Reconstruction of 2-convex polyominoes

A. Frosini, S. Rinaldi, K. Tawbe, L. Vuillon

Department of Sistemi and Informatica, University of Firenze

Department of Matematica and Informatica, University of Siena

Laboratoire de Mathématiques, Université de Savoie, CNRS UMR 5127, 73376 Le Bourget du Lac

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Abstract

A polyomino P is called 2-convex if for every two cells belonging to P, there exists a monotone path included in P with at most two changes of direction. This paper studies the tomographical aspects of 2-convex polyominoes from their horizontal and vertical projections and gives an algorithm that reconstructs all 2-convex polyominoes in polynomial time.

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1 Introduction

There are many notions of discrete convexity of polyominoes (namely hv-convex [2], Q-convex [3], L-convex polyominoes [8]) and each one leads to interesting studies. One natural notion of convexity on the discrete plane is the class of hv-convex polyominoes, that is polyominoes with consecutive cells in rows and columns. Following the works of Barcucci et al. [2] we are able to reconstruct polyominoes that are hv-convex according to their horizontal and vertical projections. In addition to that, for an hv-convex polyomino P every pair of cells of P can be reached using a path included in P with only two kinds of unit steps (such a path is called monotone). A polyomino is called k-convex if for every two cells we find a monotone path with at most k changes of direction. Obviously a k-convex polyomino is an hv-convex polyomino. Thus, the families of k-convex polyominoes for $k \in \mathbb{N}$ forms a hierarchy of hv-convex polyominoes. When the value of k is equal to 1 we have the so called L-convex polyominoes, where this terminology is motivated by the L-shape of the path

Email address: `andrea.frosini@unifi.it, rinaldi@unisi.it, khalil.tawbeh@univ-savoie.fr, laurent.vuillon@univ-savoie.fr` (A. Frosini, S. Rinaldi, K. Tawbe, L. Vuillon).

that connects any two of its cells. This notion of L-convex polyominoes has been considered by several points of view. In [5,4] combinatorial aspects of L-convex polyominoes are analyzed, giving the enumeration according to the semi-perimeter and the area. From a tomographical point of view, in [7] it is given an algorithm that reconstructs an L-convex polyomino from the set of its maximal L-polyominoes, while in [8] the same problem is solved from the size of some special paths, called bordered L-paths. The general reconstruction problem from two projections together with its related uniqueness problem have finally been solved in [6]. A different approach requires the class of 2-convex polyominoes since it is geometrically more complex to characterize. Duchi et al. enumerate in [11] this class using a purely analytical fashion, but their enumeration technique gives no idea for the tomographical reconstruction.

In this paper we furnish an algorithm to reconstruct the 2-convex polyominoes from two projections. We proceed by splitting the class of 2-convex polyominoes into three subclasses, up to symmetries, with respect to the mutual positions of the feet of their elements. Two of them have a simple geometrical characterization, and they can be reconstructed by standard algorithms, while the third one, say $\Im$, that includes all those polyominoes that are 2-convex but not 1-convex, represents the core of the problem. Our approach resembles that in [9], i.e. first we characterize the class $\Im$ in a purely geometrical fashion, then we express this characterization by means of Horn clauses which admit a quick valuation process [1].

2 Definition and notation

A planar discrete set is a finite subset of the integer lattice $\mathbb{N}^2$ defined up to translations. A discrete set S can be represented either by a set of cells, i.e. unitary squares in the cartesian plane, or by a binary matrix $A = (a_{i,j})$, whose dimensions are those of the minimal bounding rectangle of the set, and such that each 1 represents the presence of a point of the subset in the correspondent position, see Fig. 1. By convention, the positions of the points of the set S inherit the standard notation for the elements of a matrix (i.e. the point in position $(1, 1)$ of S is in the upper left position of the minimal bounding rectangle of S).

To each discrete set S, represented by a $m \times n$ binary matrix, we associate two integer vectors $H = (h_1, ..., h_m)$ and $V = (v_1, ..., v_n)$ such that for each $1 \leq i \leq m$, $1 \leq j \leq n$, h_i and v_j are the number of cells of S (elements 1 of the matrix) which lie on row i and column j, respectively. The vectors H and V are called the *horizontal* and *vertical projections* of S, respectively. As an example, the projections of the discrete set in Fig. 1 are

$$H = (3, 2, 3, 1, 1, 1, 3) \quad \text{and} \quad V = (3, 3, 1, 2, 2, 3).$$

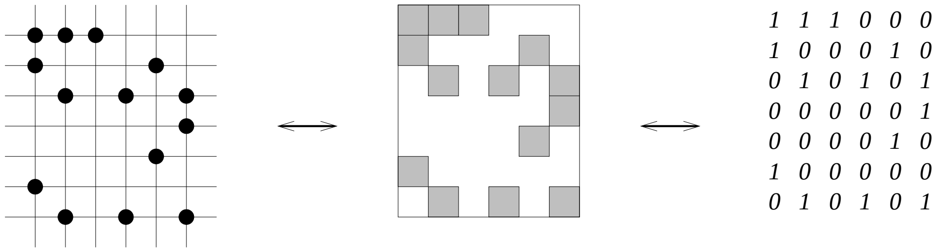

Fig. 1. A finite set of $\mathbb{N} \times \mathbb{N}$, and its representation in terms of a set of cells and of a binary matrix.

