Predicate Transformers, (co)Monads and Resolutions

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Abstract. This short note contains random thoughts about a factorization theorem for closure/interior operators on a powerset which is reminiscent to the notion of *resolution* for a monad/comonad. The question originated from formal topology but is interesting in itself.
The result holds constructively (even if it classically has several variations); but usually not predicatively (in the sense that the interpolant will no be given by a set). For those not familiar with predicativity issues, we look at a "classical" version where we bound the size of the interpolant.

Introduction

A very general theorem states that any monotonic operator $F : \mathcal{P}(X) \to \mathcal{P}(Y)$ can be factorized in the form $\mathcal{P}(X) \to \mathcal{P}(Z) \to \mathcal{P}(Y)$, where Z is an appropriate set; and the first predicate transformer commutes with arbitrary unions and the second commutes with arbitrary intersections.

We prove similar factorization for interior and closure operators on a powerset; the idea being to "resolve" the operator as is usually done for (co)monad in categories. We then look at the constructive version of those factorizations.

1 Relations and Predicate Transformers

We start by introducing the basic notions:

Definition 1. *If X and Y are sets, a (binary) relation between X and Y is a subset of the cartesian product $X \times Y$. The* converse *of a relation $r \subseteq X \times Y$ is the relation $r^\sim \subseteq Y \times X$ defined as $(y, x) \in r^\sim \equiv (x, y) \in r$.*

A predicate transformer *from X to Y is an operator from the powerset $\mathcal{P}(X)$ to the powerset $\mathcal{P}(Y)$.*

Since most of our predicate transformers will be monotonic (with respect to inclusion), we drop the adjective when no confusion is possible.

Definition 2. *Suppose r is a relation between X and Y; we define two monotonic predicate transformers from Y to X:*

$$\begin{array}{rl} \langle r \rangle : \mathcal{P}(Y) \rightarrow \mathcal{P}(X) & \\ V \quad \mapsto & \{x \in X \mid (\exists y \in Y)\ (x,y) \in r \ \&\ y \in V\} \end{array}$$

and

$$\begin{array}{rl} [r] : \mathcal{P}(Y) \rightarrow \mathcal{P}(X) & \\ V \quad \mapsto & \{x \in X \mid (\forall y \in Y)\ (x,y) \in r \Rightarrow y \in V\}\ ; \end{array}$$

and an antitonic predicate transformer:

$$\begin{array}{rl} \lfloor r \rceil : \mathcal{P}(Y) \rightarrow \mathcal{P}(X) & \\ V \quad \mapsto & \{x \in X \mid (\forall y \in V)\ (x,y) \in r\}\ . \end{array}$$

Concerning notation:

- $\langle r \rangle$ and $[r]$ are somewhat common in the refinement calculus, even though the main reference ([1]) uses $\{r\}$ instead of $\langle r \rangle$. The problem is that this clashes with set theoretic notation.
- In [2], Birkhoff uses $V^{\leftarrow}$ for $\lfloor r \rceil$, but this supposes that r is clear from the context. (The notation $V^{\rightarrow}$ would then be $\lfloor r^{\sim} \rceil(U)$.)
- The linear logic community would use $V^{\perp}$ for the same thing, but this also supposes that the relation is called "$\perp$".

We will always be in a "typed" context; *i.e.* subsets will always be subset of some ambient set. We write $\neg$ for complementation with respect to that ambient set.

Lemma 1. *Suppose r is a relation between X and Y; we have:*

1. $\langle r \rangle \cdot \neg = \neg \cdot [r]$;
2. $\langle \neg r \rangle = \neg \cdot \lfloor r \rceil$. *(where $(x,y) \in \neg r \equiv (x,y) \notin r$)*

A very interesting property is the following Galois connections:

Lemma 2. *Suppose r is a relation between X and Y; then $\langle r \rangle \vdash [r]$, and $\lfloor r \rceil$ is Galois-connected to itself:*

1. $\langle r \rangle(V) \subseteq U \Leftrightarrow V \subseteq [r^{\sim}](U)$;
2. $U \subseteq \lfloor r \rceil(V) \Leftrightarrow V \subseteq \lfloor r^{\sim} \rceil(U)$.

