Sturmian tilings of discrete planes

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Abstract

Bidimensional Sturmian words are natural extensions of Sturmian words, linked to discrete planes. This paper introduces some definitions and tools for this kind of words. We obtain an extension of the Morse-Hedlund theorem with a constructive way to tile a bidimensional Sturmian word. These tilings give some interesting tridimensional continued fraction algorithms.

Keywords: Sturmian word, bidimensional word, discret plan, tiling, continued fraction

1 Sturmian words

In combinatorics of words, Sturmian words have been extensively studied. The close link between discrete lines and these words is probably one of the main reasons of interest of these words. In [Dur98], Durant introduced the notion of return words and in [Vui01] Vuillon gave a characterization of Sturmian sequence in terms of return words. It is natural to seek analogous results for a 2-dimensional extension of Sturmian words linked to discrete planes. Priebe found a way to tile a discrete plan with seven tiles using Voronoi diagrams. After a short overview about return words and Sturmian words in this section , we introduce some tools for bidimensional words in section 2 and finally, in section 3, we give a way to extend the notion of Sturmian words in dimension 2 and to obtain a result for return words in 2-dimensional Sturmian words, which generalizes the 1-dimensional theorem of Morse-Hedlund. In the end of the section 3, we show how to obtain continued fraction algorithms from the 2-dimensional Morse-Hedlund theorem.

Let w be an infinite word, we define the complexity of w as the function $p_w : \mathbb{N} \to \mathbb{N}$ such that $p_w(n)$ is the number of distinct factors of w of length n. This function can be used to characterize some classes of words :

Theorem 1.1
An infinite word w is ultimately periodic if and only if $\exists n \in \mathbb{N}$ such that $p_w(n) \leq n$

This result is no longer true if we consider bi-infinite words like : $w = {}^{\omega}010^{\omega}$ for which $p_w(n) = n + 1 \ \forall n \in \mathbb{N}$.

Sturmian words are the non ultimately periodic words of minimal complexity ($p_w(n) = n+1$). There are many ways to characterize them. Two of them are particularly interesting for the tiling view on Sturmian words.

Definition 1.2 (Discrete line)
A discrete line of slope $\alpha > 0$, of intercept $\rho \in \mathbb{R}$ and of thickness ω is the set :

$$D_{\alpha,\rho,\omega} = \left\{(x, y) \in \mathbb{Z}^2 \mid \rho \leq \alpha y - x < \rho + \omega\right\}$$

If $\omega = 1 + \alpha$, a discrete line is a broken line and can be seen as a binary word. Such a line is called a normal discrete line.

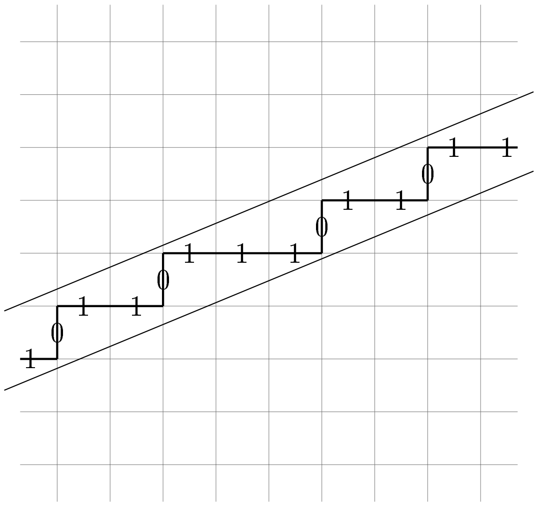

Figure 1: Example of association beetween discret line and binary word

Property 1.3
w is a Sturmian word if and only if it is associated to a normal discrete line of irrational slope.

This vision of Sturmian word allow us to construct a dynamical system which simply describes a Sturmian word :

Property 1.4 (Dynamical system of a Sturmian word)
Let w be Sturmian word associated to a discrete line $D_{\alpha,\rho,\omega}$. Let (I_0, I_1) be the following partition of $[0, 1[$:

- $I_0 = [0, 1 - \alpha[$
- $I_1 = [1 - \alpha, 1[$

and R_α be the function $R_\alpha : [0, 1[\to [0, 1[$ such that $R_\alpha(x) = x + \alpha \bmod 1$. Then

$$w_n = \begin{cases} 0 & \text{if } R_\alpha^n(\{\rho\}) \in I_0 \\ 1 & \text{if } R_\alpha^n(\{\rho\}) \in I_1 \end{cases} \quad \text{with } \{\rho\} \text{ the fractional part of } \rho$$

Vuillon in [Vui01] gave a second characterization using return words introduced by Durand in [Dur98] :

Definition 1.5 (Return word)
Let w be a finite or infinite word and u a factor of w. v is a return word of u in w if and only if v satisfies the following conditions :

1. *vu is a factor of w*
2. *u is a prefix of vu*
3. *$|vu|_u = 2$*

Example 1.6
In $w = 10000100100010100010$, the return words of 1 are : 10000, 100, 10 and 1000.

Example 1.7
In $w = 10000100100010100010$*, the return words of* 010 *are :* 010, 01 *and* 0100.

Example 1.8
In $w = 10000100100010100010$*,* 10000 *has no return word.*

Example 1.9
In $w = 100(10)^\omega$*,* 100 *has no return word.*

Remark : An infinite word can have no return word for some factors like in the last example because the factors appears only one time. To avoid this kind of situation, we can study recurrent words :

Definition 1.10 (Recurrent word)
A infinite word w *is recurrent if each factor of* w *appears infinitely often in* w*.*

The first result about return words in Sturmian words is given by Morse and Hedlund (1940) :

Theorem 1.11 (Morse-Hedlund)
If w *is a Sturmian word then there exists* $(m, n) \in \mathbb{N}^2$ *such that the return words of* 1 *are* 10^n *and* 10^{n+1} *and the return words of* 0 *are* 01^m *and* 01^{m+1}*. Further, exactly one of this two integers is zero.*

Proof : An easy study of the complexity of Sturmian word gives that 11 and 00 cannot both be factors of w. So without loss of generality, we can assume than 11 doesn't appear in w. Consequently, the return words of 0 are 0 and 01. It remains to prove that there exists $n > 0$ such that the return words of 1 are 10^n and 10^{n+1}. For that, we consider the dynamical system associated to w. Let $\rho_i = R_\alpha^i(\{\rho\})$. If $w_i = 1$ then $\rho_i \in I_1$ and $\rho_{i+1} \in R_\alpha(I_1)$. If we study the different $R_\alpha^k(I_1)$ we can deduce the return words of 1. Let n be the first first integer than $R_\alpha^{n+1}(I_1) \cap I_1 \neq$. Because 11 is not a factor of w and $|I_1| = \alpha$, $R_\alpha(I_1) \subset I_0$ so $n > 0$. Because $|I_1| = \alpha$, for all $x \in I_1$, if $R_\alpha^{n+1}(x) \notin I_1$ then $R_\alpha^{n+2}(x) \in I_1$. Therefore we can deduce that the return words of 1 are exactly 10^n and 10^{n+1}.

