Abstract

A theory for the multiscale analysis of digital shapes would be very interesting for the pattern recognition community, giving a digital equivalent of the continuous scale-space theory. We focus here on providing analytical formulae of the multiresolution of Digital Straight Segments (DSS), which is a fundamental tool for describing digital shape contours.

1. Introduction

Digital Straight Lines (DSL) and Digital Straight Segments (DSS) are useful to describe the geometry of a digital shape (coding, geometric estimators, feature detection) and this explains why they have been so deeply studied (see the survey [7] or [6]). It is well known that shapes should be studied at different scales, and this has led to the development of regular and irregular pyramids for shape analysis and scene understanding (e.g. [1]). However there exists no analytical description of the multiresolution of a digital shape, contrary to the famous scale-space analysis in the continuous world [11, 8]. One of the contribution of this paper is to give new analytical results on the multiresolution of DSL and DSS. A byproduct is new results about digital line intersection.

Figueiredo [4] first provided an analytical description of the multiresolution of 8-connected DSL. Recently, Said et al. [9] have presented analogous results for standard DSL (4-connected DSL). They have also proved that its multiresolution is also a standard DSL. For DSS, they have given a sublinear algorithm to extract its characteristics, but no analytical formula.

Analytical formulae for DSS appear to be a much harder problem: since DSS are finite parts of DSL, they are more sensitive to arithmetic peculiarites. We therefore follow an indirect path to DSS multiresolution. In Section 2, given a DSS, we build two DSL whose intersection contains it and whose main connected part has the same arithmetic characteristics as well as the same number of patterns. We note here that we propose new results about the combinatorics of such digital line intersections, that are complementary to the results of Sivignon et al. [10]. Section 3 determines the multiresolution of DSS by examining the multiresolution of the intersection of these two DSL. We give a new analytical description of this set with arithmetic inequalities.

This paper is a first step toward the multiresolution of a DSS in constant time by analytical formulae. Proofs are omitted due to space limitations.

2. Standard Digital Lines Intersection

We recall here some definitions and properties about DSL, their relations with rational fractions, and a combinatoric definition of a DSS. We restrict our study of DSS to the main connected part (say $S$) of the intersection of two well-chosen standard DSL. These two lines are related to the downward moves in the Stern-Brocot tree during a DSS recognition. We finally show that $S$ can be built so that it has the characteristics of any given DSS. Note that intersection of digital lines can be complex and may not be connected.

DSS, patterns and continued fractions. A standard line $D$ of characteristics $(a, b, \mu)$ in the fourth quadrant is the set $\{ (x, y) \in \mathbb{Z}^2, \mu \leq ax + by < \mu + a + b \}$. A pattern of characteristics $(a, b)$ is the succession of Freeman moves between any two consecutive lower leaning points of $D$. We recall that a simple continued fraction is an expression:
\[ z = \frac{b}{a} = [u_0, u_1, u_2, \ldots, u_i, \ldots, u_n] = u_0 + \frac{1}{u_1 + \frac{1}{u_2 + \frac{\delta}{u_3 + \frac{\delta}{\ddots + \frac{\delta}{u_i + \frac{\delta}{\ddots + \frac{\delta}{u_n + \frac{\delta}{u_{n+1}}}}}}}} \]

whose intermediate fractions \( z_k \) are called partial quotients and \( n \) is its complexity.

Consider \( E \) a mapping from the set of positive rational number smaller than one onto Freeman code words defined as follows. First terms are stated as \( E(z_0) = 0 \) and \( E(z_1) = 0; 3 \) and others are expressed recursively:

\[
E(z_{2i+1}) = E(z_{2i})^{u_{2i+1}} E(z_{2i-1}) \\
E(z_{2i}) = E(z_{2i-2}) E(z_{2i-1})^{u_{2i}}
\]

(1)

The role of partial quotients can be visualized with a structure called Stern-Brocot tree (see [5] for a complete definition and [12] for a link with continued fractions). In [3], Debled and Réveillès introduced the link between this tree and the recognition of digital line. Recognizing a piece of digital line is like going down the Stern-Brocot tree up to the directional vector of the line. Their online recognition algorithm \textbf{DR95} [3] (reported in [6]) updates the DSS slope when adding a point just exterior to the current line (weak exterior points). The slope evolution is analytically given by next property.