Those predicate transformers satisfy:

Lemma 3. *If r is a relation between X and Y, then*

- *$\langle r \rangle$ commutes with arbitrary unions; (i.e. it is a sup-lattice morphism[3])*
- *$[r]$ commutes with arbitrary intersections; (i.e. it is an inf-lattice morphism)*
- *$\lfloor r \rceil$ transforms arbitrary unions into intersections.*

[3] All of our lattices are complete, so we do not bother writing "complete" all the time...

and moreover:

– *any sup-lattice morphism from* $\mathcal{P}(Y)$ *to* $\mathcal{P}(X)$ *is of the form* $\langle r\rangle$ *for some* $r \subseteq X \times Y$*;*
– *any inf-lattice morphism from* $\mathcal{P}(Y)$ *to* $\mathcal{P}(Y)$ *is of the form* $[r]$ *for some* $r \subseteq X \times Y$*;*
– *any predicate transformer from* Y *to* X *taking arbitrary unions to intersections is of the form* $\lfloor r\rceil$ *for some* $r \subseteq X \times Y$*.*

Proof. The sup-lattice part is easy; and the rest is an application of Lemma 1. □

Just like it is possible to factorize any relation as the composition of a total function and the inverse of a total function, it is possible to factorize an monotonic predicate transformer as the composition of a $\langle r\rangle$ and a $[s]$ (see [3] for a detailed categorical construction).

Proposition 1. *Suppose* F *is a monotonic predicate transformer from* X *to* Y*; then there is an* X' *and there are relations* $s \subseteq X' \times X$ *and* $r \subseteq Y \times X'$ *such that* $F = \langle r\rangle \cdot [s]$*.*

Proof. Let F be a monotonic predicate transformer, and define $X' = \mathcal{P}(X)$ together with $(U, x) \in s \equiv x \in U$ and $(y, U) \in r \equiv y \in F(U)$. We have:

$$
\begin{aligned}
& y \in \langle r\rangle \cdot [s](U) \\
& \quad \Leftrightarrow \; \{ \text{ definition of } \langle r\rangle \; \} \\
& (\exists V \in X')\ (y, V) \in r \;\&\; V \in [s](U) \\
& \quad \Leftrightarrow \; \{ \text{ definition of } X' \text{ and } r \; \} \\
& (\exists V \subseteq X)\ y \in F(V) \;\&\; V \in [s](U) \\
& \quad \Leftrightarrow \; \{ \text{ definition of } [s] \; \} \\
& (\exists V \subseteq X)\ y \in F(V) \;\&\; (\forall x)\ (V, x) \in s \Rightarrow x \in U \\
& \quad \Leftrightarrow \; \{ \text{ definition of } s \; \} \\
& (\exists V \subseteq X)\ y \in F(V) \;\&\; (\forall x)\ V \subseteq U \\
& \quad \Leftrightarrow \; \{ \text{ since } F \text{ is monotonic } \} \\
& y \in F(U)
\end{aligned}
$$

The result thus holds, but the proof doesn't bring much information... □

And as a direct application of Lemma 1:

Corollary 1. *Any monotonic predicate transformer can be factorized as a* $[r]\cdot\langle s\rangle$ *or as a* $\lfloor r\rceil \cdot \lfloor s\rceil$*.*
Similarly, any antitonic predicate transformer can be written has one of $\lfloor r\rceil \cdot [s]$*,* $\lfloor r\rceil \cdot \langle s\rangle$*,* $\langle r\rangle \cdot \lfloor s\rceil$ *or* $[r] \cdot \lfloor s\rceil$*.*

2 Interior and Closure Operators

Definition 3. *If $(X, \leq)$ is a partial order, we say that $F : X \to X$ is an interior operator if:*

- *F is monotonic;*
- *F is contractive: $F(x) \leq x$;*
- *$F(x) \leq FF(x)$.*

We say that it is a closure operator if:

- *F is monotonic;*
- *F is expansive: $x \leq F(x)$;*
- *$FF(x) \leq F(x)$.*

It is well known that the composition of two Galois connected operators yield interior/closure operators, so that we have:

Lemma 4. *If r is a relation between X and Y, then*

- *$\langle r\rangle \cdot [r^\sim]$ is an interior operator on $\mathcal{P}(X)$;*
- *$[r] \cdot \langle r^\sim\rangle$ is a closure operator on $\mathcal{P}(X)$;*
- *$\lfloor r\rceil \cdot \lfloor r^\sim\rceil$ is a closure operator on $\mathcal{P}(X)$.*

Some other consequences of the Galois connection are listed below:

- *$\langle r\rangle \cdot [r^\sim] \cdot \langle r\rangle = \langle r\rangle$;*
- *$[r] \cdot \langle r^\sim\rangle \cdot [r] = [r]$;*
- *$\lfloor r\rceil \cdot \lfloor r^\sim\rceil \cdot \lfloor r\rceil = \lfloor r\rceil$.*

The problem is now to mimic Proposition 1.