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If we call w_1 and w_2 the two return words of 1, we can see that I_1 is cut into two intervals : I_{w_1} and I_{w_2}.
This result can be extended to the return words of any factor of w. This implies the following characterization of Sturmian words :

Theorem 1.12 (Vuillon)
w *is a Sturmian word if and only if each factor of* w *has exactly* 2 *distinct return words.*

Moreover, Sturmian words are recurrent so for a factor v of a Sturmian word w, the two return words (v_0, v_1) tile the word w. The word obtained by the action of the substitution $\sigma(v_i) = i$ is a new Sturmian word. The aim of this article is to obtain a similar result for some bidimensional word associated to a discrete plane.

2 Bidimensional words

Sturmian words are a good modelization of discrete lines, which transforms geometrical properties into combinatorial properties. It is natural to try to obtain similar results in higher dimension. To study the discrete plane we need to use bidimensional words. In this section, we will introduce some tools and definitions that are useful for the study of bidimensional words.

Definition 2.1 (Bidimensional words)
Let $\mathcal{A}$ be an alphabet ($\{1,2,3\}$ for example), a bidimensional word $U_{(n,m)}$ is a bidimensional sequence such that :

$$\forall (n,m) \in \mathcal{D}_U,\ U_{(n,m)} \in \mathcal{A}$$

where $\mathcal{D}_U \subset \mathbb{Z}^2$ is the domain of definition of U.

This definition raises a main problem of dimension 2 : it is difficult to define a natural concatenation for finite words. The issue comes from two problems : defining the position of a factor in a word and piecing together two finite words. First, like in dimension 1, we want to have a notion of complexity. For that, we need to define a special kind of factors :

Definition 2.2 (Rectangular factor)
Let $U_{(n,m)}$ be a bidimensional word, a rectangular factor $V_{(n,m)}$ of U is a bidimensional sequence such that :

- *There exists a pair of integers (N,M) (called the size of $\mathcal{D}_V$) such that $\mathcal{D}_V = [1,N] \times [1,M]$.*
- *There exists two integers k,l such that $\forall (n,m) \in \mathcal{D}_V,\ (n+k, m+l) \in \mathcal{D}_U$ and $V_{(n,m)} = U_{(n+k,m+l)}$*

Definition 2.3 (Complexity)
Let $U_{(n,m)}$ be a bidimensional word, the complexity of U is the following function :

$$\begin{array}{rccc} p_U : & \mathbb{N}\times\mathbb{N} & \longrightarrow & \mathbb{N} \\ & (N,M) & & \#\{\text{rectangular factors of } U \text{ of size } (N,M)\} \end{array}$$

In section 1, we have shown a way to characterize periodic words with the complexity. This problem is still open in dimension 2 :

Definition 2.4 (Periodic words)
Let $U_{(n,m)}$ be a bidimensional word with domain of definition $\mathbb{Z}^2$. U is periodic if and only if

$$\exists (k,l) \in \mathbb{Z}^2 \setminus \{(0,0)\} \mid \forall (n,m) \in \mathbb{Z}^2,\ U_{(n,m)} = U_{(n+k,m+l)}$$

Rectangular factors are not enough to locally describe bidimensional words. We need to introduce the notion of pattern, and in order to answer some questions about the localization of factors in a word, we will introduce the notion of pointed pattern.

Definition 2.5 (Pattern)
Let $\mathcal{A}$ be an alphabet, a pattern on $\mathcal{A}$ is a pair $\mathcal{M} = (M, v)$ where M is a finite subset of $\mathbb{Z}^2$ and v is a map from M to $\mathcal{A}$ called valuation.

Definition 2.6 (Equivalence of pattern up to a translation)
Let $\mathcal{M} = (M, v)$ and $\mathcal{M}' = (M', v")$ be two patterns on the same alphabet

$$\mathcal{M} \sim \mathcal{M}' \Longleftrightarrow \exists t \in \mathbb{Z}^2 \mid M' = M + t \text{ and } \forall x \in M,\ v(x) = v'(x+t)$$

Property 2.7 (Classes of patterns)
$\sim$ is a equivalence relation. By $\bar{M}$, we denote the class of a motif M up to this relation.

Proof : Let $\mathcal{M}, \mathcal{M}'$ and $\mathcal{M}''$ be three patterns on the same alphabet :

- $\mathcal{M} \sim \mathcal{M}$, just take $t = (0, 0)$.
- If $\mathcal{M} \sim \mathcal{M}'$ then there exists a translation of vector t from $\mathcal{M}$ to $\mathcal{M}'$. So there exists a translation of vector $-t$ from $\mathcal{M}'$ to $\mathcal{M}$: $\mathcal{M}' \sim \mathcal{M}$
- If $\mathcal{M} \sim \mathcal{M}'$ and $\mathcal{M}' \sim M''$ then there exists a translation of vector t from $\mathcal{M}$ to $\mathcal{M}'$ and a translation of vector t' from $\mathcal{M}'$ to $\mathcal{M}''$. So there exists a translation of vector $t + t'$ from $\mathcal{M}$ to $\mathcal{M}''$: $\mathcal{M} \sim \mathcal{M}''$.

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We can define classes of patterns under this relation :

Definition 2.8 (Sub-pattern)
Let $\mathcal{M}$ and $\mathcal{M}'$ be two patterns, $\mathcal{M}'$ is a sub-pattern of $\mathcal{M}$ if and only if :

- $M' \subset M$
- $\forall (i, j) \in M',\ v(i, j) = v'(i, j)$

This notion can be extended to classes of patterns.

Definition 2.9 (Size of a pattern)
Let $\mathcal{M}$ be a pattern, the size of $\mathcal{M}$ is the value of $|M|$.

To locate a pattern in a word, we need to chose one of the points of the pattern and locate the pattern by the coordinates of this special point.

Definition 2.10 (Pointed pattern)
Let $\mathcal{A}$ be an alphabet, a pointed pattern on $\mathcal{A}$ is a triple $\mathcal{M} = (M, v, p)$ where (M, v) is a pattern and $p \in \mathbb{Z}^2$ is called the pointer of $\mathcal{M}$.

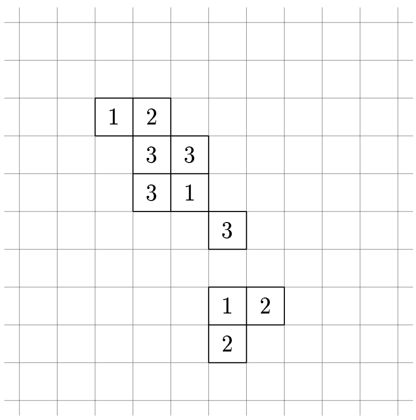

Figure 2: Example of pattern of size 10

It is important to notice that we don't require the pointer to be in M. As with classes of patterns, we can define classes of pointed patterns with the following relation :

Definition 2.11 (Equivalence of pointed patterns)
Let $\mathcal{M} = (M, v, p)$ and $\mathcal{M}' = (M', v', p')$ be two pointed patterns on the same alphabet

$$\mathcal{M} \sim \mathcal{M}' \iff \exists t \in \mathbb{Z}^2 \mid M' = M + t, \forall x \in M,\ v(x) = v'(x+t) \text{ and } p' = p + t$$

Property 2.12 (Canonical representative)
If $\mathcal{M}$ is a pointed pattern then there is an unique pointed pattern $\mathcal{M}_0$ such that :

- $\mathcal{M} \sim \mathcal{M}_0$
- $p_0 = (0, 0)$

This pattern is called the canonical representative of the equivalence class of $\mathcal{M}$.