Classes of polyominoes

A planar discrete set whose cells are connected is called a *polyomino*. A polyomino is *horizontally-convex* [resp. *vertically-convex*] if its cells lying on each column [resp. row] are connected, while it is *hv-convex* (simply *convex*), if it is both horizontally and vertically convex, see Fig. 2.

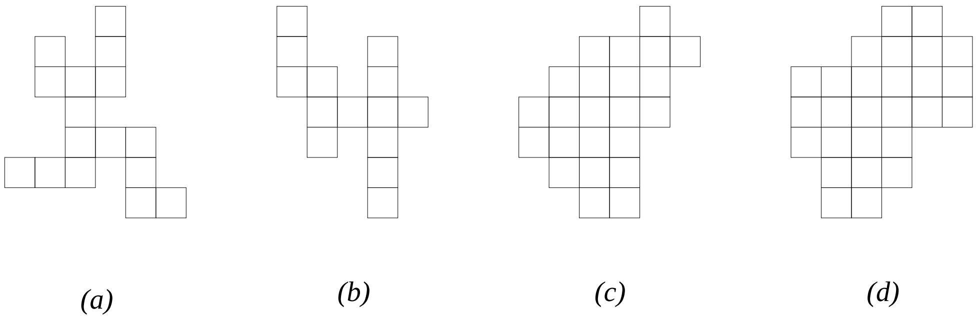

Fig. 2. A polyomino (a), a vertically convex polyomino (b), a convex polyomino (c), and an h-centered polyomino (d).

In each convex polyomino P, we can define the *N-foot* to be the set of cells of P that lie in its first row. Note that, by convexity, the cells of the N-foot form a bar, and let us indicate by $(1, m_N)$ and $(1, M_N)$ its two extremal points, and sometimes, by abuse of notation, simply m_N and M_N.

Analogously, we define the *S-foot*, *W-foot*, and *E-foot* of P, and their extremal points, as depicted in Fig. 4.

We notice that the border of P delimits four disjoint (possibly void) regions in its minimal bounding rectangle, that lie outside P. Following [9], we indicate these four regions with the letters A, B, C, and D, arranged as shown in Fig. 4.

Finally, a convex polyomino P is said to be *horizontally-centered* (briefly *h-centered*) [resp. *vertically-centered* (briefly *v-centered*)], if at least one cell of its W-foot and one cell of its E-foot [resp. N-foot and S-foot] lie the same row [resp. same column], as in Fig. 2, (d).

Now, we define the problem we are going to study

Reconstruction $(H, V, \mathcal{C})$

Input: two integer vectors H and V, and a class of discrete sets $\mathcal{C}$.
Task: reconstruct an element of $\mathcal{C}$ whose horizontal and vertical projections are H and V, respectively, if it exists, otherwise give FAILURE.

A hierarchy on convex polyominoes

For any two cells a and b in a polyomino P, a *path* $\prod_{ab}$, from a to b, is a sequence $(i_1, j_1), (i_2, j_2), ..., (i_r, j_r)$ of adjacent disjoint cells of P, with $a = (i_1, j_1)$, and $b = (i_r, j_r)$. For each $1 \leq k \leq r - 1$, we say that the two consecutive cells $(i_k, j_k), (i_{k+1}, j_{k+1})$ form

- an *east* step if $i_{k+1} = i_k$ and $j_{k+1} = j_k + 1$;
- a *north* step if $i_{k+1} = i_k - 1$ and $j_{k+1} = j_k$;
- a *west* step if $i_{k+1} = i_k$ and $j_{k+1} = j_k - 1$;
- a *south* step if $i_{k+1} = i_k + 1$ and $j_{k+1} = j_k$.

We define a path to be *monotone* if it is entirely made of only two of the four types of steps defined above.

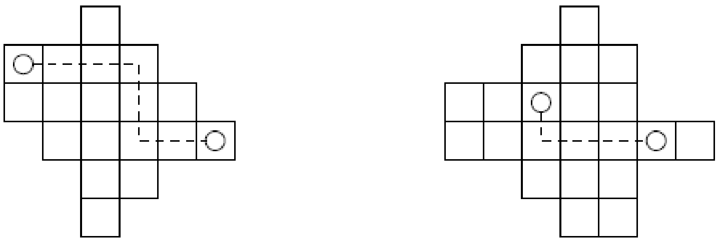

Fig. 3. The convex polyomino on the left is 2-convex, while the one on the right is L-convex. For each polyomino, two cells and a monotone path connecting them are shown.