Definition 4. *If F is an interior operator on $\mathcal{P}(X)$, a* resolution *for F is given by a set Y (called the interpolant) together with a relation $r \subseteq Y \times X$ such that $F = \langle r\rangle \cdot [r^\sim]$.*

Proposition 2. *If F is an interior operator on $\mathcal{P}(X)$, then it has a resolution.*

The proof relies on the following lemma:

Lemma 5. *Let F be an interior operator on a complete sup-lattice $(X, \leq, \bigvee)$; write $\mathbf{Fix}(F)$ for the collection of fixed-point for F. We have that $(\mathbf{Fix}(F), \leq, \bigvee)$ is a complete sup-lattice; and for any $x \in X$*

$$F(x) = \bigvee \left\{ y \in \mathbf{Fix}(F) \;\mid\; y \leq x \right\} \ .$$

Proof. That $(\mathbf{Fix}(F), \leq, \bigvee)$ is a complete sup-lattice is left as an easy exercise; for the second point, let $x \in X$;

- we know that $F(x)$ is a fixed point of F, and that $F(x) \leq x$. This implies that $F(x) \in \{y \in \mathbf{Fix}(F) \mid y \leq x\}$; and so $F(x) \leq \bigvee\{y \in \mathbf{Fix}(F) \mid y \leq x\}$;

- suppose $y \in \mathbf{Fix}(F)$ and $y \leq x$; this implies that $F(y) \leq F(x)$, *i.e.* that $y \leq F(x)$. We can conclude that $\bigvee\{y \in \mathbf{Fix}(F) \mid y \leq x\} \leq F(x)$.

$\square$

Proof (of proposition 2). Suppose F is an interior operator on $\mathcal{P}(X)$; define $Y = \mathbf{Fix}(F)$ and $(U, x) \in r \equiv x \in U$. We have:

$$
\begin{aligned}
& x \in \langle r \rangle \cdot [r^{\sim}](U) \\
& \quad \Leftrightarrow \ \{ \text{ definition of } r \text{ } \} \\
& \big(\exists V \in \mathbf{Fix}(F)\big)\ x \in V \ \&\ (\forall y)\, y \in V \Rightarrow y \in U \\
& \quad \Leftrightarrow \\
& \big(\exists V \in \mathbf{Fix}(F)\big)\ x \in V \ \&\ V \subseteq U \\
& \quad \Leftrightarrow \\
& x \in \bigcup\{V \in \mathbf{Fix}(F) \mid V \subseteq U\} \\
& \quad \Leftrightarrow \ \{ \text{ Lemma 5 } \} \\
& x \in F(U)
\end{aligned}
$$

which concludes the proof. $\square$

Just like for Proposition 1, the statement of the theorem is interesting, but the proof hardly tells us anything about the structure of F. To gain a little more information about F, we will try to "bound" the size of the interpolant set Y.

Definition 5. *If $(X, \leq, \bigvee)$ is a complete sup-lattice, we say that a family $(x_i)_{i \in I}$ of element of X is a basis if, for any $y \in X$, we have*

$$y = \bigvee\{x_i \mid x_i \leq y\} \ .$$

Corollary 2. *Suppose F is an interior operator on $\mathcal{P}(X)$; if $\big(\mathbf{Fix}(F), \subseteq, \bigcup\big)$ has a basis of cardinality κ, then we can find a resolution of F with an interpolant Y of cardinality κ.*

Proof. It is easy to see that in the above proof of Proposition 2, we can replace $\mathbf{Fix}(F)$ by a basis of $\big(\mathbf{Fix}(F), \subseteq, \bigcup\big)$. $\square$

In particular, if there is a basis of $\mathbf{Fix}(F)$ which has cardinality less that the cardinality of X; we can use X as the interpolant and use a relation $r \subseteq X \times X$ to obtain a resolution of F.

We now show that this result is optimal:

Lemma 6. *Let F be an interior operator on $\mathcal{P}(X)$; and suppose there are no basis of $\mathbf{Fix}(F)$ of cardinality κ; then there is no interpolant of cardinality less than κ.*

Proof. To show that, we will construct a basis of $\mathbf{Fix}(F)$ indexed by any interpolant for F. Suppose Y and r form a resolution for F. For any $y \in Y$, define $U_y \equiv \langle r \rangle\{y\}$. We will show that $(U_y)_{y \in Y}$ is a basis for $\mathbf{Fix}(F)$.