Proof : Let $\mathcal{M}$ be a pointed pattern, the canonical representative of the equivalence class of $\mathcal{M}$ is just the translate of $\mathcal{M}$ by the vector $-p$.

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Sometimes we will need to change the way to localize a pattern, so we will need to change the position of the pointer :

Definition 2.13 (Change of pointer)
Let $\mathcal{M}$ and $\mathcal{M}'$ be two pointed patterns. $\mathcal{M}$ and $\mathcal{M}'$ are equal up to a change of pointer if and only if $(M, v) = (M', v')$. We also say that the classes of $\mathcal{M}$ and $\mathcal{M}'$ are equal up to a change of pointer.

Definition 2.14 (Factor)
Let $\mathcal{M}$ be a pointed pattern and $U_{(n,m)}$ a bidimensional word. $\mathcal{M}$ is a factor of $U_{(n,m)}$ if and only if there is a element $\mathcal{M}'$ in the class of $\mathcal{M}$ such that :

$$\forall (i,j) \in M'\ v'(i,j) = U_{(i,j)}$$

In dimension 1, factors are required by definition to be connected. We will need a similar notion in dimension 2 :

Definition 2.15 ($8-$connectivity)
Let $\mathcal{M}$ be a pattern. $\mathcal{M}$ is $8-$connected if and only if the following graph $G = (V, E)$ is connected :

- *V is indexed by the elements of M*
- *Two vertices (n, m) and (k, l) are linked if and only if* $\max(|n-k|, |m-l|) = 1$

When we study the positions of a factor in a word, it can be useful to introduce a binary word which localizes the different iterations of this factor :

Definition 2.16 (Set of positions)
Let U be a bidimensional word and $\mathcal{M}$ be a factor of U. The set of positions of $\mathcal{M}$ in U is

$$\mathcal{P}(\mathcal{M}) = \{p' \mid \mathcal{M}' = (M', v', p') \sim \mathcal{M} \text{ and } \forall (n,m) \in M',\ v'(n,m) = U_{(n,m)}\}$$

Definition 2.17 (Localization word)
Let U be a bidimensional word and $\mathcal{M}$ be a factor of U. The localization word of $\mathcal{M}$ in U is the bidimensional binary word $\mathcal{L}(\mathcal{M})$ defined by :

$$\mathcal{L}(\mathcal{M})_{(n,m)} = \begin{cases} 1 & \text{if } (n,m) \in \mathcal{P}(\mathcal{M}) \\ 0 & \text{else} \end{cases}$$

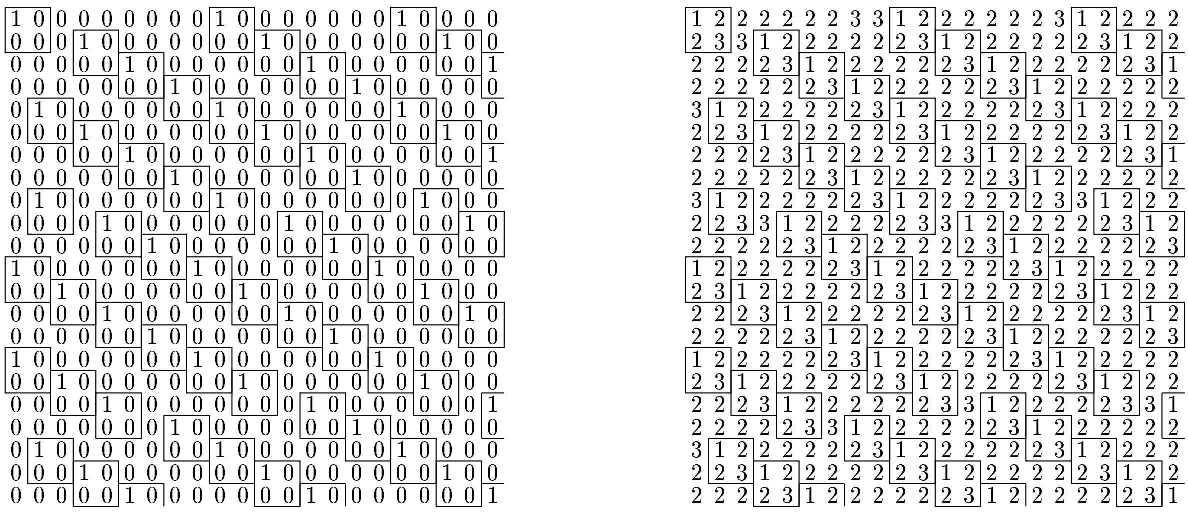

Figure 3: Localization word (left) for the pattern $\begin{matrix} 1 & 2 \\ 2 & 3 \end{matrix}$ of a bidimensional word (right).

In section 3, we will extract lines of a bidimensional word in order to study it like a unidimensional word. For that, we need to introduce directional words :

Definition 2.18 (Directional word)
Let U be a bidimensional word, $(i, j) \in \mathbb{Z}^2$ and $(z_1, z_2) \in \mathbb{Z}^2$. The word of direction (z_1, z_2) passing through (i, j) is the unidimensional word V such that :

$$V_n = U_{(i+nz_1, j+nz_2)}$$

A unidimensional word W is called a word of U of direction (z_1, z_2) if and only if there exists a point (i, j) such that V is the word of direction (z_1, z_2) passing through (i, j).

In section 1, we have seen that return words tile a sturmian word. The aim of the section 3 is to obtain a similar result in dimension 2. But we need a good definition of tiling in bidimensional words and a good definition of return words :

Definition 2.19 (Set of tiles and tiling)
Let U be a bidimensional word. A set of tiles $\bar{T} = \{\bar{T}_1 \dots \bar{T}_n\}$ is a finite set of class of patterns. We say $\bar{T}$ tiles U if there exists a infinite set of pattern T which verifies :

1. $\forall \mathcal{M} \in T, \exists i \in [1, n] \mid \mathcal{M}$ *is a representative of* T_i
2. $\forall i \in [1, n],\ \exists \mathcal{M} \in T \mid \mathcal{M}$ *is a representative of* T_i
3. $\mathcal{D}_U = \bigsqcup_{\mathcal{M} \in T} M$

4. $\forall x \in \mathcal{D}_U, \forall \mathcal{M} \in T,$ *if* $x \in M$ *then* $v(x) = U_x$

The conditions 1 and 2 ensure that $\bar{T}$ is a minimal set of tiles. The condition 3 ensure that T tiles the plan and the last condition assure that this tiling is synchronized with the bidimensional word.