**Proposition 1** [2] The slope evolution in \textbf{DR95} depends on the parity of the complexity of its slope, the type of weakly exterior point added to the right (UWE and LWE stands respectively for upper and lower weakly exterior). This is summed up in the table below, where the slope is \([0, u_1, \ldots, u_n] \), \( n = 2i \) even or \( n = 2i + 1 \) odd, \( \delta \) pattern(s) and \( \delta' \) reversed pattern(s):

<table>
<thead>
<tr>
<th>Even ( n )</th>
<th>Odd ( n )</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>UWE</strong></td>
<td>([0, u_1, \ldots, u_{2i}, \delta] )</td>
</tr>
<tr>
<td><strong>LWE</strong></td>
<td>([0, u_1, \ldots, u_{2i}, 1, 1, \delta] )</td>
</tr>
</tbody>
</table>

**Segments by digital line intersection.** In order to study a DSS composed of \( \delta \) patterns of slope \( z_0 \), we build a very similar DSS which includes it as the intersection of two DSL with carefully chosen slopes. Their patterns are placed so that one starts at the first lower leaning point and the other ends at the last lower leaning point (see Fig. 1).

**Proposition 2** The main connected part \( S \) of the intersection between \( E(2z_{2i+1}) \) with \( z_{2i+1} = [0, u_1, \ldots, u_{2i}, \delta] \) and \( E(z'_{2i+2}) \) with \( z'_{2i+2} = [0, u_1, \ldots, u_{2i}, 1, 1, \delta] \) is defined as their common part as placed below:

\[
\begin{array}{c}
\ldots E(z_{2i+1}) \end{array} \quad \begin{array}{c}
E(z_{2i+1})^{\delta}
\end{array} \quad \begin{array}{c}
E(z_{2i+2}) \end{array} \quad \begin{array}{c}
E(z'_{2i+2}) \end{array} \quad \begin{array}{c}
S
\end{array}
\]

**Figure 1.** Intersection of two patterns \( E(z_3) \) and \( E(z'_4) \), where \( S \) is the main connected part of their intersection.

The word \( S \) is exactly \( w_1 E(z_{2i})\delta w_2 \), with \( w_1 = E(z_1)u_2 \ldots E(z_{2i-1})u_{2i-2} \), \( E(z_{2i-1})^{u_{2i-1}} \), \( E(z_{2i-2})u_{2i-1} \) and \( w_2 = E(z_{2i-2})u_{2i-1} \ldots E(z_{2i-2})u_{2i} \).

We remark that it contains the pattern \( E(z_{2i}) = E(z'_{2i}) \) repeated \( \delta \) times (Figure 1 exemplifies the construction of this intersection).

**Proposition 3** The main connected part \( S \) of the intersection between \( E(z_{2i+1}) \) with \( z_{2i+2} = [0, u_1, \ldots, u_{2i+1}, \delta] \) and \( E(z'_{2i+3}) \) with \( z'_{2i+3} = [0, u_1, \ldots, u_{2i+1}, 1, 1, \delta] \) is defined as their common part as placed below:

\[
\begin{array}{c}
\ldots E(z_{2i+1})^{\delta}
\end{array} \quad \begin{array}{c}
E(z_{2i+1})
\end{array} \quad \begin{array}{c}
E(z_{2i+2}) \end{array} \quad \begin{array}{c}
S
\end{array}
\]

The word \( S \) is exactly \( w_1 E(z_{2i+1})^\delta w_2 \), with \( w_1 = E(z_{2i})u_2 \ldots E(z_{2i-1})u_{2i-2} \), \( \ldots E(z_{2i-3})u_{2i-2} \), \( E(z_{2i-1})^{u_{2i-1}} \) and \( w_2 = E(z_{2i})u_{2i+1} \ldots E(z_{2i-2})u_{2i+1} E(z_{2i-4})u_{2i-3} \ldots E(z_{2i-2})u_{2i} \ldots E(z_{2i})u_{2i+1} E(z_{2i})^{u_{2i+1}} \).