Each U_y is a fixed point for F:

$$\begin{aligned}
U_y &= \langle r\rangle\{y\} && \{\text{definition}\}\\
&= \langle r\rangle \cdot [r^\sim] \cdot \langle r\rangle\{y\} && \{\text{second part of Lemma 4}\}\\
&= F \cdot \langle r\rangle\{y\} && \{r \text{ is a resolution of } F\}\\
&= F(U_y) && \{\text{definition}\}
\end{aligned}$$

Let U be a fixed point of F:

$$\begin{aligned}
U &= \langle r\rangle \cdot [r^\sim](U) && \{U \text{ is a fixed point of } F\}\\
&= \langle r\rangle\big(\bigcup\big\{\{y\} \ \mid\ y \in [r^\sim](U)\big\}\big) && \{\text{basic logic}\}\\
&= \bigcup\big\{\langle r\rangle\{y\} \ \mid\ \{y\} \subseteq [r^\sim](U)\big\} && \{\langle r\rangle \text{ commutes with unions}\}\\
&= \bigcup\big\{\langle r\rangle\{y\} \ \mid\ \langle r\rangle\{y\} \subseteq U\big\} && \{\text{Galois connection between } \langle r\rangle \text{ and } [r^\sim]\}\\
&= \bigcup\{U_y \ \mid\ U_y \subseteq U\} && \{\text{definition}\}
\end{aligned}$$

which concludes the proof that $(U_y)_{y\in Y}$ is a basis for $\mathbf{Fix}(F)$. □

Let's look at an example of interior operator on $\mathcal{P}(X)$ which *cannot* be resolved using X as an interpolant. Let X be a countable infinite set (natural numbers for example); and define $F : \mathcal{P}(X) \to \mathcal{P}(X)$ as follows:

$$F(U) = \begin{cases} \emptyset & \text{if } U \text{ is finite}\\ U & \text{if } U \text{ is infinite}\end{cases}$$

The sup-lattice $\mathbf{Fix}(F)$ is given by the collection of infinite subsets of X; and this lattice doesn't have a countable basis. To prove that, it is enough to do it for any particular countable infinite set. Take C to be the set of finite strings over $\{0, 1\}$. If α is an infinite string of 0's and 1's, define $U_\alpha \subseteq C$ to be the set of finite prefixes of α. Each U_α is an element of $\mathbf{Fix}(F)$; but no countable family of infinite subsets can "generate" all the U_α: since $\alpha \neq \beta$ implies that $U_\alpha \cap U_\beta$ is finite, if $V_i \subseteq U_\alpha$ and $V_j \subseteq U_\beta$ then $i \neq j$. In other words, a family which generates all the U_α's needs to have the cardinality of the collection of the U_α's, *i.e.* uncountable.

Using Lemma 1, we can now extend all what has been done for interior operators for closure operators: if F is a closure operator on $\mathcal{P}(X)$, then $\neg \cdot F \cdot \neg$ is an interior operator on $\mathcal{P}(X)$.

Corollary 3. *If F is a closure operator on $\mathcal{P}(X)$, then it has a resolution as a composition $[r] \cdot \langle r^\sim\rangle$ or as $\lfloor r\rceil \cdot \lfloor r^\sim\rceil$.*

As for interior operators, the possible cardinalities of the interpolant are given by the cardinalities of the bases for the inf-lattice $\mathbf{Fix}(F)$.

In particular, for linear logicians, it is not the case that any closure operator can be written as a biorthogonal...

3 Comonad and Monads

It seems that the traditional way to look at monads in a category is to see them as a kind of generalized monoid; at least in my part of the world. Another view[4]

[4] which has given me a much better understanding of what (co)monads are

is to view them as a generalization of closure operators. This is the view taken in the introduction of [4].

Definition 6. *A monad on a category $\mathcal{C}$ is a morphism $F : \mathcal{C} \to \mathcal{C}$ together with two natural transformations $\eta : \mathbf{Id}_{\mathcal{C}} \to F$ and $\mu : FF \to F$ s.t. some diagram commute.*

If one takes the partial order category $\mathcal{P}(X)$, then a monad is an operator on $\mathcal{P}(X)$ s.t.:

- it acts on morphisms: if $i : U \subseteq V$ then $F_i : F(U) \subseteq F(V)$; *i.e.* F is monotonic;
- there is a natural transformation $\eta_U : U \subseteq F(U)$;
- there is a natural transformation $\mu_U : FF(U) \subseteq F(U)$.

