AN INTRODUCTION TO MOTIVIC INTEGRATION THEORY

Contents

1. History of motivic integration : Why? 1
2. Arcs spaces : What we will measure 2
2.1. Variety $n - jets$ of X (Greenberg) 2
2.2. Truncation maps 2
2.3. Arc space of X 3
3. Additive invariants : what interests us 3
4. Grothendieck rings : Values of the measure 3
5. Motivic measure 4
5.1. Constructible or cylinder subset of $\mathcal{L}(X)$ 4
5.2. Stable subset of $\mathcal{L}(X)$ 4
5.3. Non stable constructible subset of $\mathcal{L}(X)$: completion of $\mathcal{M}_k$! 5
5.4. Integrable function 5
5.5. Change variable formula 5
5.6. Comparison with p-adic integration 6
6. Proof's of Kontsevich theorem 6
References 7

These pages are the notes of a talk given at the Catholic University of Leuven on October 13', 2010.

For wonderful surveys about this subject, look at [5], [7], [9] and [10].

1. History of motivic integration : Why ?

The story starts with a theorem coming from strings theory :

Theoreme 1.1 (Batyrev, 95', [1]). *Let X and Y be two Calabi-Yau varieties (complex algebraic varieties, smooth and proper which admit a non vanishing regular differential form of maximal degree). If X and Y are birationally equivalent then X and Y have the same Betti numbers :*

$$\forall i \geq 0, \ \ rank\, H^i(X(\mathbb{C}), \mathbb{C}) = rank\, H^i(Y(\mathbb{C}), \mathbb{C}).$$

Proof. It uses :

(1) Hironaka's theorem,
(2) p-adic integration and its change variables formula,
(3) Weil conjectures,
(4) Comparison theorem between l-adic Betti numbers and usual Betti numbers.

□

After that :

- Kontsevich (December 7, 95') at Orsay, explained a direct approach, avoiding p-adic integration and Weil conjectures but involving arc spaces : *motivic integration.*

  He showed more : X and X' have the same Hodge numbers, $h^{p,q}(X)$ (where $h^{p,q}(X)$ is the dimension of $H^q(X, \Omega^p_X)$).

- Denef-Loeser (99') [4] constructed a motivic integration theory on arbitrary (in particular singular) algebraic varieties on a field k (car k=0).

- Loeser-Sebag (03) [8] constructed a motivic integration theory on formal schemes and rigid varieties (for an arbitrary complete discrete valuation ring with perfect residue field).

- Cluckers-Loeser (08) [3] and differently Hrushowski-Kazhdan (06) [6] constructed a general framework for motivic integration based on model theory.

2. Arcs spaces : What we will measure

Let k be a field, car k=0.
Let X be a *variety* = separated and reduced scheme of finite type over k.

2.1. Variety $n-jets$ of X (Greenberg).

For all $n \geq 0$, we denote by $\mathcal{L}_n(X)$ the k-variety which represents the functor

$$\begin{array}{ccc} k-alg & \to & \underline{Set} \\ R & \mapsto & Hom_{k-scheme}(Spec(R[t]/t^{n+1}), X) \end{array}$$

Note that "the base extension operation" $Y \mapsto Y \times_k k[t]/t^{n+1}$ is a covariant functor and it has a right adjoint $X \mapsto \mathcal{L}_n(X)$.

- $\mathcal{L}_n(X)$ is called *n-jets of X*.
- For all field K containing k : $\mathcal{L}_n(X)(K) = X(K[t]/t^{n+1})$.

Example 2.1. Let X be an affine variety :

$$X = \left\{ \begin{array}{c} f_1(x) = 0 \\ . \\ . \\ f_m(x) = 0 \end{array} \right. , \quad x = (x_1, .., x_l).$$

$\mathcal{L}_n(X)$ is given by the equations in variables $\vec{a_0}, .., \vec{a_n}$ expressing that $f_i(\vec{a_0} + ... + \vec{a_n} t^n) = 0 \, mod t^{n+1}, i = 1, ..., m$.

Example 2.2. $X = \mathbb{C}^d$, $\mathcal{L}_n(X) = \{(a_0^{(1)} + ...a_n^{(1)} t^n, ..., a_0^{(d)} + ...a_n^{(d)} t^n) \mid a_i^{(j)} \in \mathbb{C}\} \simeq \mathbb{C}^{d(n+1)}$.