Definition 2.20 (Parallelogram)
Let U be a bidimensional word and $\mathcal{M}$ be a factor of U. $\mathcal{M}$ is a parallelogram if and only if there exists a parallelogram $ABCD$ in $\mathbb{R}^2$ and $E \subset \{[AB],[BC][CD][DA]\}$ such that :

- *Each vertex of $ABCD$ is in $\mathbb{Z}^2$*
- $(n,m) \in M \Leftrightarrow (n,m) \in \overline{ABCD} \setminus E$

In this definition, E accounts for the possibility to exclude some edges of the parallelogram (to avoid overlapping problems in the tiling) as in the following examples :

Definition 2.21 (Return word)
Let U be a bidimensional word and $\mathcal{M}$ be a factor of U. A parallelogram factor of U : $\mathcal{N}$ associated to the parallelogram $ABCD$ is a return word of $\mathcal{M}$ in U if and only if

- $\forall (n,m) \in \{A,B,C,D\}, \mathcal{L}(\mathcal{M})_{(n,m)} = 1$
- $\forall (n,m) \in N \setminus \{A,B,C,D\}, \mathcal{L}(\mathcal{M})_{(n,m)} = 0$

In the following section, we will show that we can tile some special bidimensional words obtained by the projection of discrete planes with exactly 3 return words. However, the parallelogram tiles obtained, have some specific aspects, there are flat :

Definition 2.22 (Flat tile)
Let $\mathcal{M}$ be a tile of a bidimensional word U. $\mathcal{M}$ is flat if and only if there is a directional word V of U such that $\mathcal{M}$ is a factor of V.

3 Discrete plane and tiling

In this section, we extend the notion of Sturmian words by studying bidimensional words associated to discrete planes. After that, we see how to obtain a tiling of these words, with, in many cases, exactly three flat return words.

Definition 3.1 (Discrete plane)
Let S be the set of all integer translates of the unit cube which intersect the half-space defined by :

$$ax + by + cz < 0$$

The border of S is the (upper) discrete plane P of equation $ax + by + cz = 0$.

Remark : It is possible to define discrete plane with a formula similar to the one used to define discrete lines. However, This convention is more often used in the context of dynamical systems.
Let π be the projection onto the plane $x+y+z=0$ along the direction $(1,1,1)$. Let (e_1, e_2, e_3) be the canonical base of the lattice $\mathbb{Z}^3$ then :

$$\pi(p,q,r) = (p-r)\pi(e_1) + (q-r)\pi(e_2)$$

Therefore, π send the lattice $\mathbb{Z}^3$ to a the lattice $\mathbb{Z}\pi(e_1) + \mathbb{Z}\pi(e_2)$. It is possible to show that the restriction of π to P is a bijection. The projection of P gives a tiling of the plane with 3 tiles. Then we can associated with any point of $\mathbb{Z}\pi(e_1) + \mathbb{Z}\pi(e_2)$ this preimage in P. With the following rule we can associated with any point of P one of the tiles. Consequently we can create and bidimensional word U such that $U_{(n,m)}$ is the type of the tile associated to the preimage of the point $n\pi(e_1) + m\pi(e_2)$.

Property 3.2 (Association tile/point)
Let $\vec{v}$ be a vector of $\mathbb{Z}^2$ independent of directions of the edges of the tiling. There exists a bijection between tiles and vertices by associate with any vertex the first tiles see in the direction $\vec{v}$.

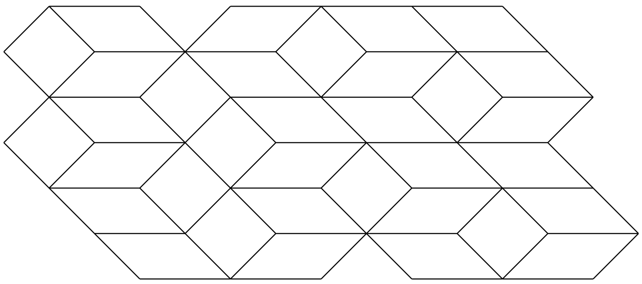

Figure 4: Local view of the projection of a discrete plane

Theorem 3.3 (Bidimensional word of discrete plane)
The bidimensional word U associated to the discret plan P of equation $ax+by+cz=0$ fulfill :

$$\begin{array}{lll} U_{(n,m)} = 1 & \text{if} & an + bm \bmod (a+b+c) \in [0, a[\\ U_{(n,m)} = 2 & \text{if} & an + bm \bmod (a+b+c) \in [a, a+b[\\ U_{(n,m)} = 3 & \text{if} & an + bm \bmod (a+b+c) \in [a+b, a+b+c[\end{array}$$

To simplify the study, we can normalize the plan by choosing an equation of the plan with $c = 1$. The word obtained is associated to the following dynamical system :

Property 3.4 (Dynamical system associated to a discrete plane)
Let P be a discrete plane of equation $ax+by+z=0$ and $U_{(n,m)}$ be the associated bidimensional word . Let (I_1, I_2, I_3) be the partition of $[0,1[$ such that :

- $I_1 = [0, \alpha[$ *with* $\alpha = \dfrac{1}{1+a+b}$
- $I_2 = [\alpha, \alpha+\beta[$ *with* $\beta = \dfrac{b}{1+a+b}$
- $I_3 = [\alpha+\beta, 1[$

Then

$$U_{(n,m)} = i \Longleftrightarrow \rho(n,m) = n\alpha + m\beta \bmod 1 \in I_i$$

Up to some elementary change of coordinates, we may assume $|I_1| \leq [I_2| \leq |I_3|$. We will make this assumption throughout the rest of the article.

Property 3.5 (Rotations)
Let $U_{(n,m)}$ be a bidimensional word associated to a discrete plane P. Let x be in $[0,1[$ and R_x be the following function :

$$R_x : \begin{array}{ccc} [0,1[& \longrightarrow & [0,1[\\ y & & y + x \ mod \ 1 \end{array}$$

R_x is called the $x-$rotation on $[0,1[$ and we have :

- $\rho(n+1,m) = R_\alpha(\rho(n,m))$
- $\rho(n,m+1) = R_\beta(\rho(n,m))$

Proof : By definition,

$$\rho(n+1,m) = (n+1)\alpha + m\beta \text{ mod } 1 = \rho(n,m) + \alpha \text{ mod } 1 = R_\alpha(\rho(n,m))$$

$$\rho(n,m+1) = n\alpha + (m+1)\beta \text{ mod } 1 = \rho(n,m) + \beta \text{ mod } 1 = R_\beta(\rho(n,m))$$

■

The two previous properties show that a bidimensional word associated to a discrete plane can be represented by the partition (I_1, I_2, I_3) and the two rotations R_α and R_β. The letter $U_{(n,m)}$ is given by the position of the point $\rho(n,m) = R_\alpha^n \circ R_\beta^m(c)$ (according to the previous notations).
In dimension 1, periodic discrete lines (and words) are associated to rational angles. The corresponding propriety in dimension 2 is the following :

Property 3.6 (Periodicity and discrete planes)
Let P be a discrete plane and U be the bidimensional word associated to P.

$$U \text{ is periodic } \Longleftrightarrow 1, \alpha, \beta \text{ are rationally dependent.}$$

Proof : U is periodic $\Leftrightarrow \exists (k,l) \in \mathbb{Z}^2 \mid \forall (n,m) \rho(n+k, m+l) = \rho(n,m)$. Thus,

$$U \text{ is periodic } \Longleftrightarrow \forall (n,m)(n+k)\alpha + (m+l)\beta = n\alpha + m\beta \text{ mod } 1 \Longleftrightarrow k\alpha + l\beta = 0 \text{ mod } 1.$$

■

Definition 3.7 (Bidimensional Sturmian words)
A non-periodic bidimensional word associated to a discrete plane is called a bidimensional Sturmian word.