Proposition 1 (Castiglione, Restivo [7]) *A polyomino P is convex if and only if every pair of cells is connected by a monotone path.*

Let us consider a polyomino P. A path in P has a change of direction in the cell (i_k, j_k), for $2 \leq k \leq r-1$, if

$$i_k \neq i_{k-1} \Longleftrightarrow j_{k+1} \neq j_k.$$

A convex polyomino such that every pair of its cells can be connected by a monotone path with at most k changes of direction is called *k-convex*.

In [7], it is proposed a hierarchy on convex polyominoes based on the number of changes of direction in the paths connecting any two cells of the polyomino. For $k = 1$, we have the first level of hierarchy, i.e. the class of 1-convex polyominoes, also denoted L-convex polyominoes for the typical shape of each path having at most one single change of direction. Tomographical aspects of L-convex polyominoes have been deeply investigated in these last few years, in particular it has been shown that they are characterized both by their horizontal and vertical projections [6], and by their maximal L shapes [7,8] and, in both cases, it has been defined a fast algorithm for their reconstruction. These results have furnished a starting point for the enumeration of the class of L-convex polyominoes according to their perimeter [5] and, successively, according to their area [4].

In the present studies, we focus our attention to the next level of the hierarchy, i.e. the class of 2-convex polyominoes (see Fig. 3), whose tomographical properties turn out to be more interesting and substantially harder to be investigated than those of L-convex polyominoes [7,8].

The following simple property links centered polyominoes and 2-convex polyominoes:

Proposition 2 *If P is a centered polyomino (either h-centered or v-centered), then it is a 2-convex polyomino.*

Centered polyominoes are also characterized by means of the shape of the monotone paths that connect their cells:

Proposition 3 (Duchi et al. [11]) *If P is a h-centered polyomino then there exists a monotone path that connects two of its cells, and that has one of the form $(north)^*(east)^*(north)^*$ or $(north)^*(west)^*(north)^*$.*

In [9], the authors study the problem *Reconstruction*$(H, V, \mathcal{C})$, with $\mathcal{C}$ being the class of convex polyominoes. In this framework, they also consider centered polyominoes as special cases, and they define a linear time algorithm to reconstruct them. So, from now on, we concentrate only on convex polyominoes which are not h-centered or v-centered. In particular, we consider the mutual positions of the feet of a polyomino, and we define the following classes (see Fig. 4) that provide a partition of the 2-convex polyominoes: let $\mathcal{C}_2$ be the class of 2-convex polyominoes

- $\Im = \{P \in \mathcal{C}_2 \mid M_N < m_S \text{ and } M_W < m_E\}$;
- $\Im' = \{P \in \mathcal{C}_2 \mid M_S < m_N \text{ and } M_E < m_W\}$;
- $\gamma = \{P \in \mathcal{C} \mid M_N < m_S \text{ and } M_E < m_W\}$;
- $\gamma' = \{P \in \mathcal{C} \mid M_S < m_N \text{ and } M_W < m_E\}$.

The classes γ and γ' can be reconstructed in a polynomial time from their horizontal and vertical projections, by means of an algorithm defined in [12].

Furthermore, the classes $\Im$ and $\Im'$ coincide up to horizontal symmetry, so they are equivalent from a tomographical perspective. In the sequel, we restrict our investigation only to one of them, i.e. the class $\Im$.

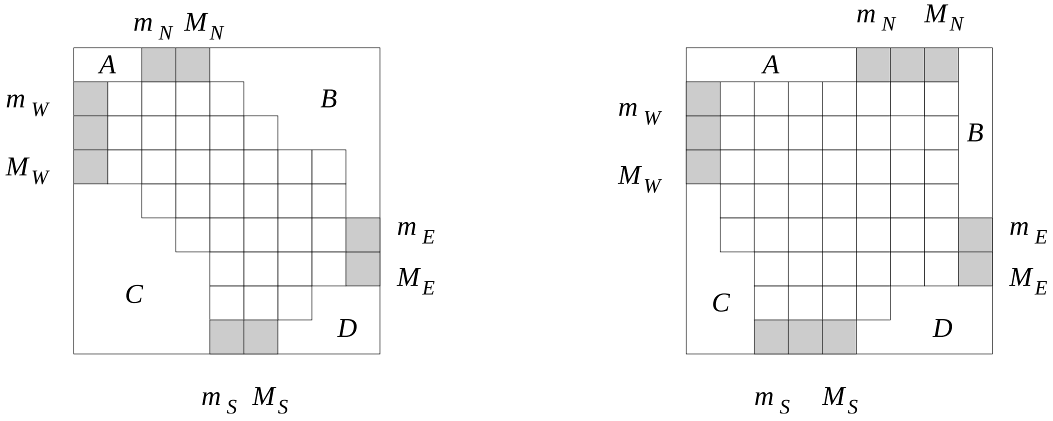

Fig. 4. An element of the class $\Im$ on the left and one of the class γ' on the right. The cells of the four feet are highlighted in both the polyominoes.