Example 2.3. Cusp :

- $X = \{y^2 - x^3 = 0\}$
- $\mathcal{L}_0(X)(\mathbb{C}) = \{(a_0, b_0) \in \mathbb{C}^2 \mid b_0^2 = a_0^3\} = X(\mathbb{C})$
- $\mathcal{L}_1(X)(\mathbb{C}) = \{(a_0 + b_1 t, b_0 + b_1 t) \in (\mathbb{C}[t]/t^2)^2 \mid (b_0 + b_1 t)^2 = (a_0 + a_1 t)^3 \;\; mod \, t^2\}$

  which implies two equations $\left\{ \begin{array}{l} b_0^2 = a_0^3 \\ 2b_0 b_1 = 3a_0^2 a_1 \end{array} \right.$.

  Note that $\mathcal{L}_1(X)(\mathbb{C}) \simeq TX(\mathbb{C})$.

Remark 2.4. Let X be an algebraic variety, we always have these isomorphisms

$$\mathcal{L}_0(X) \simeq X \quad \text{and} \quad \mathcal{L}_1(X) \simeq TX$$

2.2. Truncation maps.

For all $n \geq m$, there is a natural map induced by reducing modulo t^{m+1} and called *truncation map*

$$\pi_m^n \quad \begin{array}{c} \mathcal{L}_n(X) \\ \downarrow \\ \mathcal{L}_m(X) \end{array}$$

2.3. **Arc space of X.** We obtained in this way a projective system and we call *arc space of X* the projective limit

$$\mathcal{L}(X) := \underleftarrow{lim}\mathcal{L}_n(X).$$

It's a reduced separated scheme over k, but not (in general) of finite type over k.

Note that : it's a scheme and not a pro-scheme because the truncation maps are affine.

For all field K containing k

$$\mathcal{L}(X)(K) = X(K[[t]]).$$

Example 2.5. $X = \mathbb{C}^d$, $\mathcal{L}(X) = \{(\sum a_n^{(i)} t^n)_{i\in\{1,..d\}} \mid a_i^{(j)} \in \mathbb{C}\} \simeq \mathbb{C}[[t]]^d$.

Example 2.6. Cusp :

- $X = \{y^2 - x^3 = 0\}$
- $\mathcal{L}(X)$ is given in the infinite dimensional affine space with coordinates (a_i), (b_i), by an infinite number of equations $\begin{cases} b_0^2 = a_0^3 \\ 2b_0b_1 = 3a_0^2a_1 \\ \ldots \end{cases}$

3. Additive invariants : what interests us

Let k a field, $car k = 0$.

Definition 3.1. An *additive invariant* is a map $\lambda : Var_k \to R$ where R is a ring, such that

$$\begin{cases} \lambda(X) = \lambda(Y) & \text{when } X \simeq Y \\ \lambda(X) = \lambda(F) + \lambda(X \setminus F) & \text{for } F \text{ a closed subset of } X \\ \lambda(X \times Y) = \lambda(X) \times \lambda(Y) \end{cases}$$

Example 3.2. Euler Characteristic, $k = \mathbb{C}$

$$Eu(X) := \sum_i (-1)^i \mathrm{rank} H_c^i(X(\mathbb{C}), \mathbb{C}).$$

Jan Denef said me, that by a Grothendieck's theorem the result is the same by using not compact support cohomology.

Example 3.3. Hodge polynomial , $k = \mathbb{C}$

$$\begin{array}{lccc} H: & Var_{\mathbb{C}} & \to & Z[u, v] \\ & X & \mapsto & \sum_{i,p,q} (-1)^i h_{p,q}^i u^p v^q \end{array}$$

where $h_{p,q}^i$ is the dimension of $H_c^i(X(\mathbb{C}), \mathbb{C})^{p,q}$, the (p, q)-part of the mixed Hodge structure of Deligne on $H_c^i(X(\mathbb{C}), \mathbb{C})$.

