A commutative product for sets

Pierre Hyvernat
January 2004

If S is a set, we write $\mathcal{M}_f(S)$ for the set of all finite multisets with elements in S. There are two equivalent ways to define them:

- as functions $f : S \to \mathbf{N}$ such that $\sum_{s\in S} f(s)$ is finite;
- as families $(s_i)_{i\in I}$ with finite index I quotiented by $(a_i)_{i\in I} \simeq (b_j)_{j\in J}$ iff $(\forall i \in I)\ a_i = b_{\sigma i}$ for some bijection $\sigma : I \to J$. We write $[a_i]_{i\in I}$ for the equivalence class of $(a_i)_{i\in I}$ w.r.t. $\simeq$.

We prefer the latter as it has a more "constructive" feeling.

Definition. *Suppose $(A_i)_{i\in I}$ is a finite family of subsets of some set S, we construct the set $\bigwedge_{i\in I} A_i$ of multisections on (A_i) as follows:*

- $\bigwedge_{i\in I} A_i \subseteq \mathcal{M}_f(S)$;
- $(a_j)_{j\in J} \in \bigwedge_{i\in I} A_i$ *iff* $(\forall j \in J)\ a_j \in A_{\sigma j}$ *for some bijection* $\sigma : J \to I$.

We also use the N-ary notation $A_1 * A_2 * \dots A_N$.

It is easy to see that this operation is well defined w.r.t. the relation $\simeq$ and that is is *commutative* in the sense that if $(A_i) \simeq (B_j)$, then $\bigwedge_i A_i = \bigwedge_j B_j$. In other words $\bigwedge$ is a well defined operator $\bigwedge : \mathcal{M}_f\big(\mathcal{P}(S)\big) \to \mathcal{P}\big(\mathcal{M}_f(S)\big)$. For those familiar with categorical notions, $\bigwedge$ is even a natural transformation between the functors $\mathcal{M}_f\mathcal{P}$ and $\mathcal{P}\mathcal{M}_f$. This operation is a commutative version of the usual (finite) cartesian product $\prod$.

One property of the cartesian product is that the operation is injective on families of non-empty subsets:

Lemma. *Suppose $(A_i)_{i\in I}$ and $(B_i)_{i\in I}$ are two finite families of non-empty subsets of S; suppose moreover that $\prod_i A_i = \prod_i B_i$, then $A_i = B_i$ for all $i \in I$.*

A similar result holds for $\bigwedge$, although is not as obvious!

Proposition. *Suppose $(A_i)_{i\in I}$ and $(B_i)_{i\in I}$ are two finite families of non-empty subsets of S; suppose moreover that $\bigwedge_i A_i = \bigwedge_i B_i$, then $(A_i) \simeq (B_i)$.*

The proof goes as follows: suppose $\bigwedge A_i = \bigwedge B_j = P$;

- we first show that there is one set in common in (A_i) and B_j: $A \in (A_i)$ and $A \in (B_j)$;
- we define an operation of *division* such that $(A * \bigwedge B_j)/A = \bigwedge B_j$;
- this implies that $\bigwedge_{i\neq 1} A_i = P/A = \bigwedge_{j\neq 1} B_j$;
- a trivial induction concludes the proof.

Lemma. *Suppose $\bigwedge A_i \subseteq \bigwedge B_i$, then $\forall j\ \exists i \quad A_i \subseteq B_j$.*

proof: by contradiction, suppose that $\exists j\ \forall i \quad \neg(A_i \subseteq B_j)$. Let j_0 be such a j.
We have that $\forall i\ \exists a_i \in A_i, a_i \notin B_{j_0}$. This implies that $[a_i] \in \bigwedge A_i$, but $[a_i]$ cannot be in $\bigwedge B_j$! Contradiction.

QED

Lemma. *Suppose $\bigwedge A_i = \bigwedge B_i$, then there is a pair (i, j) s.t. $A_i = B_j$.*

proof: by the above lemma, we can construct an infinite chain $A_{i_1} \supseteq B_{j_1} \supseteq \dots A_{i_n} \supseteq B_{j_n} \dots$ Since there is only a finite number of A_i's and B_j's, there is a cycle. This imply that some $A_{i_n} = B_{j_n}$.

