)$. In fact the two translations are congruent because it leads to the same metatile up to a translation. We only investigate the translation $(0, 1, 1)$ and we construct the metatile

$$M_2 = T_2 \cup (T_2 + (0, 1, 1)).$$

This metatile contains the following 16 distinct unit cubes

$$M_2 = \{(0, 3, 0), (0, 2, 0), (0, 1, 0), (0, 0, 0), (1, 2, 1), (1, 2, 0), (1, 0, 0), (1, 0, -1)\} \\ \cup \{(0, 4, 1), (0, 3, 1), (0, 2, 1), (0, 1, 1), (1, 3, 2), (1, 3, 1), (1, 1, 1), (1, 1, 0)\}.$$

In order to prove that M_2 tiles the space, we will show that M_2 up to well chosen moduli defines a fundamental domain. We use the moduli on each coordinate given by $(2, 4, 2)$ that is we use the modulo 2 for the first coordinate, the modulo 4 for the second coordinate and the modulo 2 for the last coordinate. Thus, we must prove that the unit cubes of M_2 goes by moduli on the box $2 \times 4 \times 2$ (see Fig. 2).

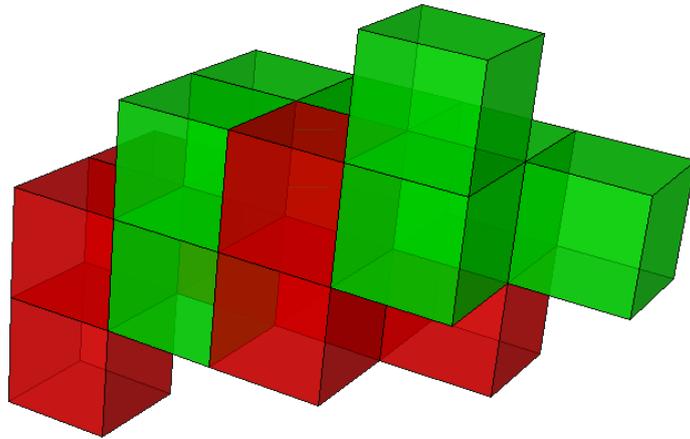


Figure 2: The polycube $T_2 \cup (T_2 + (0, 1, 1))$.

By computation of the moduli,

$$T_2 \text{ mod } (2, 4, 2) =$$

$$\{(0, 3, 0), (0, 2, 0), (0, 1, 0), (0, 0, 0), (1, 2, 1), (1, 2, 0), (1, 0, 0), (1, 0, 1)\}$$

and

$$(T_2 + (0, 1, 1)) \text{ mod } (2, 4, 2) =$$

$\{(0, 0, 1), (0, 3, 1), (0, 2, 1), (0, 1, 1), (1, 3, 0), (1, 3, 1), (1, 1, 1), (1, 1, 0)\}$.

And, we find each unit cube of M_2 one time in the box $2 \times 4 \times 2$. Indeed, $(0, j, 0)$ appears with $j = 0, 1, 2, 3$ (this is exactly the bar B modulo $(2, 4, 2)$). The unit cube $(0, j, 1)$ appears with $j = 0, 1, 2, 3$ (this is exactly the bar $B + (0, 1, 1)$ modulo $(2, 4, 2)$). The cubes $(1, j, 0)$ and $(1, j, 1)$ appear one time in $U, U + (0, 1, 1), L$ and $L + (0, 1, 1)$ modulo $(2, 4, 2)$. There is no overlapping thus M_2 defines a fundamental domain. Thus M_2 tiles the plane by integral combination of vectors $v_1 = (2, 0, 0), v_2 = (0, 4, 0)$ and $v_3 = (0, 0, 2)$. In other words M_2 forms a lattice periodic tiling with lattice vectors v_1, v_2, v_3 .

Remark that the translation $(0, 1, 1)$ is not an integral combination of v_1, v_2, v_3 and then T_2 is not lattice periodic while M_2 is lattice periodic with two copies of T_2 . Thus T_2 has an anisohedral number equals to 2.