Property 3.8 (Complexity of bidimensional Sturmian words)
If U is a bidimensional Sturmian word then :

$$p_U(n,m) = nm + n + m.$$

In the case of Sturmian words in dimension 1, using the dynamical system it is possible to associate a interval of $[0,1[$ to a factor. In dimension 2, the same result is possible but we need to be more careful.

Property 3.9 (Set of points associated to a pattern)
Let $\mathcal{M}$ be a factor of a bidimensional Sturmian word U and $\mathcal{M}_0$ be its canonical representative. The set of points associated to $\mathcal{M}$ is the set of elements $x \in [0,1[$ such that :

$$\forall (i,j) \in M_0, \ R_\alpha^i \circ R_\beta^j(x) \in I_{v_0(i,j)}$$

This set is noted $I_\mathcal{M}$. It corresponds exactly to the set of situations, in the dynamical system, which are associated to a pointer of a representative of $\mathcal{M}$.

Proof : (n,m) is the position of a pointer of a representative of $\mathcal{M}$. It is equivalent to :

$$\forall (i,j) \in M_0,\ U_{(n+i,m+j)} \in v_0(i,j)$$

Therefore :

$$x \in I_{\mathcal{M}} \Leftrightarrow \forall (i,j) \in M_0,\ x + \rho(n+i, m+j) \in I_{v_0(i,j)}$$

which can be rewritten as :

$$x \in I_{\mathcal{M}} \Leftrightarrow \forall (i,j) \in M_0,\ R_\alpha^i \circ R_\beta^j(x) \in I_{v_0(i,j)}$$

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Property 3.10 (Sub-pattern and $I_{\mathcal{M}}$)
If $\mathcal{M}$ is a factor of a bidimensional word U and $\mathcal{M}'$ is a sub-pattern of $\mathcal{M}$ then $I_{\mathcal{M}} \subset I_{\mathcal{M}'}$.

Proof : $\mathcal{M}'$ is a sub-pattern of $\mathcal{M}$ so $M'_0 \subset M_0$ and $v'_0 = v_0|_{M'_0}$

$$x \in I_{\mathcal{M}} \Rightarrow \forall (i,j) \in M_0,\ R_\alpha^i \circ R_\beta^j(x) \in I_{v_0(i,j)} \Rightarrow \forall (i,j) \in M'_0 \subset M_0,\ R_\alpha^i \circ R_\beta^j(x) \in I_{v'_0(i,j)}$$

So

$$x \in I_{\mathcal{M}} \Rightarrow x \in I'_{\mathcal{M}}$$

■

Property 3.11 (Change of pointer and $I_{\mathcal{M}}$)
If $\mathcal{M}$ and $\mathcal{M}'$ is two factors of a bidimensional word that are equal up to a change of pointer and $p' - p = (a,b)$ then

$$I_{\mathcal{M}'} = \{x + t \text{ mod } 1\ |x \in I_{\mathcal{M}}\} \text{ with } t = a\alpha + b\beta$$

($I_{\mathcal{M}'}$ is the translate of $I_{\mathcal{M}}$ by t).

Proof : By definition, $M'_0 = M_0 - t$ and $\forall x \in M_0, v_0(x) = v'_0(x-t)$. Let $t = (t_i, t_j)$

$$\begin{aligned}
x \in I_{\mathcal{M}} &\Leftrightarrow \forall (i,j) \in M_0,\ R_\alpha^i \circ R_\beta^j(x) \in I_{v_0(y)} \\
&\Leftrightarrow \forall (i,j) \in M_0,\ R_\alpha^{i-t_i} \circ R_\beta^{j-t_j}(x+t) \in I_{v_0(i,j)} \\
&\Leftrightarrow \forall (i',j') \in M'_0,\ R_\alpha^{i'} \circ R_\beta^{j'}(x+t) \in I_{v_0(i'-t_i,j'-t_j)} \\
&\Leftrightarrow \forall (i',j') \in M'_0,\ R_\alpha^{i'} \circ R_\beta^{j'}(x+t) \in I_{v'_0(i',j')} \\
&\Leftrightarrow x + t \in I_{\mathcal{M}'}
\end{aligned}$$

■

We will show that if we impose some connectivity conditions we can assume that $I_{\mathcal{M}}$ is a kind of interval :

Definition 3.12 (Interval in $[0,1[$)
Let U be the unit circle and p the natural projection of U in $[0,1[$. $I \subset [0,1[$ is an interval if and only if $p^{-1}(I)$ is connected. In other word, I is either of the form $|a,b|$ or $[0,a| \cup |b,1[$ with $a \leq b$ and $| \in \{[,]\}$

Example 3.13
$[0,0.5[\cup[0.7,1[$ *is an interval but* $[0.1,0.2[\cup[0.3,1[$ *isn't.*

Definition 3.14 (Size of an interval)
Let I be an interval of $[0,1[$. The size of I is defined as :

- *if $I = |a,b|$ then $|I| = b - a$*
- *if $I = [0,a| \cup |b,1|$ then $|I| = 1 - b + a$*

Property 3.15 (Intersection of intervals)
Let I and J be two intervals of $[0,1[$. The intersection $I \cap J$ is either an interval or the union of two intervals of $[0,1[$.
If $|I| + |J| < 1$ then $I \cap J$ is an interval and $|I \cap J| \leq \min(|I|,|J|)$.
If $\bigcap_{i\in[1,n]} I_i$ is an interval with $|I_1| < 1$ then $\forall I' \subset I_1, \bigcap_{i\in[2,n]} I_i \cap I'$ is an interval of $[0,1[$.