QED

Definition. *Let $E \subseteq \mathcal{M}_f(S)$; define:*

- *for $a \in S$: $E/a = \{\mu \mid \mu + [a] \in E\}$;*
- *for $A \subseteq S$: $E/A = \bigcap_{a\in A} E/a$.*

Lemma. *For all $B_0, B_1 \dots B_N \subseteq S$ (non empty), we have $(B_0 * B_1 * \dots * B_N)/B_0 = B_1 * \dots * B_N$.*

proof: the $\supseteq$ inclusion is immediate. Let's show the converse inclusion:
let $[b_1, \dots b_N] \in (B_0 * B_1 * \dots * B_N)/B_0$; suppose by contradiction that $[b_i] \notin \bigwedge B_i$.
Let $a \in B_0$, we have $[a, b_1, \dots b_N] \in B_0 * \bigwedge B_i$. Without loss of generality, we can suppose $b_1 \in B_0$, $a \in B_1$ and $b_i \in B_i$ for all $i \geq 2$. $(*)$
Since $b_1 \in B_0$, we have $[b_1, b_1, b_2, \dots b_N] \in B_0 * \bigwedge B_i$, *i.e.* there is a bijection $\sigma : \{0, \dots N\} \to \{0, \dots N\}$ s.t. $b_{\sigma i} \in B_i$. (To make notation simpler, we put $b_0 = b_1$.)
Define (k_i) by induction as follows:

- $k_0 = \sigma 0$;
- $k_{i+1} = \sigma k_i$.

Let $K = \min\{i \mid k_i = 0 \text{ or } k_i = 1\}$. It exists. (`TODO: more details?`); let $I = \{k_0, \dots k_K\}$.
Now, rearrange the columns of the following table:

$$\begin{array}{c} \overbrace{\begin{array}{cccccccc} B_0 & B_1 & \dots & B_l & \dots & B_{l'} & \dots & B_N \end{array}}^{\{0,\dots N\}} \\ \underbrace{\begin{array}{cccccccc} b_{\sigma 0} & b_{\sigma 1} & \dots & b_1 & \dots & b_1 & \dots & b_{\sigma N} \end{array}}_{\{1,1,\dots N\}} \end{array}$$

into

$$\begin{array}{cc} \overbrace{\begin{array}{ccccc} B_0 & B_{k_0} & \dots & B_{k_{K-1}} & B_{k_K} \end{array}}^{\{0\}\cup I=\{0,k_0,\dots k_K\}} & \overbrace{\begin{array}{ccc} \dots & B_l & \dots \end{array}}^{\{1,\dots N\}\setminus I=\overline{I}} \\ \underbrace{\begin{array}{ccccc} b_{k_0} & b_{k_1} & \dots & b_{k_K} & b_1 \end{array}}_{\{1\}\cup I=\{1,k_0,\dots k_K\}} & \underbrace{\begin{array}{ccc} \dots & b_1 & \dots \end{array}}_{\{1,\dots N\}\setminus I=\overline{I}} \end{array}$$

From this (right hand part), we can deduce that $[b_i]_{i\in\overline{I}} \in \bigwedge_{i\in\overline{I}} B_i$. By hypothesis $(*)$, we also have that $[b_i]_{i\in I} \in \bigwedge_{i\in I} B_i$ (because $1 \notin I$). This implies that $[b_i]_{i\in\{1,\dots N\}} \in \bigwedge_{i\in\{1,\dots N\}} B_i$! Contradiction.

QED

The proof of the proposition is now immediate: by induction on N.

- $N = 0$: trivial;
- $N > 0$: suppose $\bigwedge A_i = \bigwedge B_i$; this implies that $[A_i]_{i\leq N}$ and $[B_i]_{i\leq N}$ are in fact of the form $[C] + [A_i]_{i<N}$ and $[C] + [B_i]_{i<N}$. Apply the lemma to get $\bigwedge_{i<N} A_i = \bigwedge_{i<N} B_i$, then the induction hypothesis to obtain $[A_i]_{i<N} = [B_i]_{i<N}$. From this, we can easily conclude that $[C] + [A_i]_{i<N} = [C] + [B_i]_{i<N}$.

One interesting point of this operation is that it is to the cartesian product what the union is to the disjoint sum:

- if A and B are disjoint, then $A \cup B$ (disjoint union) is isomorphic to $A \oplus B$ (sum or coproduct).
- if A and B are disjoint, then $A * B$ (disjoint commutative product) is isomorphic to $A \times B$ (usual product).

For both $*$ and $\cup$, the operations are truly commutative (real equalities rather than isomorphisms) whereas both $\times$ and $\oplus$ are only commutative up to isomorphisms.

One problem remains, namely that $*$ is not really associative!

Remark. *The above proposition doesn't hold if one replaces equalities by inclusions (even though it holds for the usual cartesian product). The simplest counter-example is probably the following:* $\{1, 3\} * \{2\} \subseteq \{1, 2\} * \{2, 3\}$.