Note by M. Kontsevich

Grothendieck ring of motives and related rings

Let k be a field. Denote by $\mathcal{M} = \mathcal{M}(k)$ the abelian group generated by symbols $[X]$ where X is a scheme of finite type over k satisfying the following relations:

(1) if $X_1 \simeq X_2$ then $[X_1] = [X_2]$,
(2) if Y is a closed subset in X then $[X] = [Y] + [X \setminus Y]$.

Evidently one can define a ring structure on $\mathcal{M}$ by the formula $[X] \times [Y] := [X \times Y]$. The unit 1 in $\mathcal{M}$ is $[\mathbf{A}^0]$. By the result of H. Gillet and C. Soulé if $char(k) = 0$ then there is a homomorphism from $\mathcal{M}$ to the Grothendieck K_0 group of (pure) motives over k. If we have a good cohomology theory, like l-adic cohomology or Hodge structures then we have a homomorphism from $\mathcal{M}$ to K_0 of the corresponding abelian category. Symbol $[X]$ maps to

$$\sum_{i=0}^{2\dim X} (-1)^i [H^i_c(X)] \ .$$

If k is a finite field then there is a homomorphism $\# : \mathcal{M} \to \mathbf{Z}$ defined by the formula $\#([X]) = \#(X(k))$. Also, for any field there is a [illegible] $\lambda : \mathcal{M} \to \mathbb{Z}$ [illegible]

We will use the following elementary identity in $\mathcal{M}$.

If $X \to Y$ is a locally trivial bundle in Zariski topology with fibers equivalent to the affine space $\mathbf{A}^n$ then $[X] = [Y] \times [\mathbf{A}^n] = [Y] \times [\mathbf{A}^1]^n$.

Usually one considers the localization $\mathcal{M}'$ of $\mathcal{M}$ with respect to the element $[\mathbf{A}^1]$ (the Tate motive). We will denote by $[\mathbf{A}^d]$ for $d \in \mathbf{Z}$ the d-th power of the invertible element $\mathbf{A}^1$ in $\mathcal{M}'$.

Ring $\mathcal{M}$ has a natural filtration by non-negative integers: $\mathcal{M}_{\leq k}$ is a subgroup in $\mathcal{M}$ spanned by symbols $[X]$ where $\dim X \leq k$. We denote by $\widehat{\mathcal{M}}$ the completion of $\mathcal{M}'$ with respect to the induced filtration by all integers on $\mathcal{M}'$. For example, in the Hodge realization for the case $k = \mathbf{C}$ elements of $\widehat{\mathcal{M}}$ maps to formal combinations $\sum_{n=0}^{\infty} [H_{(n)}]$ where $H_{(n)}$ is an equivalence class of pure Hodge structures with coefficients in $\mathbf{Q}$ of weights $w_n \in \mathbf{Z}$, and w_n tends to $-\infty$ as n tends to $+\infty$.

Our calculations of motivic analogues of p-adic integrals lead naturally to the consideration of a new localization $\mathcal{M}''$ of $\mathcal{M}$. Namely, we would like to invert not only symbols of affine spaces $[\mathbf{A}^n]$ but also of all projective spaces $[\mathbf{P}^n]$, $n \geq 0$. There is a homomorphism from $\mathcal{M}''$ to $\widehat{\mathcal{M}}$:

$$([\mathbf{P}^n])^{-1} \longmapsto ([\mathbf{A}^1] - 1) \times \sum_{k=1}^{\infty} [\mathbf{A}^{-k(n+1)}] \ . \qquad [P^n] = \frac{1 - A^{n+1}}{1 - A}$$

If k is a finite field then the homomorphism $\#$ from $\mathcal{M}$ to $\mathbf{Z}$ extends uniquely to the homomorphism from $\mathcal{M}''$ to $\mathbf{Q}$. Evidently this homomorphism can be defnied on the image of $\mathcal{M}''$ in $\widehat{\mathcal{M}}$. Abusing slightly notations we will denote all these homomorphisms again by $\#$. [illegible] field [illegible]

Motivic analog of a p-adic integral

Let X be a smooth scheme of finite type over a field k and D be an effective divisor on X We will construct an element $[X : D]$ in the ring $\widehat{\mathcal{M}}(k)$. This element will be independent under certain class of birational transformations. Using the resolution of singularities in the case $char(k) = 0$ we will show that $[X : D]$ belongs to the image of $\mathcal{M}''$ in $\widehat{\mathcal{M}}$.