We remark that it contains the pattern \( E(z_{2i+1}) = E(z'_{2i+1}) \) repeated \( \delta \) times.

**Theorem 1** [2] If the DSL \( D \) of even slope \( \frac{p_{2i}}{q_{2i}} = [0, u_1, \ldots, u_{2i}] \) (or of odd slope \( \frac{p_{2i+1}}{q_{2i+1}} = [0, u_1, \ldots, u_{2i+1}] \)) is the common part of two standard digital lines \( D_1 \) and \( D_2 \), then their slopes are:

<table>
<thead>
<tr>
<th>( D_1 )</th>
<th>( D_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( s_{2i} )</td>
<td>( s_{2i} )</td>
</tr>
<tr>
<td>( (0+1)p_{2i} - P_{2i-1} )</td>
<td>( (0+1)p_{2i} - P_{2i-1} )</td>
</tr>
</tbody>
</table>

**Lemma 1** On the arithmetic straight lines \( D_1 \) and \( D_2 \), if \( D(a, b, \mu) \) has an even complexity with the remainder \( \mu = a(x - x_0) + b(y - y_0) \) where \( (x, y) \) is the first point
of $D$ and $(x_0, y_0)$ defines the origin of the pixels in $\mathbb{Z}^2$, 
then the points of $D_1$ and $D_2$ of remainders $\delta_1 \mu + \mu - \delta 
and \delta_2 \mu$ respectively (if $D$ has an odd complexity, then 
the points of $D_1$ and $D_2$ of remainders $\delta_2 \mu$ and $\delta_1 \mu + 
\mu - \delta$ respectively) have their coordinates given by (An 
illustration of this lemma is given in Figure 1) (top row: 
even complexity, bottom row: odd complexity):

<table>
<thead>
<tr>
<th>$D_1$</th>
<th>$D_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(\mu(b − q_{2i−1}) − \delta q_{2i−1}, \mu(p_{2i−1} − a) + \delta p_{2i−1}) + k(−\delta_q_{2i−1} − q_{2i−1}, δp_{2i−1})$</td>
<td>$(\mu(b − q_{2i−1}), \mu(p_{2i−1} − a)) + k(−(\delta + 1)q_{2i−1} + q_{2i−1}, (δ + 1)p_{2i−1} − p_{2i−1})$</td>
</tr>
<tr>
<td>$(\mu q_{2i}, \mu p_{2i})$</td>
<td>$(\mu q_{2i+1} − \delta q_{2i}, \mu p_{2i+1} + \delta p_{2i})$</td>
</tr>
</tbody>
</table>

**Proposition 4** Let $D_1(a_1, b_1, \mu_1)$ and $D_2(a_2, b_2, \mu_2)$ 
be two standard DSLs of slopes $\frac{a_1}{b_1} = [0, u_1, \ldots, u_{2i+1}, \delta]$ 
and $\frac{a_2}{b_2} = [0, u_1, \ldots, u_{2i+1}, \delta]$ with $\mu_1 = \delta \mu + \mu - \delta$ 
and $\mu_2 = \delta \mu$. Then their main connected part is a DSS 
of slope $z_{2i+1}$ with $\delta$ patterns and shift $\mu$.