We can now check the fact that T_2 is the smallest in number of unit cubes by a direct examination of the enumeration of polycubes of size 1 to 7. A more geometrical explanation comes from the fact that a gap is constructed by at least 5 cubes in a same layer surrounding the gap (namely the cubes in positions : $(1, 2, 0), (0, 2, 0), (0, 1, 0), (0, 0, 0), (1, 0, 0)$) and to block the 3 directions in space, we must add three extra cubes in positions $(0, 3, 0), (1, 2, 1)$ and $(1, 0, -1)$. Thus, we find exactly T_2 by this reasoning. \square

3 A k -anisohedral tile

In this section, we construct a k -anisohedral tile for each $k \geq 2$ by generalization of the construction of T_2 . We take a fixed k in $\mathbb{N} - \{0, 1\}$. The polycube T_k is constructed by the union of an horizontal bar of $2k$ unit cubes

$$B_k = \{(0, 2k - 1, 0), (0, 2k - 2, 0), \dots, (0, 0, 0)\}$$

with two extra vertical bars of k unit cubes

$$U_k = \{(1, k, k - 1), (1, k, k - 2), \dots, (1, k, 0)\}$$

and

$$L_k = \{(1, 0, 0), (1, 0, -1), \dots, (1, 0, -k + 1)\}.$$

Theorem 3.1. *The polycube $T_k = B_k \cup U_k \cup L_k$ has an anisohedral number equals to k .*

Proof. We have to prove that T_k tiles the space in a non lattice periodic way. In fact by construction of T_k there is a gap with $k - 1$ cubes

between the two vertical bars U_k and L_k , namely the unit cubes with positions $(1, k-1, 0), (1, k-2, 0), \dots, (1, 1, 0)$. Thus we must fill these positions by $k-1$ copies of T_k . There is only two ways of filling these gaps by copies of T_k . Either we take a copy of T_k by the translations $(0, j, j)$ with $j = 1, 2, \dots, k-1$ or by the translations $(0, -j, -j)$ with $j = 1, 2, \dots, k-1$. In fact the two translations are congruent because it leads to the same metatile up to a translation. We only investigate the translations $(0, j, j)$ with $j = 1, 2, \dots, k-1$ and we construct the metatile

$$M_k = T_k \bigcup_{j=1}^{k-1} (T_k + (0, j, j)).$$

By construction, this metatile M_k contains $4k^2$ distinct unit cubes. Remark, that we are obliged to use $k-1$ copies in order to fill the $k-1$ gaps of the origin tile.

In order to prove that M_k tiles the space, we will show that M_k up to well chosen moduli defines a fundamental domain. We use the moduli $(2, 2k, k)$ that is we use the modulo 2 for the first coordinate, the modulo $2k$ for the second coordinate and the modulo k for the last coordinate. We must prove that the unit cubes of M_k goes by moduli on the box $2 \times 2k \times k$.

By computation of the moduli,

$$T_k \pmod{(2, 2k, k)} = B_k \cup U_k \cup \{(1, 0, 0), (1, 0, k-1), \dots, (1, 0, 1)\}$$

and

$$(T_k + (0, 1, 1)) \pmod{(2, 2k, k)} = B_k + (0, 0, 1) \cup U_k + (0, 1, 0)$$

$$\cup \{(1, 1, 0), (1, 1, k-1), \dots, (1, 1, 1)\},$$

⋮

$$(T_k + (0, k-1, k-1)) \pmod{(2, 2k, k)} = B_k + (0, 0, k-1) \cup U_k + (0, k-1, 0)$$

$$\cup \{(1, k-1, 0), (1, k-1, k-1), \dots, (1, k-1, 1)\}.$$

And, we find each unit cube of $M_k \pmod{(2, 2k, k)}$ one time in the box $2 \times 2k \times k$, indeed, $(0, j, \ell)$ appears with $j = 0, 1, \dots, 2k-1$ and $\ell = 0, 1, \dots, k-1$ (this is exactly the bar $B_k + (0, \ell, \ell)$ modulo $(2, 2k, k)$). The cubes $(1, j, 0), (1, j, 1), \dots, (1, j, k)$ with $j = 0, 1, \dots, 2k-1$, appear one time in $U_k, U_k + (1, 1, 0), \dots, U_k + (0, k-1, k-1), L_k, L_k + (1, 1, 0), \dots, L_k + (0, k-1, k-1)$ modulo $(2, 2k, k)$. There is no overlapping thus M_k defines a fundamental domain. The metatile M_k tiles the plane by integral combination of vectors $v_1 = (2, 0, 0)$, $v_2 = (0, 2k, 0)$ and $v_3 = (0, 0, k)$. In other words M_k forms a lattice periodic tiling with lattice vectors v_1, v_2, v_3 .

Remark that the translation $(0, j, j)$ for each $j = 1, 2, \dots, k - 1$ is not an integral combination of v_1, v_2, v_3 and then T_k is not lattice periodic while M_k is lattice periodic with k copies of T_k . Thus T_k has an anisohedral number equals to k . \square

4 Decidability result, experimental results and open questions

In 2D, many problems of tilings are undecidable [6], thus in 3D it will be hard to find decidable results. Nevertheless, we find that it is decidable for a finite polycube to tile the space by translation in a periodic way :

Theorem 4.1. *It is decidable to show that a polycube P with finite volume tiles the space by translation in a lattice periodic way.*

Proof. First, we make all the surrounding of the origin tile P by translated copies of P . For each surrounding, we verify that it cover the surface of P with no overlapping between two translated copies of P . If the vector v is used to find one translated tile in the surrounding of P then $-v$ is also used in the surrounding of P , indeed, in a lattice periodic tiling if v is a vector of the lattice then all integral combinations of v are used and in particular $-v$ is used.