4. Grothendieck rings : Values of the measure

There exists an universal additive invariant

$$[-] : Var_k \to K_0(Var_k)$$

such that for all additive invariant $\lambda : Var_k \to R$, there exists a unique ring morphism $\tilde{\lambda} : K_0(Var_k) \to R$ such that the following diagram

$$\begin{array}{ccc} Var_k & \xrightarrow{[]} & K_0(Var_k) \\ {\scriptstyle\lambda}\downarrow & \swarrow_{\tilde{\lambda}} & \\ R & & \end{array}$$

is commutative.

Construction of $K_0(Var_k)$: It's a ring with the presentation :

- generators : isomorphism classes $[S]$, $S \in Var_k$

- relations :
- $[S] = [S'] + [S \setminus S']$ for all S' closed subset of S
- $[S \times S'] = [S].[S']$.

Remark 4.1. If $[X] = [X']$ then $\lambda(X) = \lambda(X')$, for all additive invariant λ. For instance, same Euler characteristic and same Hodge-Deligne polynomial. In particular, same Hodge numbers and thus same Betti numbers.

We denote by $\mathbb{L}$ the class of the affine line $[\mathbb{A}^1]$. In the following, we will use $\mathcal{M}_k := K_0(Var_k)[\mathbb{L}^{-1}]$ the localisation of $K_0(Var_k)$ with respect to $\mathbb{L}$.

Remark 4.2. Some remarks :

- Poonen proved that $K_0(Var_k)$ is not a domain, for k a field with $car\ k = 0$.
- In the same way, $\mathcal{M}_k$ is not a domain.
- It's not known if the localisation morphism $K_0(Var_k) \to \mathcal{M}_k$ is injective.
- There is an alternative description of $K_0(Var_k)$ given by Bittner [2] :
- generators : isomorphism classes $[V]$ of non-singular projective varieties
- relations :
  (1) $[\emptyset] = 0$
  (2) $[\tilde{V}] - [E] = [V] - [Z]$, for $(\tilde{V}, E)$ a blow-up of (V, Z).
  But : it uses the weak factorisation theorem !

5. Motivic measure

Let k be a field, $car\ k = 0$.
Let X be an algebraic variety over k of pure dimension d. Let X_{sing} denote the singular locus of X.

5.1. Constructible or cylinder subset of $\mathcal{L}(X)$.

Definition 5.1. A subset $A \subset \mathcal{L}(X)$ is *constructible* or a *cylinder* if and only if $A = \pi_n^{-1}(C)$ with C a constructible subset of $\mathcal{L}_n(X)$, for some $n \in \mathbb{N}$.

5.2. Stable subset of $\mathcal{L}(X)$.

Definition 5.2. A subset $A \subset \mathcal{L}(X)$ is *stable* if and only if A is constructible and $A \cap \mathcal{L}(X) = \emptyset$.

Proposition 5.3. *If $A \subset \mathcal{L}(X)$ is stable then $[\pi_n(A)]\mathbb{L}^{-nd}$ in $\mathcal{M}_k$ stabilizes for n big enough. We denote*

$$\mu(A) := [\pi_n(A)]\mathbb{L}^{-nd},\ n >> 1$$

and call it motivic measure of the stable subset A.

Proof. In the non-singular case : X smooth.

(1) By Hensel lemma :
  – for all $n \geq m$, π_m^n is a locally trivial fibration for the Zariski topology with fiber $\mathbb{A}^{(m-n)d}$.
  – for all n, π_n is surjective.

(2) If $E \to B$ is a locally trivial fibration for the Zariski topology with fiber F then $[E] = [F][B]$.

(1)+(2) If

$$\begin{array}{ccccc} A = \pi_n^{-1}(C) & \subset & \mathcal{L}(X) & \\ \downarrow & & \downarrow & \pi_n \\ \pi_n(A) & & \mathcal{L}_n(X) & \\ \downarrow & & \downarrow & \pi_m^n \\ \underset{\text{constructible}}{C} & \subset & \mathcal{L}_m(X) & \end{array}$$

then $[\pi_n(A)] = \mathbb{L}^{(n-m)d}[C]$.

So for all $n \geq m$,

$$\frac{[\pi_n(A)]}{\mathbb{L}^{nd}} = \frac{[C]}{\mathbb{L}^{md}}.$$

□

Remark 5.4. If X is smooth then $\mu(\mathcal{L}(X)) = [\pi_0(\mathcal{L}(X))] = [X]$.