**Proposition 5** Let $D_1(a_1, b_1, \mu_1)$ and $D_2(a_2, b_2, \mu_2)$ 
be two standard DSLs of slopes $\frac{a_1}{b_1} = [0, u_1, \ldots, u_{2i+1}, \delta]$ 
and $\frac{a_2}{b_2} = [0, u_1, \ldots, u_{2i+1}, \delta]$ with $\mu_1 = \delta \mu + \mu - \delta$ 
and $\mu_2 = \delta \mu$ and $\mu_2 = \delta \mu$. Then their main connected part is a DSS 
of slope $z_{2i+1}$ with $\delta$ patterns and shift $\mu$.

### 3. Multiscale of digital lines intersection

We are now in position to study the multisolution of 
a DSS defined by the intersection of two DSL, as 
specified in Proposition 4. We again denote these two 
DSL by $D_1(a_1, b_1, \mu_1)$ and $D_2(a_2, b_2, \mu_2)$ with $\mu_1 = \delta \mu + \mu - \delta$ 
and $\mu_2 = \delta \mu$.

The tiling generated by $S(h, v)$ on $\mathbb{Z}^2$ induces a new 
coordinate system where coordinates $(X, Y)$ are related 
to the canonical coordinates of $\mathbb{Z}^2$ by the obvious relations 
$X = \left[\frac{X}{h}\right]$ and $Y = \left[\frac{y}{h}\right]$, where $\left[\frac{X}{h}\right]$ is the quotient 
of the euclidean division of $X$ by $h$. Furthermore 
we denote by $\left[\frac{X}{h}\right]$ the remainder of this division. An 
$(h, v)$-covering of a set of points of $\mathbb{Z}^2$ is the set of tiles 
of $S(h, v)$ which intersect it. Let $\Delta_1(a_1', b_1', \mu_1')$ and 
$\Delta_2(a_2', b_2', \mu_2')$ be the two digital straight lines that are 
the $(h, v)$-covering of $D_1$ and $D_2$ respectively. 
Theorem 1 of [9] states that these two lines are standard and 
gives their arithmetic inequalities:

$$-p^i + Q_2^i − Q_1^i + SS^i \leq a'_1 X + b'_1 Y < Q_3^i − Q_2^i + SS^i \quad (\Delta_1)$$
$$-p^i + Q_2^i − Q_2^i + SS^i \leq a'_2 X + b'_2 Y < Q_3^i − Q_2^i + SS^i \quad (\Delta_2)$$

where, for $i \in \{1, 2\}, p^i = a'_i + b'_i, g^i = gcd(a_i, h, b_i)$, 
$a'_i = \frac{a_i}{g^i}$, $b'_i = \frac{b_i}{g^i}$, $\mu_i = \delta \mu + (2 - i)(\mu - \delta)$,

for $k = 1, 2, 3$, $Q_k^i = \left[\frac{(k-1)\mu + k(b_i + b'_i) - 1}{g^i}\right]$ \quad $R_k^i = \left\{\frac{(k-1)\mu + k(b_i + b'_i) - 1}{g^i}\right\}$

$$SI^i = \begin{cases} 0 & \text{if } R_k^i \leq R_l^i \\ SS^i & \text{otherwise} \end{cases}$$

To simplify equations, we set $A_1^i = -p^i + Q_2^i − Q_1^i + 
SI^i$ and $B_1^i = Q_3^i − Q_2^i + SS^i$ for $i = 1, 2$. To find their 
intersection we have thus to solve the following system of equations:

$$A_1^i \leq a'_1 X + b'_1 Y < B_1^i$$
$$A_2^i \leq a'_2 X + b'_2 Y < B_2^i$$

Since $a'_1$ and $b'_1$ are relatively prime, there exist $u_1$ and $v_1$ such that 
a'_1 u_1 + b'_1 v_1 = 1$. We introduce $U = \begin{pmatrix} u_1 & -b'_1 \\ v_1 & a'_1 \end{pmatrix}$ and the change of coordinates 
$X' = U^{-1} X$, $Y' = Y$. Thus Equation (2) can be rewritten as:

$$\begin{pmatrix} A_1^i \\ B_1^i \end{pmatrix} \leq \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} a'_1 u'_2 + v'_2 b'_1 \\ a'_1 b'_2 - a'_2 b'_1 \end{pmatrix} \begin{pmatrix} X' \\ Y' \end{pmatrix} < \begin{pmatrix} B_1^i \\ B_2^i \end{pmatrix}$$