Let X be smooth scheme over a number field K, vol be a section of the canonical bundle, $vol \neq 0$. Let us choose a model X' of X and vol over $Spec(\mathcal{O}_K)$. Then for almost all finite points v we have

$$\#([X_v : D_v]) = \int_{X'(\mathcal{O}_v)} |vol|_v$$

where X_v and D_v are good reductions at v of X and $D := Div(vol)$.

The basic example: D is divisor with normal crossings. For the sake of simplicity we assume that reduced irreducible components $(D_i)_{i \in I}$ of D are smooth. Denote by $n_i > 0$ the multiplicity of D_i in D. Then we have

$$[X : D] = [\mathbf{A}^{-\dim X}] \times \sum_{J \subset I} \left((-1)^{\#(J)} \left[\bigcap_{j \in J} D_j \right] \times \prod_{j \in J} \left(\frac{1}{[\mathbf{P}^{n_j}]} - 1 \right) \right)$$

First we give an heuristic definition of $[X : D]$ using infinite-dimensional manifolds and after that we will comment on the rigorous definition.

Definition of motivic integral.

The analog of the set $X'(\mathcal{O}_v)$ will be the pro-algebraic variety of formal parametrized paths on X

$$\mathcal{L}_+X := \underline{Map}(Spec(k[[t]]), X)$$

Scheme $\mathcal{L}_+X$ is the projective limits of finite-dimensional smooth schemes $\mathcal{L}_nX := \underline{Map}(Spec(k[t]/(t^{n+1}), X)$:

$$X = \mathcal{L}_0X \longleftarrow TX = \mathcal{L}_1X \longleftarrow \mathcal{L}_2X \longleftarrow \cdots \longleftarrow \mathcal{L}_+X$$

All forgetting maps $\mathcal{L}_{k+1}X \longrightarrow \mathcal{L}_kX$ are locally trivial bundles in Zariski topology with fibers equivalent to affine spaces.

We consider $\mathcal{L}_+X$ as an algebraic conterpart of the space of parametrized holomorphic discs on a complex manifold which is embedded into the free loop space $\mathcal{L}X$.

We ~~denote by~~ define δ the constructible function on $\mathcal{L}_+X$ with values in $\mathbf{Z}_{\geq 0} \cup \{+\infty\}$ by the formula

$$\delta(\phi) = ord_{t=0}\phi^*(f)$$

where $\phi : Spec(k[[t]]) \longrightarrow X$ is a k-point of $\mathcal{L}_+X$ and f is a local equation of the divisor D. Function h takes value $+\infty$ only on the subspace in $\mathcal{L}_+X$ consisting of paths in D. This subspace has infinite codimension in $\mathcal{L}_+X$ and we will ignore it.

Each stratum $\delta^{-1}(k)$ where $k \in \mathbf{Z}_{\geq 0}$ is a locally closed subscheme in $\mathcal{L}_+X$ of a finite codimension. Also it can be startified by finitely many pieces each of which is fibered over a finite-dimensional scheme with fibers equivalent to infinite-dimensional affine spaces.

We want to regularize dimensions of all strata and make them finite. In order to do it we will describe linearized situation on the tangent space to $\mathcal{L}_+X$ at each point ϕ such that $\delta(\phi) \neq +\infty$. It is clear that

$$T_\phi \mathcal{L}_+ X \simeq \Gamma(Spec(k[[t]], \phi^* T_X)$$

is a lattice in the vector space $\Gamma\left(Spec\left(k((t))\right), \phi^* T_X\right)$ over local field $K := Speck((t))$.

Volume element and dimensions.

Let V be a finite dimensional vector space over K. Then there is a natural class of compact vector subspaces over k in V consisting of $U \subset V$ such that there exist lattices U_+, U_-, $U_- \subset U \subset U_+$. For any two compact subspaces U_1, U_2 we define their relative dimension by the formula

$$\dim\,(U_1, U_2) = \dim_k (U_1/(U_1 \cap U_2)) - \dim_k (U_2/(U_1 \cap U_2))$$

There is no canonical way to define the regularized dimension $\dim\,(U) \in \mathbf{Z}$ such that $\dim\,(U_1, U_2) = \dim\,(U_1) - \dim\,(U_2)$, the set all such functions forms a torsor over $\mathbf{Z}$.

In order to resolve the ambiguity one has to fix something. We claim that this data is a lattice in 1-dimensional vector space over K

$$det(V) := \wedge^{\dim V}(V)$$

If $e_1, \ldots, e_N$ is a base of V over K such that $\wedge_i e_i$ generate a given lattice in $det(V)$ then we define regularized dimension of $\oplus_i k[[t]]e_i$ to be equal to 0. One can check easily that this definition is independent on the choice of base e_i.

