Predicate Transformers, (co)Monads and Resolutions

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Abstract. This short note contains random thoughts about a factorization theorem for closure/interior operators on a powerset which is reminiscent to the notion of resolution for a monad/comonad. The question originated from formal topology but is interesting in itself. The result holds constructively (even if it classically has several variations); but usually not predicatively (in the sense that the interpolant will no be given by a set). For those not familiar with predicativity issues, we look at a “classical” version where we bound the size of the interpolant.

Introduction

A very general theorem states that any monotonic operator $F : \mathcal{P}(X) \to \mathcal{P}(Y)$ can be factorized in the form $\mathcal{P}(X) \to \mathcal{P}(Z) \to \mathcal{P}(Y)$, where $Z$ is an appropriate set; and the first predicate transformer commutes with arbitrary unions and the second commutes with arbitrary intersections.

We prove similar factorization for interior and closure operators on a powerset; the idea being to “resolve” the operator as is usually done for (co)monad in categories. We then look at the constructive version of those factorizations.

1 Relations and Predicate Transformers

We start by introducing the basic notions:

\textbf{Definition 1.} If $X$ and $Y$ are sets, a (binary) relation between $X$ and $Y$ is a subset of the cartesian product $X \times Y$. The converse of a relation $r \subseteq X \times Y$ is the relation $r^\sim \subseteq Y \times X$ defined as $(y, x) \in r^\sim \iff (x, y) \in r$.

A predicate transformer from $X$ to $Y$ is an operator from the powerset $\mathcal{P}(X)$ to the powerset $\mathcal{P}(Y)$.

Since most of our predicate transformers will be monotonic (with respect to inclusion), we drop the adjective when no confusion is possible.
Definition 2. Suppose \( r \) is a relation between \( X \) and \( Y \); we define two monotonic predicate transformers from \( Y \) to \( X \):

\[
\langle r \rangle : \mathcal{P}(Y) \rightarrow \mathcal{P}(X) \quad V \mapsto \{ x \in X \mid (\exists y \in Y) (x, y) \in r \land y \in V \}
\]

and

\[
[r] : \mathcal{P}(Y) \rightarrow \mathcal{P}(X) \quad V \mapsto \{ x \in X \mid (\forall y \in Y) (x, y) \in r \Rightarrow y \in V \} ;
\]

and an antitonic predicate transformer:

\[
\lceil r \rceil : \mathcal{P}(Y) \rightarrow \mathcal{P}(X) \quad V \mapsto \{ x \in X \mid (\forall y \in V) (x, y) \in r \} .
\]

Concerning notation:

- \( \langle r \rangle \) and \( [r] \) are somewhat common in the refinement calculus, even though the main reference ([1]) uses \( \{ r \} \) instead of \( \langle r \rangle \). The problem is that this clashes with set theoretic notation.
- In [2], Birkhoff uses \( V^- \) for \( [r] \), but this supposes that \( r \) is clear from the context. (The notation \( V^- \) would then be \( [r^-](U) \).)
- The linear logic community would use \( V^- \) for the same thing, but this also supposes that the relation is called “\( \bot \)”. We will always be in a “typed” context; i.e. subsets will always be subset of some ambient set. We write \( \neg \) for complementation with respect to that ambient set.

Lemma 1. Suppose \( r \) is a relation between \( X \) and \( Y \); we have:

1. \( \langle r \rangle \cdot \neg = \neg \cdot [r] ; \)
2. \( (\neg r) = \neg \cdot [r] . \) \hspace{1cm} (where \( (x, y) \in \neg r \equiv (x, y) \notin r \))

A very interesting property is the following Galois connections:

Lemma 2. Suppose \( r \) is a relation between \( X \) and \( Y \); then \( \langle r \rangle \vdash [r] \), and \( [r] \) is Galois-connected to itself:

1. \( \langle r \rangle (V) \subseteq U \iff V \subseteq [r^-](U) ; \)
2. \( U \subseteq [r](V) \iff V \subseteq [r^-](U) . \)

Those predicate transformers satisfy:

Lemma 3. If \( r \) is a relation between \( X \) and \( Y \), then

- \( \langle r \rangle \) commutes with arbitrary unions; \hspace{1cm} (i.e. it is a sup-lattice morphism\(^3\))
- \( [r] \) commutes with arbitrary intersections; \hspace{1cm} (i.e. it is an inf-lattice morphism)
- \( \lceil r \rceil \) transforms arbitrary unions into intersections.

