TD 4, exercice 3, correction :

e) $\big[\exists x\,(A \to B)\big] \;\leftrightarrow\; \big[(\forall x\, A) \to (\exists x\, B)\big]$

Remarque : j'utilise volontairement un nouveau nom de variable x_0 lors de l'utilisation de la règle $\exists$ elim. Ceci permet d'utiliser la règle, même s'il y avait un contexte Γ contenant la variable libre x. (Il faudrait alors choisir la variable x_0 correctement.)

Première direction :

$\vdash\; \big[\exists x\,(A \to B)\big] \;\to\; \big[(\forall x\, A) \to (\exists x\, B)\big]$ . . . . $\to$ intro $\times 2$

$\exists x\,(A \to B)\,,\, \forall x\, A \;\vdash\; \exists x\, B$ . . . . $\exists$ elim

(1.) $\underline{\exists x\,(A \to B)}\,,\, \forall x\, A \;\vdash\; \underline{\exists x_0\, A[x := x_0] \to B[x := x_0]}$ . . axiome (α-conversion)

(2.) $\exists x\,(A \to B)\,,\, \forall x\, A\,,\, A[x := x_0] \to B[x := x_0] \;\vdash\; \exists x\, B$ . . . affaiblissement

$\forall x\, A\,,\, A[x := x_0] \to B[x := x_0] \;\vdash\; \exists x\, B$ . . . . . . $\exists$ intro pour $t := x_0$

$\forall x\, A\,,\, A[x := x_0] \to B[x := x_0] \;\vdash\; B[x := x_0]$ . . . . $\to$ elim

(2.1.) $\forall x\, A\,,\, A[x := x_0] \to B[x := x_0] \;\vdash\; A[x := x_0] \to B[x := x_0]$ . . . axiome

(2.2.) $\forall x\, A\,,\, A[x := x_0] \to B[x := x_0] \;\vdash\; A[x := x_0]$ . . . . $\forall$ elim pour $t := x_0$

$\forall x\, A\,,\, A[x := x_0] \to B[x := x_0] \;\vdash\; \forall x\, A$ . . . . axiome

Deuxième direction :

$\vdash\; \big[(\forall x\, A) \to (\exists x\, B)\big] \;\to\; \big[\exists x\,(A \to B)\big]$ . . . . $\to$ intro

$(\forall x\, A) \to (\exists x\, B) \;\vdash\; \exists x\,(A \to B)$ . . . . $\vee$ elim

(1.) $(\forall x\, A) \to (\exists x\, B) \;\vdash\; (\forall x\, A) \vee \neg(\forall x\, A)$ . . . . tiers exclu, *cf.* TD 3

(2.) $(\forall x\, A) \to (\exists x\, B)\,,\, \forall x\, A \;\vdash\; \exists x\,(A \to B)$

(3.) $(\forall x\, A) \to (\exists x\, B)\,,\, \neg\forall x\, A \;\vdash\; \exists x\,(A \to B)$

(2.) $(\forall x\, A) \to (\exists x\, B)\,,\, \forall x\, A \;\vdash\; \exists x\,(A \to B)$ . . . . coupure, *cf.* TD 3

(2.1.) $(\forall x\, A) \to (\exists x\, B)\,,\, \forall x\, A \;\vdash\; \exists x\, B$

(2.2.) $(\forall x\, A) \to (\exists x\, B)\,,\, \forall x\, A\,,\, \exists x\, B \;\vdash\; \exists\,(A \to B)$

(2.1.) $(\forall x\, A) \to (\exists x\, B)\,,\, \forall x\, A \;\vdash\; \exists x\, B$ . . . . $\to$ elim

(2.1.1.) $(\forall x\, A) \to (\exists x\, B)\,,\, \forall x\, A \;\vdash\; (\forall x\, A) \to (\exists x\, B)$ . . . . axiome

(2.1.2.) $(\forall x\, A) \to (\exists x\, B)\,,\, \forall x\, A \;\vdash\; \forall x\, A$ . . . . axiome

(2.2.) $(\forall x\, A) \to (\exists x\, B)\,,\, \forall x\, A\,,\, \exists x\, B \;\vdash\; \exists x\,(A \to B)$ . . . . $\exists$ elim

(2.2.1.) $(\forall x\, A) \to (\exists x\, B)\,,\, \forall x\, A\,,\, \underline{\exists x\, B} \;\vdash\; \underline{\exists x_0\, B[x := x_0]}$ . . axiome (α-conversion)

(2.2.2.) $(\forall x\, A) \to (\exists x\, B)\,,\, \forall x\, A\,,\, \exists x\, B\,,\, B[x := x_0] \;\vdash\; \exists x\,(A \to B)$. affaiblissement

$B[x := x_0] \;\vdash\; \exists x\,(A \to B)$ . . . . $\exists$ intro pour $t := x_0$

$B[x := x_0] \;\vdash\; A[x := x_0] \to B[x := x_0]$ . . . . $\to$ intro

$B[x := x_0]\,,\, A[x := x_0] \;\vdash\; B[x := x_0]$ . . . . axiome

(3.) $(\forall x\, A) \to (\exists x\, B)\,,\, \neg\forall x\, A \;\vdash\; \exists x\,(A \to B)$ . . . . affaiblissement

$\neg\forall x\, A \;\vdash\; \exists x\,(A \to B)$ . . . . règle dérivée, *cf.* TD 3 ; et loi de de Morgan

$\exists x\, \neg A \;\vdash\; \exists x\,(A \to B)$ . . . . $\exists$ elim

(3.1.) $\exists x\, \neg A \;\vdash\; \exists x_0 \neg A[x := x_0]$ . . . . axiome (α conversion)

(3.2.) $\exists x\, \neg A\,,\, \neg A[x := x_0] \;\vdash\; \exists x\,(A \to B)$ . . . . $\exists$ intro pour $t := x_0$

$\exists x\, \neg A\,,\, \neg A[x := x_0] \;\vdash\; A[x := x_0] \to B[x := x_0]$ . . . . $\to$ intro

$\exists x\, \neg A\,,\, \neg A[x := x_0]\,,\, A[x := x_0] \;\vdash\; B[x := x_0]$ . . . . $\bot$ elim

$\exists x\, \neg A\,,\, \neg A[x := x_0]\,,\, A[x := x_0] \;\vdash\; \bot$ . . . . $\neg$ elim

(3.2.1.) $\exists x\, \neg A\,,\, \neg A[x := x_0]\,,\, A[x := x_0] \;\vdash\; \neg A[x := x_o]$ . . . . . . axiome

(3.2.1.) $\exists x\, \neg A\,,\, \neg A[x := x_0]\,,\, A[x := x_0] \;\vdash\; A[x := x_o]$ . . . . axiome