3 Further properties of 2-convex polyominoes

Let P be an element of $\Im$ and let $Bor'(P) = \{(i_1, j_1), \ldots, (i_r, j_r)\}$ be the set of cells of P such that $(i_1, j_1) = (m, M_S)$, $(i_r, j_r) = (M_E, n)$, and for $2 \leq k \leq r-1$, let (i_k, j_k) be a cell of the border of P that delimits the zone D of the exterior of P, and sharing one side with the cells (i_{k-1}, j_{k-1}) and (i_{k+1}, j_{k+1}).

Now let $R = \{R_1, \ldots, R_r\}$ be the set of maximal rectangles entirely contained in P, and whose lower rightmost cells correspond to the elements of $Bor'(P)$. Let the upper rightmost cells of $R_1, \ldots, R_r$ be $i_1, \ldots, i_r$, respectively. Figure 5, (a) shows a polyomino in $\Im$, and the cell d_3 that belongs to Bor'; the rectangle R_3, and its upper rightmost cell i_3 are also highlighted.

We define $R' = \{R'_1, \ldots, R'_r\}$ to be the set of rectangles whose lower rightmost cells are $i_1, \ldots, i_r$, respectively, and whose bases and heights extend till reaching the border of P. For each $1 \leq k \leq r$, we indicate with c_k, b_k, and a_k the lower leftmost cell, the upper rightmost cell, and the upper leftmost cell

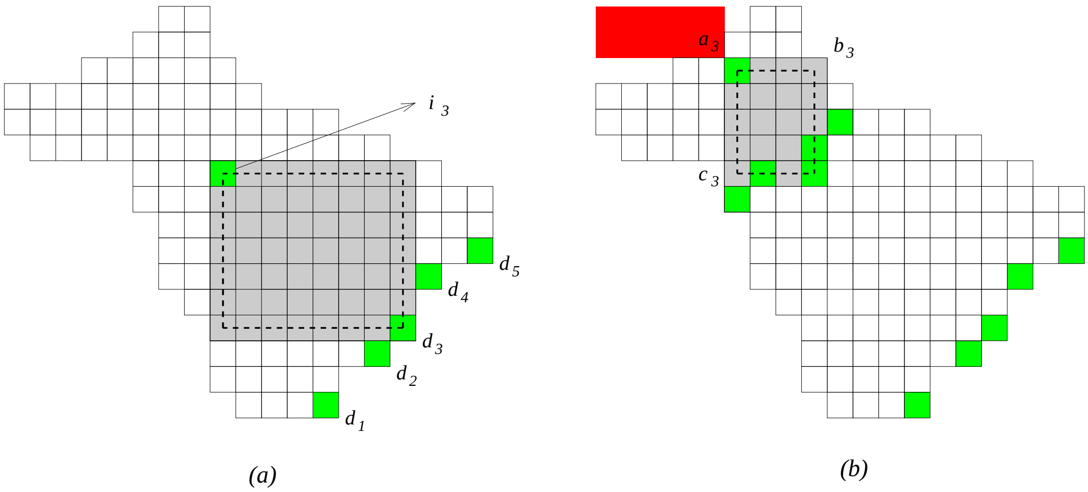

Fig. 5. In (a), a convex polyomino where the $south-east$ corners and the rectangle R_3 corresponding to the corner d_3 are shown. In (b), for the same polyomino, they are highlighted the interior points $i_1, \ldots, i_5$, and the rectangles R'_3 and F_3.

of R'_k, respectively, as depicted in Fig. 4, (b). Note that, even in a 2-convex polyomino, each rectangle R'_k has not to be entirely contained in P. Finally, we indicate with $F_1, \ldots, F_k$ the rectangles having $a_1, \ldots, a_k$ as lower leftmost cells, and that extend till reaching the sides of the minimal bounding rectangle containing P. In Fig. 5, (b) one of these rectangles, i.e. that related to the cell d_3, is also highlighted. We indicate the set of cells $F(P) = \bigcup_{k=1}^{r} F_k$ as *forbidden set*.

Proposition 1 *A convex polyomino P is 2-convex if and only if the set $F(P)$ does not contain any cell of P.*

Proof. ($\Rightarrow$) Let us proceed by contradiction assuming that P is a 2-convex polyomino, and (i, j) is a cell in $P \cap F(P)$. By definition there exists at least one rectangle F_k that contain (i, j), with $1 \leq k \leq r$, and r being the number of elements of $Bor'(P)$.