\(^3\) All of our lattices are complete, so we do not bother writing “complete” all the time...
and moreover:

- any sup-lattice morphism from $\mathcal{P}(Y)$ to $\mathcal{P}(X)$ is of the form $\langle r \rangle$ for some $r \subseteq X \times Y$;
- any inf-lattice morphism from $\mathcal{P}(Y)$ to $\mathcal{P}(Y)$ is of the form $[r]$ for some $r \subseteq X \times Y$;
- any predicate transformer from $Y$ to $X$ taking arbitrary unions to intersections is of the form $[r]$ for some $r \subseteq X \times Y$.

**Proof.** The sup-lattice part is easy; and the rest is an application of Lemma 1. \(\square\)

Just like it is possible to factorize any relation as the composition of a total function and the inverse of a total function, it is possible to factorize a monotonic predicate transformer as the composition of a $\langle r \rangle$ and a $[s]$ (see [3] for a detailed categorical construction).

**Proposition 1.** Suppose $F$ is a monotonic predicate transformer from $X$ to $Y$; then there is an $X'$ and there are relations $s \subseteq X' \times X$ and $r \subseteq Y \times X'$ such that $F = \langle r \rangle \cdot [s]$.

**Proof.** Let $F$ be a monotonic predicate transformer, and define $X' = \mathcal{P}(X)$ together with $(U, x) \in s \iff x \in U$ and $(y, U) \in r \iff y \in F(U)$. We have:

$$y \in \langle r \rangle \cdot [s](U)$$

$$\iff \{ \text{definition of } \langle r \rangle \}$$

$$\exists V \in X' \ (y, V) \in r \ & V \in [s](U)$$

$$\iff \{ \text{definition of } X' \text{ and } r \}$$

$$\exists V \subseteq X \ y \in F(V) \ & V \in [s](U)$$

$$\iff \{ \text{definition of } [s] \}$$

$$\exists V \subseteq X \ y \in F(V) \ & (\forall x) \ (V, x) \in s \Rightarrow x \in U$$

$$\iff \{ \text{definition of } s \}$$

$$\exists V \subseteq X \ y \in F(V) \ & (\forall x) \ V \subseteq U$$

$$\iff \{ \text{since } F \text{ is monotonic } \}$$

$$y \in F(U)$$

The result thus holds, but the proof doesn’t bring much information... \(\square\)

And as a direct application of Lemma 1:

**Corollary 1.** Any monotonic predicate transformer can be factorized as a $[r] \cdot \langle s \rangle$ or as a $[r] \cdot [s]$.

Similarly, any antitone predicate transformer can be written has one of $[r] \cdot [s]$, $[r] \cdot \langle s \rangle$, $\langle r \rangle \cdot [s]$ or $[r] \cdot [s]$. 
2 Interior and Closure Operators

Definition 3. If \((X, \leq)\) is a partial order, we say that \(F : X \rightarrow X\) is an interior operator if:

- \(F\) is monotonic;
- \(F\) is contractive: \(F(x) \leq x\);
- \(F(x) \leq FF(x)\).

We say that it is a closure operator if:

- \(F\) is monotonic;
- \(F\) is expansive: \(x \leq F(x)\);
- \(FF(x) \leq F(x)\).

It is well known that the composition of two Galois connected operators yield interior/closure operators, so that we have:

Lemma 4. If \(r\) is a relation between \(X\) and \(Y\), then

- \(\langle r \rangle \cdot [r^\sim]\) is an interior operator on \(P(X)\);
- \([r] \cdot \langle r^\sim \rangle\) is a closure operator on \(P(X)\);
- \([r^\sim] \cdot [r^\sim]\) is a closure operator on \(P(X)\).