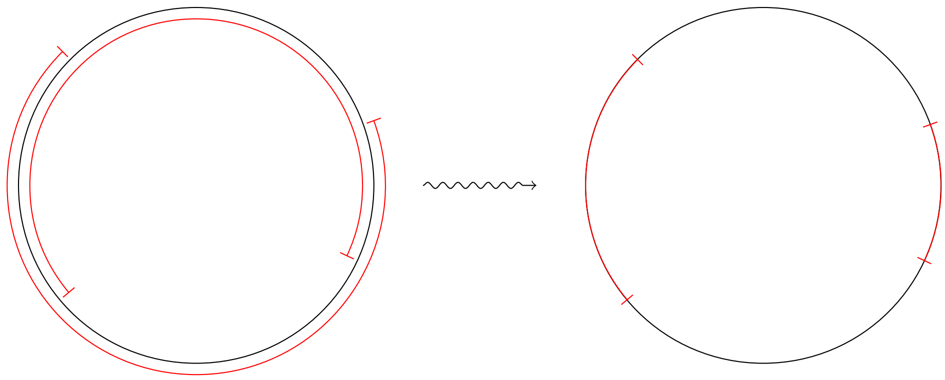


Figure 5: Intersection of two big intervals

Property 3.16 (Description of $I_{\mathcal{M}}$)
If $\mathcal{M}$ is a pointed pattern and $\mathcal{M}_0$ is its canonical representative then :

$$I_{\mathcal{M}} = \bigcap_{(i,j)\in M_0} R_\alpha^{-i} \circ R_\beta^{-j}(I_{v_0(i,j)})$$

with $I_1 = [0,\alpha[$, $I_2 = [\alpha, \alpha+\beta[$ and $I_3 = [\alpha+\beta, 1[$.

The previous property allows us to prove that $I_{\mathcal{M}}$ is at least the union of intervals of $[0,1[$ and there are cases in which $I_{\mathcal{M}}$ is not a single interval. But as we required factors to be connected in dimension 1, we will require connectivity conditions in order to restrict the number of intervals in $I_{\mathcal{M}}$.

Property 3.17 (Connectivity conditions)
If $|I_3| < 1/2$ then for every pointed pattern $\mathcal{M}$, $I_{\mathcal{M}}$ is an interval and $|I_{\mathcal{M}}| < 1/2$.
If $|I_3| > 1/2$ then for every 8-connected pointed pattern $\mathcal{M}$, $I_{\mathcal{M}}$ is an interval and $|I_{\mathcal{M}}| \leq |I_3|$.

Proof : If $|I_3| < 1/2$ then the size of each interval appearing in the formula of $I_{\mathcal{M}}$ given in the previous property is smaller than $1/2$. We can induce the result : if $|M_0| = 1$ the result is obvious. Let (a,b) be an element of M_0 :

$$I_{\mathcal{M}} = \bigcap_{(i,j)\in M_0\setminus\{(a,b)\}} R_\alpha^{-i} \circ R_\beta^{-j}(I_{v_0(i,j)}) \cap R_\alpha^{-a} \circ R_\beta^{-b}(I_{v_0(a,b)})$$

By induction $\bigcap_{(i,j)\in M_0\setminus\{(a,b)\}} R_\alpha^{-i} \circ R_\beta^{-j}(I_{v_0(i,j)})$ is an interval of size is smaller than $1/2$. Using the first part of the property 3.15 ,we obtain the result.

If $|I_3| > 1/2$. By the same kind of induction, we can show that if the pattern doesn't consist merely of 3's, $I_{\mathcal{M}}$ is an interval. The only problematic case is when $\forall (i,j) \in M_0, v_0(i,j) = 3$. We will make an induction on the number of elements of M_0. If $|M_0| = 1$, the result is obvious. If $|M_0| > 1$, the we can define the following graph $G = (V, E)$:

- The vertices are labeled by the element of M_0
- $(i,j)(k,l) \in E \Leftrightarrow \max(|i-k|, |j-l|) = 1$

G is connected because $\mathcal{M}$ is 8-connected, so it have a spanning tree T. Let (a,b) be a leaf of this tree and (k,l) its parent, $G \setminus (a,b)$ is still connected because $T \setminus (a,b)$ is one of its spanning trees. So $\mathcal{M}_0 \setminus (a,b)$ is still 8-connected. By induction $\bigcap_{(i,j)\in M_0 \setminus \{(a,b)\}} R_\alpha^{-i} \circ R_\beta^{-j}(I_{v_0(i,j)})$ is an interval. Because (k,l) is a parent of (a,b), we have

$$\max(|k-a|, |l-b|) = 1$$

so we can prove that $R_\alpha^{-a} \circ R_\beta^{-b}(I_{v_0(i,j)} \cap R_\alpha^{-k} \circ R_\beta^{-l}(I_{v_0(i,j)}$ by studying only 8 different cases :

$$(k,l) \in \Big\{(a-1,b-1),(a-1,b),(a-1,b+1),(a,b-1),(a,b),(a,b+1),(a+1,b-1),(a+1,b),(a+1,b+1)\Big\}$$

For each case, by a simple calculation we prove that this intersection is an sub-interval of $R_\alpha^{-k} \circ R_\beta^{-l}(I_{v_0(i,j)})$.
To obtain $I_{\mathcal{M}}$, we want to replace $R_\alpha^{-k} \circ R_\beta^{-l}(I_{v_0(i,j)})$ by this sub-interval in the formula :

$$\bigcap_{(i,j)\in M_0 \setminus \{(a,b)\}} R_\alpha^{-i} \circ R_\beta^{-j}(I_{v_0(i,j)})$$

But $|R_\alpha^{-k} \circ R_\beta^{-l}(I_{v_0(i,j)}| < 1$, we can use the second part of the property 3.15 to obtain the result.

■

In the rest of the article, we will always assume that one of the connectivity conditions is fulfilled.
According to the connectivity conditions, we can define the size of $I_{\mathcal{M}}$.
In the rest of the article, we will use the notation $z_1\alpha + z_2\beta$ as the (z_1, z_2) direction, like in directional words for example.

Property 3.18 (Sturmian direction)
If $\mathcal{M}$ is a factor of a bidimensional sturmian word U then there exists a unique triple $(n, z_1, z_2) \in \mathbb{N} \times \mathbb{Z}^2$ such that :

- $|I_{\mathcal{M}}| = |n - (z_1\alpha + z_2\beta)|$
- $z_1\alpha + z_2\beta > 0$

$d_{\mathcal{M}} = z_1\alpha + z_2\beta$ is called the Sturmian direction associated to $\mathcal{M}$.

Proof : According to the property 3.16, we can describe $I_{\mathcal{M}}$ like the intersection of intervals of $[0,1[$. We will show that each of these intervals have boundaries in the form $m - a\alpha - b\beta$ with $(m,a,b) \in \mathbb{Z}^3$. After that, we will prove that the intersection of an union of intervals of $[0,1[$ with an intervals of $[0,1[$ is an union of intervals of $[0,1[$ with boundaries in the form

$m - a\alpha - b\beta$ with $(m, a, b) \in \mathbb{Z}^3$.

$$I_{\mathcal{M}} = \bigcap_{(i,j)\in M_0} R_\alpha^{-i} \circ R_\beta^{-j}(I_{v_0(i,j)})$$

