Equivalence between $n$-surfaces and regular $n$-$G$-maps

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Outline

- Background and motivations
- Models description
- Main ideas underlying the proof
- Future work
Background

- **Topological representation of space subdivisions**
  - Geometric modeling, Computational geometry, Image analysis
  - Dedicated structures: incidence graphs, combinatorial maps, generalized maps, cell-tuples, simplicial complexes, simplicial sets, orders...
  - Specific tools and algorithms: construction operators, topological operators...
Motivations

- Transfer tools and notions from one model to another
Motivations

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  - $n$-surface (Image analysis): marching-cube like algorithms, homotopic thinning...
  - $n$-$G$-map (Topological modeling): efficient data structures, construction operators...
Motivations

- Transfer tools and notions from one model to another
- Design a general framework to represent the topology of subdivisions
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- Use several models in a single processing sequence
Motivations

- Transfer tools and notions from one model to another
- Design a general framework to represent the topology of subdivisions
- Use several models in a single processing sequence
  - Obtain an $n$-surface from an image
  - Transform it into an $n$-$G$-map
  - Handle it with $n$-$G$-maps operators
Motivations

- Transfer tools and notions from one model to another
- Design a general framework to represent the topology of subdivisions
- Use several models in a single processing sequence

⇒ Compare these structures
⇒ Highlight their similarities and specificities
Previous work

- quad-edge, facet-edge, cell-tuples, $n$-dimensional map (generalized or not) (Brisson 89, Lienhardt 91)
- dual graphs, combinatorial maps (Brun and Kropatsch 01)
- subclass of orders, cell complexes (Alayrangues and Lachaud 02)
\( n \)-surfaces and \( n-G \)-maps

- \( n \)-surfaces (subclass of orders)
  - Image analysis
  - Subclass of pseudo-manifolds without boundary
  - Recursive definition

- Generalized maps
  - Geometric and topological modeling
  - Quasi-manifolds with or without boundary, oriented or not
$n$-surfaces and $n$-$G$-maps

- $n$-surfaces (subclass of orders)
  - subclass of pseudo-manifolds without boundary
- Generalized maps
  - Quasi-manifolds $\subset$ pseudo-manifolds
Orders and $n$-$G$-maps

Order $|X| = (X, \alpha)$

$X$ set of elements equipped with the order relation $\alpha$
Orders and $n$-$G$-maps

- Order $|X| = (X, \alpha)$
  - $X$ set of elements equipped with the order relation $\alpha$
  - $X$ Countable
Orders and \( n-G \)-maps

- \( \text{CF- Order } |X| = (X, \alpha) \)
- \( X \) set of elements equipped with the order relation \( \alpha \)
- \( X \) Countable and \textit{locally Finite}
Orders and $n$-$G$-maps

- CF- Order $|X| = (X, \alpha)$
- Notation: $\theta = \alpha \alpha^{-1}$

$\Rightarrow$ may be represented by a DAG
Orders and $n$-$G$-maps

**CF- Order** $\mid X \mid = (X, \alpha)$

- Notation: $\theta = \alpha \alpha^{-1}$
- $\Rightarrow$ may be represented by a DAG

**$n$-$G$-map** $G = (D, \alpha_0, \cdots, \alpha_n)$
- $D$ set of darts,
- $\alpha_i$, $i \in \{0, \cdots, n\}$, involutions
- $\alpha_i \alpha_j$ involution, $i \leq j - 2$
First comparison difficulty
First comparison difficulty
First comparison difficulty
First comparison difficulty
First comparison difficulty
First comparison difficulty
First comparison difficulty
First difficulty
First difficulty
First difficulty

\[ F_1 \]

\[ F_2 \]

\[ F_3 \]
First difficulty
First difficulty
First difficulty
Second difficulty

- \( n \)-surface: subclass of connected orders
  - \( n \)-surface, \( n > 0 \), \( \theta(x) \setminus \{x\} \) \((n - 1)\)-surface
- Recursive definition

- \( n-G \)-maps
  - Constructive definition

⇒ How to characterize a subclass of \( n-G \)-maps equivalent to \( n \)-surfaces?
Methodology

$n$-surfaces $\iff$ subclass of $n$-$G$-maps
Methodology

$n$-surfaces

$\subseteq$

subclass of

$\subseteq$

$n$-$G$-maps

subclass of

incidence graphs
Augmented Incidence Graphs (AIG)

- Incidence graphs of subdivided $d$-manifolds (Brisson 89)
  - subdivided $d$-manifolds $\subset$ quasi-manifold
  - each cell belongs to at least one maximal chain
  - local property called `switch` property:
Augmented Incidence Graphs (AIG)

- Incidence graphs of subdivided $d$-manifolds (Brisson 89)
- subdivided $d$-manifolds $\subset$ quasi-manifold
- each cell belongs to at least one maximal chain
- local property called \textit{switch} property:

\[
\begin{align*}
C^i & \rightarrow C^{i+1} \\
C^i & \leftarrow C^{i-1} \\
C'^i & \rightarrow X \\
C'^i & \leftarrow X \\
C''^i & \rightarrow C''^i
\end{align*}
\]
Augmented Incidence Graphs (AIG)

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  - local property called switch property: $\Rightarrow$ allows to define involutions between maximal chains of the graph
Augmented Incidence Graphs (AIG)

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- local property called switch property :
  $\Rightarrow$ allows to define involutions between maximal chains of the graph

$\Rightarrow$ But no complete characterization of such graphs
AIG and $n$-surfaces

- Recursive characterization of AIG
  - an incidence graph which is everywhere an AIG also is an AIG
  - an AIG is locally everywhere an AIG
- an AIG of dimension 0 is isomorphic to a 0-surface

$\Rightarrow$ Equivalence between $AIG$ and $n$-surface

- Note: proof not fully completed in the paper
$n$-surface

Consequence: switch property on $n$-surfaces
\( n \text{-surface} \)

Consequence: switch property on \( n \)-surfaces
$n$-surface

Consequence: switch property on $n$-surfaces
Consequence: switch property on \( n \)-surfaces
AIG and $n$-G-maps
AIG and $n$-$G$-maps
AIG and \( n-G \)-maps
Conclusion and Future work

**Achievement:**

- Characterization of a subclass of $n$-$G$-maps equivalent to $n$-surfaces

**Future work:**

- Effectively use this equivalence
- Study $n$-$G$-maps with boundary, oriented or not
  - Define such notions on orders
  - Focus on a wider range of objects
  - Chains of maps (Elter, Lienhardt)