5.3. **Non stable constructible subset of $\mathcal{L}(X)$: completion of $\mathcal{M}_k$!** Let $A \subset \mathcal{L}(X)$ be a constructible and not stable subset. The quotient $\frac{[\pi_n(A)]}{\mathbb{L}^{nd}}$ will not always stabilize.

Example 5.5. $X = \{xy = 0\}$ show that $\frac{[\pi_n(\mathcal{L}(X))]}{\mathbb{L}^{nd}} = 2\mathbb{L} - \frac{1}{\mathbb{L}^n}$.

But the limit

$$\mu(A) := \lim_{n\to\infty} \frac{[\pi_n(A)]}{\mathbb{L}^{nd}}$$

exists in the completed Grothendieck group $\hat{\mathcal{M}}_k$ and it's called *motivic measure of the constructible set A.*

$\hat{\mathcal{M}}_k$ is the completion of $\mathcal{M}_k$ with respect to the filtration $(F^m\mathcal{M}_k)_{m\in\mathbb{Z}}$ where

$$F^m\mathcal{M}_k := < [S]\mathbb{L}^{-i}, i - \dim S \geq m > .$$

It's a ring filtration $F^{m+1}\mathcal{M}_k \subset F^m\mathcal{M}_k$, $F^mF^n \subset F^{m+n}$ and

$$\hat{M}_k := \lim_{\leftarrow} \mathcal{M}_k/F^m\mathcal{M}_k.$$

This yields a σ-additive measure μ on the Boolean algebra of constructible subsets then

$$\mu(\sqcup A_i) = \sum \mu(A_i) \in \hat{\mathcal{M}}_k.$$

There are more generally measurable subsets of $\mathcal{L}(X)$. In particular :

- the semi-algebraic subsets of $\mathcal{L}(X)$ are measurable,
- If S is a subset of X and $S \neq X$ then $\mathcal{L}(S)$ is measurable and $\mu(\mathcal{L}(S)) = 0$.

Remark 5.6. Some remarks :

(1) The completed Grothendieck ring was introduced first by Kontsevich.
(2) It's not known whether the canonical morphism $\mathcal{M}_k \to \hat{\mathcal{M}}_k$ is injective or not !
(3) Nevertheless, one can show that Euler Characteristic and Hodge polynomial factor through the image $\overline{\mathcal{M}_k}$ of $\mathcal{M}_k$ in $\hat{\mathcal{M}}_k$.

5.4. **Integrable function.** Let $A \subset \mathcal{L}(X)$ a measurable set and $\alpha : A \to \mathbb{Z} \cup \{\infty\}$ be a function such that all its fibers are measurable, $\mathbb{L}^{-\alpha}$ is integrable if the series

$$\int_A \mathbb{L}^{-\alpha} d\mu := \sum_{n\in\mathbb{Z}} \mu(A \cap \alpha^{-1}(n))\mathbb{L}^{-n}$$

is convergent in $\hat{M}_k$.

Example 5.7. If $\mathcal{I}$ is a sheaf of ideals on X then we define

$$\begin{array}{cccc} ord_t\mathcal{I} & : & \mathcal{L}(X) & \to & \mathbb{N} \cup \{+\infty\} \\ & & \varphi & \mapsto & \min_{g\in\mathcal{I}_{\pi_0(\varphi)}} ord_t g(\varphi) \end{array} .$$

5.5. **Change variable formula.** This theorem is due to Kontsevich in the smooth case and Denef-Loeser for the general case.

Theoreme 5.8. *Let*

(1) *X be an algebraic variety over k, with $\dim X = d$,*
(2) *Y be a smooth algebraic variety over k, with $\dim Y = d$,*
(3) *$h : Y \to X$ be a proper and birationnal map,*
(4) *$A \subset \mathcal{L}(X)$ be a constructible (also true for A semi-algebraic),*
(5) *$\alpha : A \to \mathbb{Z} \cup \{\infty\}$ be such that $\mathbb{L}^{-\alpha}$ is integrable on A,*

Then

$$\int_A \mathbb{L}^{-\alpha} d_{\mu_X} = \int_{h^{-1}(A)} \mathbb{L}^{-\alpha\circ h - ord_t Jac_h} d_{\mu_Y}.$$

With

- if X is non-singular, $Jac_c h$ is the ideal sheaf locally generated by the ordinary Jacobian determinant with respect to local coordinates on X and Y.