By definition of the rectangles R_k and R'_k, there is no monotone path entirely contained in P, having two changes of directions at most, and connecting (i, j) to (i_k, j_k), against the assumption on P.

($\Leftarrow$) Let P be a convex polyomino such that $M_N < m_S$ and $M_W < m_E$, and having no cells in $F(P)$. We consider two of its cells (i_1, j_1) and (i_2, j_2), and we show that there exists a monotone path connecting them and having at most two changes of direction. Some cases arise:

i) the two cells belong to two elements of F. It is immediate to check that there exists a monotone path connecting them, and having at most two changes of direction;
ii) at least one of the two cells, say (i_1, j_1) belongs to an element of F, say

F_k. By definition, from each cell of F_k one can reach the cell i_k with two monotone paths having at most one change of direction. From the point i_k, one of these monotone paths can continue till reaching all the cells of P except those in F_k with, at most, a further change of direction. By hypothesis $(i_2, j_2) \notin F_k$, and so the thesis;

iii) none of the two cells belong to an element of R. Let us consider a rectangle $R_k \in R$; since the two cells do not belong to F_k by hypothesis, so they can be reached from i_k with monotone paths having at most one change of direction, furthermore these paths either intersect or at least one of them runs along one side of R_k. In both cases, a monotone path connecting the two starting cells and having at most one change of direction can be easily computed. $\square$

Proposition 2 *Let P be a convex polyomino and (i_k, j_k) and (i_{k+1}, j_{k+1}) be two cells in $Bor'(P)$. If $i_k = i_{k+1}$ [resp. $j_k = j_{k+1}$], then $F_k \subseteq F_{k+1}$ [resp. $F_{k+1} \subseteq F_k$].*

The proof directly follows from the definitions of F_k and F_{k+1}.

Proposition 2 allows us to check the 2-convexity of a polyomino using only few cells of the set $Bor'(P)$, i.e. those cells that are also corners of the polyomino (see Fig. 5, (a), cells $d_1, \ldots, d_5$). We indicate the set of all these cells with $Bor(P)$.

4 Handling hv-convex polyominoes

In [9], the authors defined a quick method to reconstruct an hv-convex polyomino compatible with two vectors $H = (h_1, \ldots, h_m)$ and $V = (v_1, \ldots, v_n)$ of horizontal and vertical projections, if it exists: their idea relies on possibility of using a $2-SAT$ formula (a boolean expression in conjunctive normal form with at most two literals in each clause) to express the geometrical characterization of an hv-convex polyomino, i.e. the presence, in its bounding rectangle, of four disjoint zones, indicated with the letters A, B, C, and D in Fig.4, whose union forms the exterior of the polyomino, and such that each zone is hv-convex, and contains exactly one corner of the rectangle or no cells. The conjunction of the $2-SAT$ formulas used in [9] is indicated with $F_{k,l}(H, V)$. In the next paragraph, we define more of them in order to strengthen the constraint till obtaining the 2-convexity.

To make the formulas clear as much as possible to the reader, we point out that, for each $1 \leq i \leq m$ and $1 \leq j \leq n$, the variables $A_{i,j}$ [resp. $B_{i,j}$, $C_{i,j}$, and $D_{i,j}$] determine the zone A [resp. B, C, and D] of the minimal bounding rectangle of a polyomino P consistent with H and V, i.e. the valuation *true* of $A_{i,j}$ [resp. $B_{i,j}$, $C_{i,j}$, and $D_{i,j}$] means that the cell in position (i, j) belongs to the zone A [resp. B, C, and D], the valuation $false$ otherwise.

The dependance of $F_{k,l}(H, V)$ from the parameters k and l concerns an initial guess of the positions of a cell in the E-foot and in the W-foot of P. So, in general, a different formula $F_{k,l}(H, V)$ is considered for each of the m^2 possible values of k and l.

The existence of an evaluation for at least one $F_{k,l}(H, V)$ directly implies the existence of an hv-convex polyomino P having H and V as projections and such that $P = \overline{A \cup B \cup C \cup D}$. The formulas are the following:

$$Cor = \bigwedge_{i,j} \left\{ \begin{array}{l} A_{i,j} \Rightarrow A_{i-1,j} \; B_{i,j} \Rightarrow B_{i-1,j} \; C_{i,j} \Rightarrow C_{i+1,j} \; D_{i,j} \Rightarrow D_{i+1,j} \\ A_{i,j} \Rightarrow A_{i,j-1} \; B_{i,j} \Rightarrow B_{i,j+1} \; C_{i,j} \Rightarrow C_{i,j-1} \; D_{i,j} \Rightarrow D_{i,j+1} \end{array} \right\}$$