(3)

Let $\lambda_1 = u'_1 a'_2 + v'_2 b'_1$ and $\lambda_2 = a'_1 b'_2 - a'_2 b'_1$. The solution of the intersection can be computed with a double 
loop in $X'$ and $Y'$ and formulated as expressions given below.

**Theorem 2** The digital intersection of two digital 
straight lines $\Delta_1$ and $\Delta_2$ of $S(h, v)$ covering 
respectively the two digital straight lines $D_1$ and $D_2$ of $\mathbb{Z}^2$ is 
defined by:

$$A_1^i \leq X' < B_1^i$$

The expression of the boundaries of $Y'$ depends on the sign of $\lambda_2$:

$$\lambda_2 > 0, -\left[\frac{-A_2^i + \lambda_1 X'}{\lambda_2}\right] \leq Y' < -\left[\frac{-B^i + \lambda_1 X'}{\lambda_2}\right]$$

(5)

$$\lambda_2 < 0, \left[\frac{B^i - \lambda_1 X'}{\lambda_2}\right] + 1 \leq Y' < \left[\frac{A_2^i - \lambda_1 X'}{\lambda_2}\right] + 1$$

(6)

**Example.** Let $D(2, 3, 2)$ be a standard digital line of 
slope $\frac{3}{2} = [0, 1, 2]$. For instance, suppose $\delta = 1$ 
and $(h, v) = (2, 2)$. Proposition 4 gives $D_1(3, 4, 3)$ 
and $D_2(3, 5, 2)$. Their $(2, 2)$-covering is the two lines 
$\Delta_1(3, 4, -2)$ and $\Delta_2(3, 5, -3)$. We apply Theorem 2 
to determine the set of points of their intersection. 
Hence $-2 \leq X' < 5$, since $\lambda_2 = 3 > 0$, then the value of 
$Y'$ is given by the equation below:

$$-\left[\frac{3-X'}{3}\right] \leq Y' < -\left[\frac{-5-X'}{3}\right]$$

(7)

Finally, we have applied the unimodular matrix $U = 
\begin{pmatrix} 3 & -4 \\ -2 & 3 \end{pmatrix}$ on $(X', Y')$ to get the final result given in 
the table below and illustrated on Fig. 3.
and every point of \( P \) of \( S \) have calculated the characteristics (see Figure 2). Let \( P \) be a starting point \((h, v) = (2, 2)\), i.e. a subset of the main connected part \( S' \) of the intersection of \( \Delta_1 \) and \( \Delta_2 \) (\( S' \) is a covering of \( S \) by the same tiling). Therefore, the first and last points of \( P' \) are \( (X_f, Y_f) = (0, 0) \) and \( (X_i, Y_i) = (2, -1) \), and every point of \( P' \) is shadowed in the table.

Let \( \Delta \) be a covering of \( D \) by the same tiling. We have calculated the characteristics \((a, b, \mu)\) of \( \Delta \) that is equal to \((3, 4, -5)\) by using Theorem 1 of [9]. Finally, using Algorithm SmartDSS of [9] to compute the characteristics of \( P'' \) that is some subset of a DSL \( \Delta \), given a starting point \((X_f, Y_f)\) and an ending point \((X_i, Y_i)\) \( ((X_f, Y_f), (X_i, Y_i) \in \Delta) \), and are equal to \((1, 1, 0)\).

### References