- if X is general, then the sheaf of regular differential d-forms $h^*(\Omega^d_X)$ is still a submodule of Ω^d_Y but not necessary generated by one element. Taking locally a generator ω_Y of Ω^d_Y, each $h^*(\omega)$ for $\omega \in \Omega^d_X$ can be written as $h^*(\omega) = g_\omega \omega_Y$. We define Jac_h as the ideal sheaf generated by these g_ω.

5.6. **Comparison with p-adic integration.** We have the following comparison :

(1) The tabular :

	p-adic	motivic
integrate over	$\mathbb{Z}_p^m,\ \mathbb{Z}_p = \varprojlim \mathbb{Z}/p^m\mathbb{Z}$	$k[[t]]^m, k[[t]] = \varprojlim k[t]/(t^m)$
value rings	$\mathbb{Z}$	$K_0(Var_k)$
	$\mathbb{Z}[1/p]$	$\mathcal{M}_k$
	$\mathbb{R}$	$\hat{\mathcal{M}}$

(2) Let M be a d-dimensional submanifold of $\mathbb{Z}_p^m$ defined algebraically.
Denote $\pi_n : \mathbb{Z}_p^m \to (\mathbb{Z}_p/p^{n+1}\mathbb{Z}_p)^m = (\mathbb{Z}/p^{n+1}\mathbb{Z})^m$.
Then $\frac{Card\,\pi_n(M)}{p^{nd}} \in \mathbb{Z}[\frac{1}{p}]$ is constant for n big enough and is called the *volume* $\mu_p(M)$ of M.

(3) For a singular d-dimensional subvariety Z of $\mathbb{Z}_p^m$ one defines its *volume* as

$$\mu_p(Z) := \lim_{\varepsilon \to 0} \mu_p(Z \setminus T_\varepsilon(Z_{sing}))$$

where $T_\varepsilon(Z_{sing})$ is a tubular neighborhood.

Osterlé proved : $\mu_p(Z) = \lim_{n\to\infty} \frac{Card\,\pi_n(Z)}{p^{nd}}$.

6. Proof's of Kontsevich theorem

Theoreme 6.1. *Let X and Y be two Calabi-Yau manifolds. If X and Y are birationnaly equivalent then*

$$[X] = [Y] \in \overline{\mathcal{M}_k} \subset \hat{\mathcal{M}}_k.$$

So X and Y have the same Hodge numbers, hence same Betti numbers (=Batyrev's theorem).

Proof. Steps of the proof

(1) By Hironaka's theorem, there exists a non singular proper complex algebraic variety Z and birationnal morphisms $h_X : Z \to X$ and $h_Y : Z \to Y$.
(2) There exists $c \in \mathbb{C}^*$ ($c \neq 0$ because ω_X has no zeroes) such that

$$ch_X^*\omega_X = h_Y^*\omega_Y.$$

(3) So on $\mathcal{L}(Z)$

$$ord_t Jac_{h_X} = ord_t Jac_{h_Y}.$$

(4) Then now :

$$\begin{array}{lcl} [X] & \underset{\text{(smoothness)}}{=} & \mu_X(\mathcal{L}(X)) = \int_{\mathcal{L}(X)} 1 d\mu_X \\ & \underset{\text{(change variables formula)}}{=} & \int_{\mathcal{L}(Z)} \mathbb{L}^{-ord_t Jac_{h_X}} d\mu_Z \\ & \underset{\text{(CY-hypothesis)}}{=} & \int_{\mathcal{L}(Z)} \mathbb{L}^{-ord_t Jac_{h_Y}} d\mu_Z \\ & \underset{\text{(change variables formula)}}{=} & \int_{\mathcal{L}(Y)} 1 d\mu_Y = \mu_Y(\mathcal{L}(Y)) \\ & \underset{\text{(smoothness)}}{=} & [Y]. \end{array}$$

□

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