Some other consequences of the Galois connection are listed below:

- \(\langle r \rangle \cdot [r^\sim] \cdot \langle r \rangle = \langle r \rangle\);
- \([r] \cdot \langle r^\sim \rangle \cdot [r] = [r]\);
- \([r] \cdot [r^\sim] \cdot [r] = [r]\).

The problem is now to mimic Proposition 1.

Definition 4. If \(F\) is an interior operator on \(P(X)\), a resolution for \(F\) is given by a set \(Y\) (called the interpolant) together with a relation \(r \subseteq Y \times X\) such that \(F = \langle r \rangle \cdot [r^\sim]\).

Proposition 2. If \(F\) is an interior operator on \(P(X)\), then it has a resolution.

The proof relies on the following lemma:

Lemma 5. Let \(F\) be an interior operator on a complete sup-lattice \((X, \leq, \bigvee)\); write \(\text{Fix}(F)\) for the collection of fixed-point for \(F\). We have that \((\text{Fix}(F), \leq, \bigvee)\) is a complete sup-lattice; and for any \(x \in X\)

\[
F(x) = \bigvee \{ y \in \text{Fix}(F) \mid y \leq x \}.
\]

Proof. That \((\text{Fix}(F), \leq, \bigvee)\) is a complete sup-lattice is left as an easy exercise; for the second point, let \(x \in X\);

- we know that \(F(x)\) is a fixed point of \(F\), and that \(F(x) \leq x\). This implies that \(F(x) \in \{ y \in \text{Fix}(F) \mid y \leq x \}\); and so \(F(x) \leq \bigvee \{ y \in \text{Fix}(F) \mid y \leq x \}\);
– suppose \( y \in \text{Fix}(F) \) and \( y \leq x \): this implies that \( F(y) \leq F(x) \), i.e. that \( y \leq F(x) \). We can conclude that \( \bigvee \{ y \in \text{Fix}(F) \mid y \leq x \} \leq F(x) \).

\[ \square \]

Proof (of proposition 2). Suppose \( F \) is an interior operator on \( \mathcal{P}(X) \); define \( Y = \text{Fix}(F) \) and \( (U, x) \in r \equiv x \in U \). We have:

\[
\begin{align*}
x \in \langle r \rangle \cdot [r^{-1}][U] \\
\Leftrightarrow \{ \text{definition of } r \} \\
(\exists V \in \text{Fix}(F)) \ x \in V \land (\forall y) y \in V \Rightarrow y \in U \\
\Leftrightarrow \ (\exists V \in \text{Fix}(F)) \ x \in V \land V \subseteq U \\
\Leftrightarrow \ x \in \bigcup \{ V \in \text{Fix}(F) \mid V \subseteq U \} \\
\Leftrightarrow \{ \text{Lemma 5} \} \\
x \in F(U)
\end{align*}
\]

which concludes the proof. \[ \square \]

Just like for Proposition 1, the statement of the theorem is interesting, but the proof hardly tells us anything about the structure of \( F \). To gain a little more information about \( F \), we will try to “bound” the size of the interpolant set \( Y \).

**Definition 5.** If \( (X, \leq, \bigvee) \) is a complete sup-lattice, we say that a family \( (x_i)_{i \in I} \) of element of \( X \) is a basis if, for any \( y \in X \), we have

\[
y = \bigvee \{ x_i \mid x_i \leq y \} .
\]

**Corollary 2.** Suppose \( F \) is an interior operator on \( \mathcal{P}(X) \); if \( (\text{Fix}(F), \subseteq, \bigcup) \) has a basis of cardinality \( \kappa \), then we can find a resolution of \( F \) with an interpolant \( Y \) of cardinality \( \kappa \).

**Proof.** It is easy to see that in the above proof of Proposition 2, we can replace \( \text{Fix}(F) \) by a basis of \( (\text{Fix}(F), \subseteq, \bigcup) \). \[ \square \]

In particular, if there is a basis of \( \text{Fix}(F) \) which has cardinality less that the cardinality of \( X \); we can use \( X \) as the interpolant and use a relation \( r \subseteq X \times X \) to obtain a resolution of \( F \).