All the "coherence" conditions are trivially satisfied in a partial order category, since every diagram commute! What we've just shown is that a monad on $\mathcal{P}(X)$ is nothing more than a closure operator on $\mathcal{P}(X)$; and vice and versa. Similarly, a comonad corresponds to an interior operator.

In categories, a *resolution* for a monad corresponds to factorizing the functor F as the composition of two adjoint functors. Adjointness $H \vdash G$ between the functors $G : \mathcal{D} \to \mathcal{C}$ and $H : \mathcal{C} \to \mathcal{D}$ in a locally small category means that $\mathcal{C}[A, G(B)] \simeq \mathcal{D}[H(A), B]$ which, in the case of a partial order category simplifies to "$U \subseteq G(V)$ iff $H(U) \subseteq V$" *i.e.* is exactly the Galois connection $H \vdash G$.

Category theory tells us that there always is a resolution since we have two degenerate resolutions: something like $F = \mathbf{Id} \cdot F$ and $F = F \cdot \mathbf{Id}$. The first one is given by the Eleinberg-Moore category, and is available if one restrict the interpolant category to partial orders: if F is a closure operator on X, take $\mathcal{E}(X, F)$ to be the the collection of fixed points for F, with the ordering inherited from X.[5] We have two monotonic operators $F : X \to \mathcal{E}(X, F)$ and $\mathbf{Id} : \mathcal{E}(X, F) \to X$ which are adjoint: $F \vdash \mathbf{Id}$.

$x \leq \mathbf{Id}(f)$

$\Rightarrow$ { F is monotonic }

$F(x) \leq F(f)$

$\Leftrightarrow$ { $f \in \mathcal{E}(X, F)$, *i.e.* f is a fixed point for F }

$F(x) \leq f$

and $F(x) \leq f \Rightarrow x \leq f = \mathbf{Id}(f)$ since $x \leq F(x)$.

The second resolution ($F \cdot \mathbf{Id}$) is obtained via the Kleisli category, but it doesn't seem to make much sense in a partial order setting. What is important to us is that those resolutions are trivial and uninteresting. More abstract nonsense states that there is a category of resolutions for any given monad; and that those two trivial resolutions are respectively initial/terminal. What we have done with Proposition 2 and Corollary 3 corresponds to finding a non-trivial *resolution* for

[5] In a partial order, an algebra for F is just a post fixed point: $x \leq F(x)$. If F is a closure, then it is also a fixed point for F.

the monad corresponding to the interior/closure operator. It is even more than a resolution, since the interpolant is itself a complete and cocomplete category (and one of the functors preserves limits while the other one preserves colimits; but this is a general fact about adjoints). The feeling is that this resolution lies "exactly in the middle" between the initial and terminal resolutions. I don't know if this kind of "strong" resolution has been considered in category theory.[6]

The resolution constructed here is quite different from the Eleinberg-Moore resolution (even though it uses the fixed-points as a basis). We do not construct functors from $\mathcal{P}(X)$ to $\mathcal{E}(X, F)$ and back; but from $\mathcal{P}(X)$ to $\mathcal{P}\big(\mathcal{E}(X, F)\big)$ and back. In particular, as noted above, the interpolant is complete and cocomplete;[7] which is not the case for the Eleinberg-Moore category: fixed points for an interior are closed under unions but not under intersections; and conversely for closure operators.

4 Revisiting Section 2 in a Constructive Setting

4.1 Impredicative

In the previous section we used Lemma 1 to generalize results on interior to closures. It is thus natural to ask whether (1) we can make the original proof constructive; (2) we can avoid using this lemma and prove the result for closure constructively. I will not into the details but just provide some hints about that.