Thus we make this first surrounding, we take all the vectors of translation in the first surrounding and we try to extend the tiling to the whole space. In fact, we try to make a surrounding of the first surrounding using the vectors of translation given by the first surrounding. If there is no gaps and no overlapping in this second surrounding then the whole space can be tiled in a lattice periodic way. As the volume of P is finite there is a finite number of first surrounding and then a finite number of verification for the surrounding of the first surrounding. In summary, it is decidable to show that a polycube P with finite volume tiles the space by translation in a lattice periodic way. \square

Now, the general problem of proving prove the decidability of tiling by a polycube P is still open because the number of surrounding should be not bounded (see [5]).

In fact, Theorem 4.1 is true in each dimension d with $d \geq 2$. Because in dimension $d = 2$, it comes from the Theorem of Beauquier-Nivat [1] and this is the key argument for proving the characterization for polyomino that tiles the plane. In the original article of Beauquier-Nivat, they use a double counting of the perimeter of the boundary of

the first surrounding and the interior of the second surrounding in order to prove the tiling property. In dimension 3, we don't have analogous of such characterization but we are able to manage when we focus on lattice periodic tilings (see [5]). In particular, our algorithm that constructs all the lattice periodic tilings checks the double surrounding of P in order to construct all the lattice tilings of P . In higher dimension the proof is the same:

Theorem 4.2. *It is decidable to show that a hyperpolycube P with finite volume tiles the hyperspace of dimension $d \geq 4$ by translation in a lattice periodic way.*

Proof. Same arguments as Theorem 4.1. Because, we use the vectors in a first surrounding (with the constraint that if v is used in the surrounding then $-v$ is also used in the surrounding) in order to find a valid surrounding of this first surrounding with no gaps and no overlapping. \square

We now give some experimental results about the number of lattice periodic surrounded of polyominoes. The first column gives the volume of the polyominoes. The second column counts the number of polycubes. The third column counts the number of polycubes that give lattice periodic tilings.

Nb. of cubes	Nb. of polycubes	Nb. of surrounded polycubes	Quotient
2	1	1	1
3	2	2	1
4	7	7	1
5	23	20	0.87
6	112	96	0.86
7	607	403	0.66
8	3,811	2,472	0.65
9	25,413	10,666	0.42
10	178,083	57,187	0.32
11	1,279,537	180,096	0.14

We remark that the number of polycubes which could be surrounded in order to give a lattice tiling is really smaller than the number of polycubes when the number of cubes is increasing. And experimentally the quotient of the number of polycubes which could be surrounded divided by the number of polycubes seems to go to 0 when the number of cubes of the polycubes goes to infinity.

We are also able to compare the number of polycubes that give lattice periodic tilings (second column) with the number of distinct lattice periodic tilings (third column).

Nb. of cubes	Nb. of surrounded polycubes	Nb. of lattice periodic tilings	Quotient
2	1	4	4.00
3	2	12	6.00
4	7	56	8.00
5	20	97	4.85
6	96	579	6.03
7	403	991	2.46
8	2,472	7,314	2.96
9	10,666	19,206	1.80
10	57,187	99,939	1.75
11	180,096	212,760	1.18

We remark that the number of polycubes that give lattice periodic tilings is not so far from the number of distinct lattice periodic tilings. And experimentally the quotient of polycubes that give lattice periodic tilings divided by the number of distinct lattice periodic tilings seems to go to 1 when the number of cubes of the polycubes goes to infinity. By direct computation, we see that the bar with k cubes has k^2 distinct lattice periodic surrounding. While the experimental results seem to show that when k is increasing, there are more and more polycubes with only one lattice periodic surrounding. We would like to prove that for $k \gg 0$ if we take a polycube at random in the set of lattice periodic surrounding polycubes of size k then random polycubes are, with probability equals to $1-\epsilon$, polycubes with only one lattice periodic surrounding.

Many other interesting problems are still open problems. For example, we think that the polycubes in Theorem 3.1 are in fact minimal that is no polycube with anisohedral number equals to k and with volume less than $4k$. In dimension 2, it is still a challenge to find a single tile with fractal boundary that tile the space in an aperiodic way. In fact, we think that such object does not exists in dimension 2. And in dimension 3, it's an open problem to find a single tile (polycube or not) that tiles the space by translation in an aperiodic way.

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