We now show that this result is optimal:

**Lemma 6.** Let \( F \) be an interior operator on \( \mathcal{P}(X) \); and suppose there are no basis of \( \text{Fix}(F) \) of cardinality \( \kappa \); then there is no interpolant of cardinality less than \( \kappa \).

**Proof.** To show that, we will construct a basis of \( \text{Fix}(F) \) indexed by any interpolant for \( F \). Suppose \( Y \) and \( r \) form a resolution for \( F \). For any \( y \in Y \), define \( U_y \equiv \langle r \rangle \{ y \} \). We will show that \( (U_y)_{y \in Y} \) is a basis for \( \text{Fix}(F) \).
Each $U_y$ is a fixed point for $F$:

$$U_y = \langle r \rangle \{ y \} \quad \text{[definition]}$$

$$= \langle r \rangle \cdot [r^-] \cdot \langle r \rangle \{ y \} \quad \text{[second part of Lemma 4]}$$

$$= F \cdot \langle r \rangle \{ y \} \quad \{ r \text{ is a resolution of } F \}$$

$$= F(U_y) \quad \text{[definition]}$$

Let $U$ be a fixed point of $F$:

$$U = \langle r \rangle \cdot [r^-](U) \quad \{ U \text{ is a fixed point of } F \}$$

$$= \langle r \rangle \left( \bigcup \{ \{ y \} \mid y \in [r^-](U) \} \right) \quad \{ \langle r \rangle \text{ commutes with unions} \}$$

$$= \bigcup \{ \langle r \rangle \{ y \} \mid \langle r \rangle \{ y \} \subseteq U \} \quad \{ \text{Galois connection between } \langle r \rangle \text{ and } [r^-] \}$$

$$= \bigcup \{ U_y \mid U_y \subseteq U \} \quad \text{[definition]}$$

which concludes the proof that $(U_y)_{y \in Y}$ is a basis for $\text{Fix}(F)$. \qed

Let’s look at an example of interior operator on $\mathcal{P}(X)$ which cannot be resolved using $X$ as an interpolant. Let $X$ be a countable infinite set (natural numbers for example); and define $F : \mathcal{P}(X) \to \mathcal{P}(X)$ as follows:

$$F(U) = \begin{cases} \emptyset & \text{if } U \text{ is finite} \\ U & \text{if } U \text{ is infinite} \end{cases}$$

The sup-lattice $\text{Fix}(F)$ is given by the collection of infinite subsets of $X$; and this lattice doesn’t have a countable basis. To prove that, it is enough to do it for any particular countable infinite set. Take $C$ to be the set of finite strings over \{0, 1\}. If $\alpha$ is an infinite string of 0’s and 1’s, define $U_{\alpha} \subseteq C$ to be the set of finite prefixes of $\alpha$. Each $U_{\alpha}$ is an element of $\text{Fix}(F)$; but no countable family of infinite subsets can “generate” all the $U_{\alpha}$’s since $\alpha \neq \beta$ implies that $U_{\alpha} \cap U_{\beta}$ is finite, if $V_i \subseteq U_{\alpha}$ and $V_j \subseteq U_{\beta}$ then $i \neq j$. In other words, a family which generates all the $U_{\alpha}$’s needs to have the cardinality of the collection of the $U_{\alpha}$’s, i.e. uncountable.

Using Lemma 1, we can now extend all what has been done for interior operators for closure operators: if $F$ is a closure operator on $\mathcal{P}(X)$, then $\neg \cdot F \cdot \neg$ is an interior operator on $\mathcal{P}(X)$.

**Corollary 3.** If $F$ is a closure operator on $\mathcal{P}(X)$, then it has a resolution as a composition $[r] \cdot [r^-]$ or as $\neg \cdot F \cdot \neg$.

As for interior operators, the possible cardinalities of the interpolant are given by the cardinalities of the bases for the inf-lattice $\text{Fix}(F)$.

In particular, for linear logicians, it is not the case that any closure operator can be written as a biorthogonal...