$$Dis = \bigwedge_{i,j} \left\{ X_{i,j} \Rightarrow \bar{Y}_{i,j} : X, Y \in \{A, B, C, D\}, X \neq Y \right\}$$

$$Con = \bigwedge_{i,j} \left\{ A_{i,j} \Rightarrow \bar{D}_{i+1,j+1} \; B_{i,j} \Rightarrow \bar{C}_{i+1,j-1} \right\}$$

$$Anc = \left\{ \bar{A}_{k,1} \wedge \bar{B}_{k,1} \wedge \bar{C}_{k,1} \wedge \bar{D}_{k,1} \wedge \bar{A}_{l,n} \wedge \bar{B}_{l,n} \wedge \bar{C}_{l,n} \wedge \bar{D}_{l,n} \right\}$$

$$LBC = \bigwedge_{i,j} \left\{ \begin{array}{l} A_{i,j} \Rightarrow \bar{C}_{i+v_j,j} \; A_{i,j} \Rightarrow \bar{D}_{i+v_j,j} \\ B_{i,j} \Rightarrow \bar{C}_{i+v_j,j} \; B_{i,j} \Rightarrow \bar{D}_{i+v_j,j} \end{array} \right\} \wedge \bigwedge_j \left\{ \bar{C}_{v_j,j}, \; \bar{D}_{v_j,j} \right\}$$

$$UBR = \bigwedge_j \left\{ \begin{array}{ll} \wedge_{i \leq min\{k,l\}} \bar{A}_{i,j} \Rightarrow B_{i,j+h_i} & \wedge_{k \leq i \leq l} \bar{C}_{i,j} \Rightarrow B_{i,j+h_i} \\ \wedge_{l \leq i \leq k} \bar{A}_{i,j} \Rightarrow D_{i,j+h_i} & \wedge_{max\{k,l\} \leq i} \bar{C}_{i,j} \Rightarrow D_{i,j+h_i} \end{array} \right\}$$

Briefly, each set of clauses defines a specific geometrical property of the polyomino P using the four zones A, B, C, and D, in particular

Cor defines the hv-convexity of the four zones outside P and, for each non void one of them, forces the correspondent *corner* of the minimal bounding rectangle to belong to it;
Dis requires the four zones outside P to be *disjoint*;
Con asks for the *connectedness* of P;
Anc sets the E-foot and the W-foot of P to be *anchored* at the cells $(k, 1)$ and (n, l), respectively;
LBC imposes a *lower bound* to the elements of P for each of its *columns*, according with the vertical projections;
UBR imposes an *upper bound* to the elements of P for each of its *rows*, according with the horizontal projections.

So, $F_{k,l}(H,V)$ turns out to be $Cor \wedge Dis \wedge Con \wedge Anc \wedge LBC \wedge UBR$. All variables with indices outside the set $\{1,...,m\} \times \{1,...,n\}$ are assumed to have value 1.

The reconstruction of the polyomino P is summarized by the following

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Algorithm1
Input: $H \in \mathbb{N}^m, V \in \mathbb{N}^n$
W.l.o.g assume:$\forall i : h_i \in [1,n], \forall j : v_j \in [1,m], \sum_i h_i = \sum_j v_j$ *and* $m \leq n$.
For $k,l = 1,...,m$ **do begin**
If $F_{k,l}(H,V)$ is satisfiable,
then output $P = \overline{A \cup B \cup C \cup D}$ and **halt**
end
output FAILURE

Theorem 1 (Chrobak, Dürr [9]) *$F_{k,l}(H,V)$ is satisfiable if and only there exists an hv-convex polyomino P having H and V as horizontal and vertical projections.*

Each formula $F_{k,l}(H,V)$ has size $O(mn)$ and can be defined in time $O(mn)$. Since 2SAT can be solved in linear time [1,**?**], it holds the following result.

Theorem 2 (Chrobak, Dürr [9]) *Algorithm 1 solves the reconstruction problem for hv-convex polyominoes in time $O(mn \min(m^2, n^2))$.*

5 New clauses to characterize the set $\Im$

In the fashion of [9], we give a characterization of the polyominoes in $\Im$ adding to some of the clauses for *hv*-convex polyominoes, new ones to express the geometrical constraints given in Proposition 1. In addition to that we give our clauses in the form of negative Horn clauses which are of the following forms:

a) a conjunction of positive variables that implies one positive variable;
b) a conjunction of positive variables that implies one negative variable;
c) one single positive variable;
d) one single negative variable.