For each $(i, j) \in M_0$, $(I_{v_0(i,j)}$ has its boundaries in $\{0, 1, \alpha, \alpha + \beta\}$. Thus $R_\alpha^{-i} \circ R_\beta^{-j}(I_{v_0(i,j)})$ has this boundaries in $\{0, 1, -i\alpha - j\beta, 1 - i\alpha - j\beta, (1 - i)\alpha - j\beta, (1 - i)\alpha + (1 - j)\beta\}$.
Let I and J be two intervals of $\mathbb{R}$ include in $[0, 1[$ with boundaries in the form $m + a\alpha + b\beta$ with $(m, a, b) \in \mathbb{Z}^3$. Then $I \cap J$ is an intervals of $\mathbb{R}$ include in $[0, 1[$ with boundaries in the same form.
Let $\bar{I}$ be an union of intervals of $[0, 1[$ and $\bar{J}$ be an intervals of $[0, 1[$. Then $\bar{I}$ can be see like an union of intervals of $\mathbb{R}$ include in $[0, 1[$ and $\bar{J}$ like an union of two intervals of $\mathbb{R}$.

$$\bar{I} = \bigcup_{i\in[1,n]} I_i \quad \bar{J} = J_1 \cup J_2$$

Therefore

$$\bar{I} \cap \bar{J} = \bigcup_{i\in[1,n],\ j\in[1,2]} I_i \cap J_j$$

Consequently $\bar{I} \cap \bar{J}$ is an union of intervals of $[0, 1[$ with boundaries in the desired form. By induction, $I_{\mathcal{M}}$ is an union of intervals of $[0, 1[$ with boundaries in the desired form but with the connectivity conditions, we can assume than $I_{\mathcal{M}}$ is an interval with boundaries in the form $m - a\alpha - b\beta$. Therefore the size of $I_{\mathcal{M}}$ is in the desired form.

■

Property 3.19 (Sturmian direction)
Let $\mathcal{M}$ be a factor of a bidimensional Sturmian word U, $\mathcal{L}(\mathcal{M})$ be the localization word of $\mathcal{M}$ in U, $z_1\alpha + z_2\beta$ be the Sturmian direction associated to $\mathcal{M}$. Then :

- *If V is a word of $\mathcal{L}(\mathcal{M})$ of direction $z_1\alpha + z_2\beta$, then V is Sturmian.*
- *If V and W are two words of $\mathcal{L}(\mathcal{M})$ of direction $z_1\alpha + z_2\beta$, then there are associated to two discrete lines with the same angle.*

Proof : To prove this result, we just need to notice that V is associated to the dynamical system linked to the discrete line of slope $|I_{\mathcal{M}}|$.

■

Remark : This Sturmian direction is the only direction which fulfills these two conditions.
With this result, we show that the study of the localization word is close to the study of a unidimensional sturmian word. However, sometimes the Sturmian direction cannot be seen as a real direction. In fact, in the case of a Sturmian direction such as $2\alpha + 2\beta$, we obtain two words nested in the same "line". The natural direction must be $\alpha + \beta$. In such cases, the study of the localization word is much harder.

Definition 3.20 (Primality)
Let $\mathcal{M}$ be a factor of a bidimensional Sturmian word and $z_1\alpha + z_2\beta$ the Sturmian direction associated to $\mathcal{M}$. $\mathcal{M}$ is a prime factor of U if and only if $z_1 \wedge z_2 = 1$.

Afterwards we will consider a bidimensional Sturmian word U, a prime factor $\mathcal{M}$ with a Sturmian direction $d_{\mathcal{M}}$. In this direction, 1 have exactly two return words in $\mathcal{L}(\mathcal{M})$: w_1, w_2. We can assume than w_1^2 does not appear. We will construct a tiling of $\mathcal{L}(\mathcal{M})$ with three flat tiles, one with w_1 inside and the two others with w_2 inside as in the following picture :

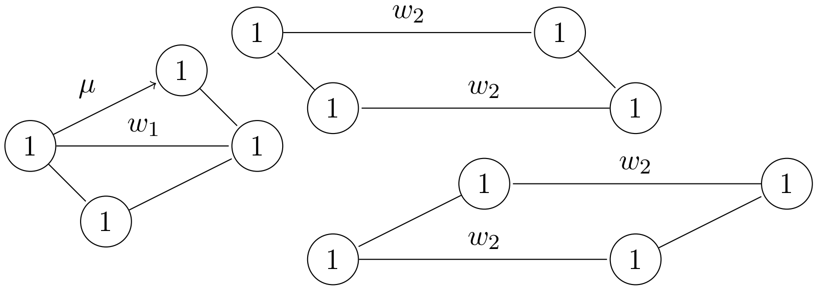


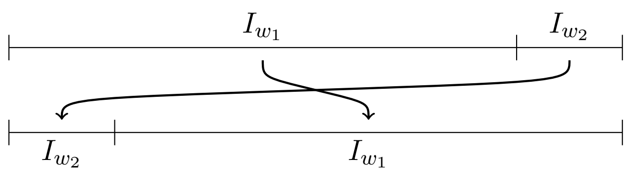
Figure 6: Aspect of the flat tiling

Theorem 3.21 (Flat tiling of $\mathcal{L}(\mathcal{M})$)
It is possible to tile $\mathcal{L}(\mathcal{M})$ with three flat tiles of direction $d_{\mathcal{M}}$.

Proof : To prove this result, we only need to find a direction μ which fulfills the following conditions :

1. The first tile is flat
2. A 1 which starts w_1 is send by μ to an other 1 of $\mathcal{L}(\mathcal{M})$
3. A 1 which ends w_1 is send by $-\mu$ to an other 1 of $\mathcal{L}(\mathcal{M})$

Let (I_{w_1}, I_{w_2}) be the partition of $I_{\mathcal{M}}$ given by w_1 and w_2 like in the Morse-Hedlund theorem in the first section.
Condition 1 is easily obtained by taking a Bezout vector u and taking $\mu \in \{u + kd_{\mathcal{M}}\}$.
Condition 2 is satisfied by choosing μ such that $|\mu| < |I_{\mathcal{M}}| - |I_{w_1}|$. It is possible because $\{u + kd_{\mathcal{M}}\}$ is dense in $[0, 1[$ and $|I_{\mathcal{M}}| - |I_{w_1}| > 0$. Condition 3 can be reformulated thanks to a classical result on Sturmian words. If we call f the function from $I_{\mathcal{M}}$ to $I_{\mathcal{M}}$ which sends a point associated to a 1 to the point associated to the next 1 in the direction $d_{\mathcal{M}}$, then f is an exchange of the intervals I_{w_1} and I_{w_2} :



So the last condition is equivalent to : $-\mu$ sends a point of $f(I_{w_1})$ to a point of $I_{\mathcal{M}}$, which is satisfied if $|\mu| < |I_{\mathcal{M}}| - |I_{w_1}|$, which is exactly the same condition than in 2.

■

If we transpose directly this tiling to U, we obtain a tiling of U with 4 tiles. To obtain a tiling of U with exactly 3 tiles we need to change the position of the pointer :

Theorem 3.22 (**3 return words**)
If $\mathcal{M}$ be a prime factor of U then, up to a change of pointer, $\mathcal{M}$ have exactly 3 returns words in the previous tiling.