For example, any nonzero element *vol* of $det(V)$ defines a dimension function $\dim_{vol}$ on the Grassmanian of compact subspaces. We will use the lattice in $det(V)$ generated by *vol*. (locally D = div(merom. volume el-t on X) $\longrightarrow$ any section of $det T^*$ gives such el-t

does ~~not~~ depend on the choice of vol

Formula.

Let us choose a stratification of $\mathcal{L}_+X$ by pieces X_α such that δ is constant on each startum and codimensions of strata tend to infinity. Also we may assume that each startum is the preimage of a locally closed smooth subscheme of $\mathcal{L}_kX$ under the tautological projection from $\mathcal{L}_+X$. Such a startification we will call ***admissible***. According to the previous construction we have a regularized dimension of each stratum. This regularized dimension is equal to the

element of $det(Spec(k((t)), \varphi^* T_X))$ (if D = div(merom. vol. el-t

$$\dim_{reg}(X_\alpha) := -\mathrm{codim}\,(X_\alpha) - \delta(X_\alpha)$$

We see that it is nonpositive and tends to $+\infty$.

The main idea is to shift the actual (infinite) dimension of each stratum by an infinite negative number making it equal to the regularized dimension. In the algebra $\mathcal{M}$ it means multiplication by $[A^{-\infty}]$.

Let us give a more precise definition. If X_α is the pullback of a closed subscheme X'_α in $\mathcal{L}_{k_\alpha}X$ then we define the regularized contribution of X_α as

$$[X_\alpha]_{reg} := [X'_\alpha] \times [\mathbf{A}^{-\delta(X_\alpha)-(k+1)\dim X}]$$

The definition of the motivic integral is

$$[X : D] := \sum_\alpha [X_\alpha]_{reg}$$

This series is convergent in the topology of projective limit in $\widehat{\mathcal{M}}$ because $[X_\alpha]_{reg} \in \widehat{\mathcal{M}}_{\leq \dim_{reg}(X_\alpha)}$.

Theorem. *The virtual motive $[X : D]$ is well-defined and independent on the choice of an admissible stratification.*

This is almost evident because every two admissible startifications have a common refinement and the change of the degree of approximation k_α results in the passing to the total space of a locally trivial bundle with fibers equal to affine spaces.

Birational invariance of the motivic integral.

Now we want imitate the invariance of p-adic integrals in our general setting. Let X, D be a smooth scheme and an effective divisor as above and $\pi : Y \longrightarrow X$ be a proper morphism of degree 1 from a smooth scheme Y. We define an effective divisor on Y by the formula

$$D' := \pi^*(D) + div(Jac(\pi))$$

where $Jac(\pi)$ is the Jacobian of the map π considered as a section of the line bundle $det(T_Y)^* \otimes \pi^*(det(T_X))$.

If vol is a section of the canonical bundle $det(T_X)^*$ such that $div(vol) = D$ then $div(\pi^*(vol)) = D'$.

Theorem. $[X : D] = [Y : D']$

This fact is essentially evident from the previous discussion. Spaces $\mathcal{L}_+X$ and $\mathcal{L}_+Y$ can be identified as sets after throwing away a piece of infinite codimension. We can choose an admissible stratification of $\mathcal{L}_+X$ such that its pullback on $\mathcal{L}_+Y$ is again admissible. Then the regularized dimension of each stratum and its isomorphic image coincide because they both can be defined via (local) meromorphic volume element.

Igusa's integrals.

One can easily generalize the definition of the virtual motive $[X : D]$ to the case of fractional divisors. We propose two versions:

(1) for $\mathbf{Q}$-divisors, (locally, zero divisors of sections of positive powers of the canonical bundle)
(2) analogues of Igusa's integrals

$$\int_{X'(\mathcal{O}_v)} |f|_v^s |vol|_v$$

where f is a section of $\mathcal{O}_X$ and s is a complex or formal parameter.

In the first case we extend $\widehat{\mathcal{M}}$ by adding roots of $[\mathbf{A}^1]$. Also one can extend the definition of the dimension function on the Grassmanian of compact subspaces. Instead of a lattice in the determinant line we can use a norm on this line over the local field. In the Hodge realization the natural candidate for $[\mathbf{A}^d]$ where d is not an integer is one-dimensional space $\mathbf{Q}(-d)$ over $\mathbf{Q}$ with the bigrading by rational numbers of $\mathbf{C} \otimes \mathbf{Q}(-d)$ equal to (d, d).

In the second case we can add a formal variable $[\mathbf{A}^s]$ to the ring $\widehat{\mathcal{M}}$.