## 3 Comonad and Monads

It seems that the traditional way to look at monads in a category is to see them as a kind of generalized monoid; at least in my part of the world. Another view\footnote{which has given me a much better understanding of what (co)monads are}...
is to view them as a generalization of closure operators. This is the view taken in the introduction of [4].

**Definition 6.** A monad on a category $\mathcal{C}$ is a morphism $F : \mathcal{C} \to \mathcal{C}$ together with two natural transformations $\eta : \text{Id}_\mathcal{C} \to F$ and $\mu : FF \to F$ s.t. some diagram commute.

If one takes the partial order category $\mathcal{P}(X)$, then a monad is an operator on $\mathcal{P}(X)$ s.t.:

- it acts on morphisms: if $i : U \subseteq V$ then $F_i : F(U) \subseteq F(V)$; i.e. $F$ is monotonic;
- there is a natural transformation $\eta_U : U \subseteq F(U)$;
- there is a natural transformation $\mu_U : FF(U) \subseteq F(U)$.

All the “coherence” conditions are trivially satisfied in a partial order category, since every diagram commute! What we’ve just shown is that a monad on $\mathcal{P}(X)$ is nothing more than a closure operator on $\mathcal{P}(X)$; and vice and versa. Similarly, a comonad corresponds to an interior operator.

In categories, a resolution for a monad corresponds to factorizing the functor $F$ as the composition of two adjoint functors. Adjointness $H \vdash G$ between the functors $G : D \to C$ and $H : C \to D$ in a locally small category means that $\mathcal{C}[A, G(B)] \simeq \mathcal{D}[H(A), B]$ which, in the case of a partial order category simplifies to “$U \subseteq G(V)$ iff $H(U) \subseteq V$” i.e. is exactly the Galois connection $H \vdash G$.

Category theory tells us that there always is a resolution since we have two degenerate resolutions: something like $F = \text{Id} \cdot F$ and $F = F \cdot \text{Id}$. The first one is given by the Eilenberg-Moore category, and is available if one restrict the interpolant category to partial orders: if $F$ is a closure operator on $X$, take $E(X, F)$ to be the collection of fixed points for $F$, with the ordering inherited from $X$.\(^5\) We have two monotonic operators $F : X \to E(X, F)$ and $\text{Id} : E(X, F) \to X$ which are adjoint: $F \dashv \text{Id}$.

\[^5\] In a partial order, an algebra for $F$ is just a post fixed point: $x \leq F(x)$. If $F$ is a closure, then it is also a fixed point for $F$.\]
the monad corresponding to the interior/closure operator. It is even more than a resolution, since the interpolant is itself a complete and cocomplete category (and one of the functors preserves limits while the other one preserves colimits; but this is a general fact about adjoints). The feeling is that this resolution lies “exactly in the middle” between the initial and terminal resolutions. I don’t know if this kind of “strong” resolution has been considered in category theory.6

The resolution constructed here is quite different from the Eleinberg-Moore resolution (even though it uses the fixed-points as a basis). We do not construct functors from \( P(X) \) to \( E(X, F) \) and back; but from \( P(X) \) to \( P(E(X, F)) \) and back. In particular, as noted above, the interpolant is complete and cocomplete;7 which is not the case for the Eleinberg-Moore category: fixed points for an interior are closed under unions but not under intersections; and conversely for closure operators.

4 Revisiting Section 2 in a Constructive Setting

4.1 Impredicative

In the previous section we used Lemma 1 to generalize results on interior to closures. It is thus natural to ask whether (1) we can make the original proof constructive; (2) we can avoid using this lemma and prove the result for closure constructively. I will not into the details but just provide some hints about that.