Proof : With a change of pointer, we can assume than the upper bound of $I_{\mathcal{M}}$ is 1. If we look to the orbit of $I_{\mathcal{M}}$ by the rotation $R_{d_{\mathcal{M}}}$, we can see that $I_{\mathcal{M}}$ will be cut into 4 intervals by the bounds of I_1, I_2 and $I_{\mathcal{M}}$.
To reduce the number of intervals, we will just translate $I_{\mathcal{M}}$ to put it in the first place in the reversed orbit of $I_{\mathcal{M}}$ by the rotation $R_{d_{\mathcal{M}}}$ where he meet the upper bound of I_2. Then by construction we obtain 3 return words for $I_{\mathcal{M}}$ in the direction $d_{\mathcal{M}}$ of the form :

$$\begin{array}{llllll} p & 3\ldots3 & 1\ldots1 & 1 & 2\ldots2 & \\ p & 3\ldots3 & 1\ldots1 & 2 & 2\ldots2 & \\ p & 3\ldots3 & 1\ldots1 & 2 & 2\ldots2 & 2 \end{array}$$

or :

$$\begin{array}{llllll} p & 3\ldots3 & 1\ldots1 & 1 & 2\ldots2 & \\ p & 3\ldots3 & 1\ldots1 & 1 & 2\ldots2 & 2 \\ p & 3\ldots3 & 1\ldots1 & 2 & 2\ldots2 & 2 \end{array}$$

where p represents the pointer. We use the position of the pointer to hide one of the cut.

■

However, there are no synchronization between the 3 tiles in the tiling of $\mathcal{L}(\mathcal{M})$ and this three return words. We obtain two different ways to recode the Sturmian bidimensional word. The first way is given by the tiling of $\mathcal{L}(\mathcal{M})$ and is linked to one of the two following dynamical systems :

The second way is given by the three return words and is linked to the following dynamical system (up to an interversion of boundaries) :

with $x \in \{\alpha, \beta, 1-(\alpha+\beta)\}$ such that $|I_{\mathcal{M}}| \neq x$.
Both of them are associated to a discrete plan but to switch from one to the other, we need to make a distortion of the tilings.
However, in some cases, it is possible to match the two dynamical system.

Theorem 3.23 (Morse-Hedlund in dimension 2)
For a pattern associated to 1, 2 *or* 3, *the flat tilling of $\mathcal{L}(\mathcal{M})$ gives a tiling of U with exactly* 3 *tiles.*

Proof : In these specific cases, we have synchronization between the two dynamical systems. We respectively obtain the following dynamical systems (up to an interversion of boundaries) :

The values of μ are respectively : $\beta - \lfloor\frac{\beta}{\alpha}\rfloor\alpha$, $1-\alpha-\beta-\lfloor\frac{1-\alpha-\beta}{\beta}\rfloor\beta$ and α.
It is interesting to remark that in these cases, the "trick" used to obtain less return word by moving the pointer is useless because the direct translation of the tiling in $\mathcal{L}(\mathcal{M})$ give only three return words instead of four.
If we don't use normalized plans and normalized dynamical system, we obtain the following inductions (continued fraction algorithms) :

Property 3.24 (Induction by 1**)**
Let P be the discret plan of equation $ax+by+cz=0$. The tiling obtained by taking the pattern restricted to the isolated letter, we obtain a new discret plan P' of equation $a'x+b'y+c'z=0$ with :

- *if $(b+c)$ mod a + b mod $a < a$ then*

$$\begin{aligned} a' &= (b+c) \ mod \ a \\ b' &= b \ mod \ a \\ c' &= a - a' - b' \end{aligned}$$

- *else*

$$\begin{aligned} a' &= a - \Big((b+c) \ mod \ a\Big) \\ b' &= a - (b \ mod \ a) \\ c' &= a - a' - b' \end{aligned}$$

Proof : The normalized dynamical system obtained is (up to an interversion of boundaries) :

with any $k=\lfloor\frac{\beta}{\alpha}\rfloor$. We can deduce by a homothety of a that the the linear dynamical system is (up to an interversion of boundaries) :

with

$$u = a + b + c - \lfloor \frac{a+b+c}{a} \rfloor a$$

and

$$v = a - \{\beta - \lfloor \frac{\beta}{\alpha} \rfloor \alpha\}(a + b + c)$$

We obtain :

$$v = a + \lfloor \frac{b}{a} \rfloor - b$$

We can deduce the induction. The two cases come from the possible intervesion of u and v.

■

By the same kind of demonstration, we can obtain the two following results :

Property 3.25 (Induction by 2)
Let P be the discret plan of equation $ax + by + cz = 0$. The tiling obtained by taking the pattern restricted to letter 2 , we obtain a new discret plan P' of equation $a'x + b'y + c'z = 0$ with :

- *if $(a + c) \bmod b + c \bmod b < b$ then*

$$\begin{aligned} a' &= (a + c) \bmod b \\ b' &= c \bmod b \\ c' &= b - a' - b' \end{aligned}$$

- *else*

$$\begin{aligned} a' &= b - \Big((a + c) \bmod b\Big) \\ b' &= b - (c \bmod b) \\ c' &= b - a' - b' \end{aligned}$$

Property 3.26 (Induction by 3)
Let P be the discret plan of equation $ax + by + cz = 0$. The tiling obtained by taking the pattern restricted to letter 3 , we obtain a new discret plan P' of equation $a'x + b'y + c'z = 0$ with :

- *if $(a + b) \bmod a + a < c$ then*

$$\begin{aligned} a' &= (a + b) \bmod c \\ b' &= a \\ c' &= c - a' - b' \end{aligned}$$

- *else*

$$\begin{aligned} a' &= c - \Big((a + b) \bmod c\Big) \\ b' &= c - a \\ c' &= c - a' - b' \end{aligned}$$

Property 3.27 (Unimodular)
The matrices associated to the previous inductions are unimodular.

Proof : This result can be prove just by computation. For exemple, the matrices of the first induction are in the form :

$$\begin{pmatrix} -a_1 & 1 & 1 \\ -a_2 & 1 & 0 \\ a_1+a_2+1 & -2 & -1 \end{pmatrix} \text{ with } (a_1, a_2) \in \mathbb{Z}^2$$

So by computation, we have a determinant equal to 1.

■

Remark : It is interesting to observe that the induction by the letter 3 stabilizes the tribonacci sequence ($u_{n+3} = u_{n+2} + u_{n+1} + u_n$) by maping ($u_n, u_{n+1}, u_{n+2}$) to ($u_n, u_{n+1}, u_{n-1}$) for all $n > 1$.

4 Conclusion

In this article, we have introduced some interesting tools to study bidimensional sturmian words. The localization word allowed us to obtain a tiling and a way to recode a discrete plane into another. However the synchronization with the 3-letters words is not directly possible in many cases. We need to make a distortion to switch from the tiling of the localization word to a tiling of the 3-letters words, and the recoding seems to be different. We obtain an extension of the Morse-Hedlund theorem but not yet of the Vuillon theorem. However the distortion must be useful to understand how construct the tiles of the 3-letters words with maybe a new condition to μ. The Morse-Hedlund theorem allow us to describe an induction on discrete plans.

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