We choose to maintain the sets of clauses Cor, Dis, and Con, while we slightly modify Anc into Anc_2 to fix the exact positions of all and four feet of P. Different sets of clauses are defined for each possible position of m_N [resp. m_S, m_W, and m_E]. The correspondent values of $M_N = m_N + h_1 - 1$ [resp. $M_S = m_s + h_m - 1$, $M_W = m_W + v_1 - 1$, and $M_E = m_E + v_n - 1$] are also computed.

$$Anc_2 = \left\{ \begin{array}{l} \overline{A}_{1,m_N} \wedge \overline{A}_{m_W,1} \wedge \overline{B}_{1,M_N} \wedge \overline{B}_{m_E,n} \wedge \\ \overline{C}_{M_W,1} \wedge \overline{C}_{m,m_S} \wedge \overline{D}_{m,M_S} \wedge \overline{D}_{M_E,n} \end{array} \right\}$$

The clauses in Pos set the positions of the feet in order to avoid polyominoes that do not belong to $\Im$:

$$Pos = \left\{ B_{1,m_S} \wedge C_{m_E,1} \right\}$$

Now we define sets of clauses to determine the zone $F(P)$ where no elements of P are admitted, in accordance with what stated in Proposition 1. The first one is Ext, where we use the new variables $ExB_{i,j}$ [resp. $ExC_{i,j}$] to identify the elements of B [resp. C] in position (i,j) that are immediate *external* to the polyomino P:

$$Ext = \left\{ \begin{array}{llll} \wedge_{i,j} & ExA_{i,j} \Rightarrow A_{i,j} & ExA_{i,j} \Rightarrow B_{i,j+h_i+1} & (A_{i,j} \wedge B_{i,j+h_i+1}) \Rightarrow ExA_{i,j} \\ \wedge_{i,M_N<j<m_S} & ExB_{i,j} \Rightarrow B_{i,j} & ExB_{i,j} \Rightarrow C_{i+v_j+1,j} & (B_{i,j} \wedge C_{i+v_j+1,j}) \Rightarrow ExB_{i,j} \\ \wedge_{i,m_S\leq j\leq M_S} & ExB_{m-v_i,j} & & \\ \wedge_{i,j>M_S} & ExB_{i,j} \Rightarrow B_{i,j} & ExB_{i,j} \Rightarrow D_{i+v_j+1,j} & (B_{i,j} \wedge D_{i+v_j+1,j}) \Rightarrow ExB_{i,j} \\ \wedge_{M_W<i<m_E,j} & ExC_{i,j} \Rightarrow C_{i,j} & ExC_{i,j} \Rightarrow B_{i,j+h_i+1} & (C_{i,j} \wedge B_{i,j+h_i+1}) \Rightarrow ExC_{i,j} \\ \wedge_{m_E\leq i\leq M_E,j} & ExC_{i,n-h_j} & & \\ \wedge_{i>M_E} & ExC_{i,j} \Rightarrow C_{i,j} & ExC_{i,j} \Rightarrow D_{i,j+h_i+1} & (C_{i,j} \wedge D_{i,j+h_i+1}) \Rightarrow ExC_{i,j} \end{array} \right\}$$

Then, we identify the elements of $Bor(P)$, i.e. those cells of $Bor'(P)$ that are corners, and whose contribution is essential to identify the forbidden region $F(P)$ as stated in Proposition 2; here a new set of variables $Bor_{i,j}$ is introduced.

$$Bor = \bigwedge_{i,j} \left\{ \begin{array}{c} Bor_{i,j} \Rightarrow C_{i,j-h_i} \quad Bor_{i,j} \Rightarrow D_{i,j+1} \quad Bor_{i,j} \Rightarrow D_{i+1,j} \\ (C_{i,j-h_i} \wedge D_{i,j+1} \wedge D_{i+1,j}) \Rightarrow Bor_{i,j} \end{array} \right\}$$

The clauses that assure the 2-convexity of the polyomino P can now be stated.

$$\textit{2-conv} = \bigwedge_{i,j} \left\{ \bigwedge_{s<m_W,t<m_N} (Bor_{i,j} \wedge ExB_{s,j-h_i+1} \wedge ExC_{i-v_j+1,t}) \Rightarrow A_{s,t} \right\}$$

Note that, for each element d in $Bor_{i,j}$, we exactly know the position of the correspondent cell i, i.e. $(i - v_j + 1, j - h_i + 1)$, while we do not for the correspondent b and c, so we need to check all their possible positions using the parameters l and k. Imposing the presence in position (l,k) of the zone A prevents any cell of P from being in the forbidden rectangle related to d.