- Galois connections from Lemma 2 are constructive.
- Lemma 3: the first part is trivial. The second part is easy: if F commutes with unions, take $(x, y) \in r$ iff $y \in F\{x\}$; if F commutes with intersections, take $(x, y) \in r$ iff $(\forall U)\, y \in F(U) \Rightarrow x \in U$; and if F transforms unions into intersections, take $(x, y) \in r$ iff $y \in F\{x\}$.
- Proposition 1: the proof is constructive.
- Lemma 3 is constructive.
- Lemma 5 and the corresponding lemma for closure operator are constructive.
- Proposition 2 is constructive.
- we can mimic the proof of Proposition 2 to obtain a resolution of a closure operator as $F = \lfloor r \rceil \cdot \lfloor r^\sim \rceil$, but *not* to obtain a resolution as $F = [r] \cdot \langle r^\sim \rangle$.
- I doubt we can constructively obtain a resolution of a closure operator as $F = [r] \cdot \langle r^\sim \rangle$.
- all of the lemmas about the size of interpolant are constructive at least if we read then as "if B is a basis for $\mathbf{Fix}(F)$ then we can use B as an interpolant".

As a proof of concept, all this (except the last point) has been proved in the proof assistant COQ.[8]

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[6] *i.e.* if $\mathcal{C}$ is a complete and cocomplete category and F is a monad on $\mathcal{C}$, a "strong resolution" is a resolution with a complete and cocomplete interpolant. Any reference to something similar in the literature would be most welcome.

[7] *i.e.* is a complete lattice since we deal with partial orders

[8] proof scripts available from `http://iml.univ-mrs.fr/~hyvernat/academics.html`

4.2 Predicative

In a predicative setting like Martin-Löf type theory (see [5]) or CZF (constructive ZF, see [6]) set theory, many of the results are not provable. The reason being that we do not allow quantification on a power-set. The main result about resolution would become something like:

Proposition 3. *Suppose* $\mathbf{Fix}(F)$ *(proper type) has a set-indexed basis, then* F *as a resolution as* $\langle r\rangle\cdot[r^\sim]$ *(if* F *is an interior) or as* $\lfloor r\rceil\cdot\lfloor r^\sim\rceil$ *(if* F *is a closure). If* F *as a resolution, then* $\mathbf{Fix}(F)$ *as a set-index basis.*

Note that having a set-indexed basis is equivalent to being "set presented" in the terminology of P. Aczel ([6, 7]). Note that in the case of an interior operator, this implies that the predicate transformer is "set-based" (in the sense that it has a factorization as in Proposition 1, where the interpolant X' is a set). It doesn't seem that the existence of a resolution for a closure operator implies that the original predicate transformer is itself set-presented.

Conclusion

Nothing revolutionary has really been done, but the statement of Propositions 1 and 2 is, in an abstract setting, quite neat. However, as the proofs show, this is mostly abstract nonsense. The best example is probably Proposition 1, where $y \in F(U)$ is factorized as "there is a V such that $s \in F(V)$ and $V \subseteq U$". The proof of Proposition 2 is slightly subtler, but is hardly interesting. In the end, the most interesting and informative thing is probably Lemma 6, in its "positive" version: if F has Y as an interpolant, then $\mathbf{Fix}(F)$ has a basis indexed by Y, which is hardly a breakthrough in mathematics...

I do nevertheless hope that it might interest some people, since while Proposition 1 is known to many (especially the refinement calculus people), it seems that Proposition 2 isn't stated anywhere. I also hope the link between monad and closure operator (together with the Eleinberg-Moore category being the partial order of fixed points) will gain in popularity, as I see it as a much better way of seeing monads, at least as far as intuition is concerned.

A final word about the motivation for this: the starting point was the question about whether it is possible to represent any "basic topology" (see the forthcoming [8]) as the formal side of a basic pair, impredicatively speaking. A basic topology is a structure $(X, \mathcal{A}, \mathcal{J})$ where X is a set, and $\mathcal{A}$ and $\mathcal{J}$ are closure and interior operators on $\mathcal{P}(X)$ such that $\mathcal{A}(U)\between\mathcal{J}(V) \Rightarrow U\between(\mathcal{J}(V)$.[9] The answer is obviously *no* since the $\mathcal{A}$ and $\mathcal{J}$ arising from a formal pair are classically dual (*i.e.* $\mathcal{A}\cdot\neg = \neg\cdot\mathcal{J}$) and there are basic topologies which are provably not dual. The question then turned into: "can any interior operator be written as the formal interior of a basic pair?" and similar for closure operators. The answers are *yes* (Proposition 2) and *I don't know* (the constructive resolution of a closure is of the form $\lfloor r\rceil\cdot\lfloor r^\sim\rceil$, and not of the form $[r]\cdot\langle r^\sim\rangle$.)

[9] $U)\between V$ is the constructive version of $U \cap V \neq \emptyset$.

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