– Galois connections from Lemma 2 are constructive.
– Lemma 3: the first part is trivial. The second part is easy: if \( F \) commutes with unions, take \((x, y) \in r \iff y \in F\{x}\); if \( F \) commutes with intersections, take \((x, y) \in r \iff \forall U \ y \in F(U) \Rightarrow x \in U\); and if \( F \) transforms unions into intersections, take \((x, y) \in r \iff y \in F\{x\}\).
– Proposition 1: the proof is constructive.
– Lemma 3 is constructive.
– Lemma 5 and the corresponding lemma for closure operator are constructive.
– Proposition 2 is constructive.
– we can mimic the proof of Proposition 2 to obtain a resolution of a closure operator as \( F = [r] \cdot \langle r^\sim \rangle \), but \not to obtain a resolution as \( F = [r] \cdot \langle r^\sim \rangle \).
– I doubt we can constructively obtain a resolution of a closure operator as \( F = [r] \cdot \langle r^\sim \rangle \).
– all of the lemmas about the size of interpolant are constructive at least if we read then as “if \( B \) is a basis for \( \text{Fix}(F) \) then we can use \( B \) as an interpolant”.

As a proof of concept, all this (except the last point) has been proved in the proof assistant COQ.8

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6 i.e. if \( C \) is a complete and cocomplete category and \( F \) is a monad on \( C \), a “strong resolution” is a resolution with a complete and cocomplete interpolant. Any reference to something similar in the literature would be most welcome.

7 i.e. is a complete lattice since we deal with partial orders

8 proof scripts available from http://iml.univ-mrs.fr/~hyvernat/academics.html
4.2 Predicative

In a predicative setting like Martin-Löf type theory (see [5]) or CZF (constructive ZF, see [6]) set theory, many of the results are not provable. The reason being that we do not allow quantification on a power-set. The main result about resolution would become something like:

**Proposition 3.** Suppose $\text{Fix}(F)$ (proper type) has a set-indexed basis, then $F$ as a resolution as $\langle r \rangle \cdot \lfloor r \sim \rfloor$ (if $F$ is an interior) or as $\lfloor r \rfloor \cdot \lfloor r \sim \rfloor$ (if $F$ is a closure).

If $F$ as a resolution, then $\text{Fix}(F)$ as a set-index basis.

Note that having a set-indexed basis is equivalent to being “set presented” in the terminology of P. Aczel ([6, 7]). Note that in the case of an interior operator, this implies that the predicate transformer is “set-based” (in the sense that it has a factorization as in Proposition 1, where the interpolant $X'$ is a set). It doesn’t seem that the existence of a resolution for a closure operator implies that the original predicate transformer is itself set-presented.

**Conclusion**

Nothing revolutionary has really been done, but the statement of Propositions 1 and 2 is, in an abstract setting, quite neat. However, as the proofs show, this is mostly abstract nonsense. The best example is probably Proposition 1, where $y \in F(U)$ is factorized as “there is a $V$ such that $s \in F(V)$ and $U \subseteq V$”. The proof of Proposition 2 is slightly subtler, but is hardly interesting. In the end, the most interesting and informative thing is probably Lemma 6, in its “positive” version: if $F$ has $Y$ as an interpolant, then $\text{Fix}(F)$ has a basis indexed by $Y$, which is hardly a breakthrough in mathematics...

I do nevertheless hope that it might interest some people, since while Proposition 1 is known to many (especially the refinement calculus people), it seems that Proposition 2 isn’t stated anywhere. I also hope the link between monad and closure operator (together with the Eilenberg-Moore category being the partial order of fixed points) will gain in popularity, as I see it as a much better way of seeing monads, at least as far as intuition is concerned.

A final word about the motivation for this: the starting point was the question about whether it is possible to represent any “basic topology” (see the forthcoming [8]) as the formal side of a basic pair, impredicatively speaking. A basic topology is a structure $(X, A, J)$ where $X$ is a set, and $A$ and $J$ are closure and interior operators on $P(X)$ such that $A(U) \subseteq J(V) \Rightarrow U \subseteq J(V)$. The answer is obviously no since the $A$ and $J$ arising from a formal pair are classically dual (i.e. $A \cdot \neg = \neg \cdot J$) and there are basic topologies which are provably not dual. The question then turned into: “can any interior operator be written as the formal interior of a basic pair?” and similar for closure operators. The answers are yes (Proposition 2) and I don’t know (the constructive resolution of a closure is of the form $[r] \cdot [r \sim]$, and not of the form $[r] \cdot \langle r \sim \rangle$).

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9 $U \cap V$ is the constructive version of $U \cap V \neq \emptyset$. 
References