Furthermore, the exact knowledge of the positions of the four feet of P allows us to impose in a slightly different way the upper bound [resp. lower bound] to the number of cells for each column [resp. row] of P according to its projections. For sake of clarity, we repeat in UBR_2 some clauses already stated in Ext, and that are relevant for setting the lower bound to the number of cells on each row of P.

$$LBC_2 = \bigwedge_i \left\{ \begin{array}{ll} \wedge_{j<m_N} A_{i,j} \Rightarrow \overline{C}_{i+v_j,j} & \wedge_{m_N \leq j \leq M_N} C_{v_j+1,j} \\ \wedge_{M_N<j<m_S} B_{i,j} \Rightarrow \overline{C}_{i+v_j,j} & \wedge_{m_S \leq j \leq M_S} B_{m-v_j,j} \\ \wedge_{j>M_S} B_{i,j} \Rightarrow \overline{D}_{i+v_j,j} & \end{array} \right\} \wedge \bigwedge_j \left\{ \overline{C}_{v_j,j} \ \overline{D}_{v_j,j} \right\}$$

$$UBR_2 = \bigwedge_j \left\{ \begin{array}{ll} \wedge_{i<m_W} ExA_{i,j} \Rightarrow B_{i,j+h_i+1} & \wedge_{m_W \leq i \leq M_W} B_{i,h_i+1} \\ \wedge_{M_W<i<m_E} ExC_{i,j} \Rightarrow B_{i,j+h_i+1} & \wedge_{m_E \leq i \leq M_E} C_{i,n-h_i} \\ \wedge_{i>M_E} ExC_{i,j} \Rightarrow D_{i,j+h_i+1} & \end{array} \right\}$$

In order to reconstruct 2-convex polyominoes, we apply Algorithm 1 to the class $\Im(H,V)$, defined as follows:

$$\Im(H,V) = Cor \wedge Dis \wedge Con \wedge Anc_2 \wedge Pos \wedge Ext \wedge Bor \wedge \textit{2-conv} \wedge LBC_2 \wedge UBR_2.$$

Theorem 3 *$\Im(H,V)$ is satisfiable if and only if there exists a polyomino P in $\Im$ having H and V as horizontal and vertical projections, respectively.*

Proof. ($\Rightarrow$) let us consider the set of cells $P = \overline{A \cup B \cup C \cup D}$. By Theorem 1, P is a convex polyomino. By definition, each variable $Bor_{i,j}$ involved in the clauses Bor has value 1 if and only if there is a cell d of $Bor(P)$ in position (i,j). Since the polyomino belongs to the class $\Im$, for each $d \in Bor(P)$ in position (i,j), there exist two indexes $s < m_W$ and $t < m_N$ such that the variables $ExtB_{s,j-h_i+1}$ and $ExtC_{i-v_j+1,t}$ have value 1 (clauses Ext); such two variables correspond to the elements b and c related to d. Finally the positions of b and c determine the position (s,t) of a, and so the clauses *2-conv* impose the constraint for the forbidden zone related to a, as required by the characterization of 2-convex polyominoes given in Proposition 1. The clauses LBC_2 and UBR_2 get into the 2-convexity context those in LBC and UBR.

($\Leftarrow$) Let P be a 2-convex polyomino in $\Im$. By Theorem 1 all the clauses for convex polyominoes are satisfied by P, and the same holds for the sets Anc_2, LBC_2 and UBR_2. The elements of P in $Bor(P)$ allow the correspondent variables $Bor_{i,j}$ to have value 1. By the characterization given in Proposition 1, for each element d_k in $Bor(P)$ there exist two indexes s and t that determine the cells b_k, c_k and a_k, and consequently the forbidden zone F_k, completely contained in the zone A, so also the related clauses in Ext and *2-conv* are satisfied. $\square$

Theorem 4 *If there exists a valuation for the formula $\Im(H,V)$, then it can be computed in $O(m^4n^4)$ time.*

We compute the complexity of finding a valuation for $\Im(H,V)$ starting from that of $F_{k,l}(H,V)$ for convex polyominoes, i.e. $O(m\,n\min\{m^2,n^2\})$. Since in $\Im(H,V)$ we impose the exact knowledge of all and four the feet of the polyomino, the complexity increases to $O(m^3n^3)$, then we consider all the possible row and column indexes s and t when imposing *2-conv*, reaching the complexity of $O(m^4n^4)$. Since the clauses are on negative Horn-SAT forms and Horn-SAT is a tractable problem with linear complexity in the size of the formula [10], the final complexity remains $O(m^4n^4)$.

6 Final comments

The characterization obtained for 2-convex polyominoes in Proposition 1 can be generalized to k-convex ones: in particular the idea of climbing up along a k-convex polyomino using k maximal internal rectangles, till reaching an extremal forbidden zone seems exactly what needed to this purpose. So, for each k, we can translate the k-convexity constraint into Horn clauses, and then solve the related reconstruction problem from two projections.

Obviously the number of such clauses (and so the computational complexity), increases till becoming exponential in the limit. However such an approach turns out to be useful once we have set un upper bound to the class of convexity to which at least one solution of the reconstruction problem belongs: in particular, given a couple of projections, we can use the algorithm in [9] to find a convex polyomino compatible with them, then we compute its level of convexity, say k, and finally run the reconstruction algorithm for k'-convex polyominoes, for each $k' < k$. This procedure allows us to define the concept of convexity level of a couple of projections.

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