# Fair testing vs. must testing in a fair setting

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# Goal

Reconcile, in the particular case of Milner's CCS,

- Joyal, Nielsen, and Winskel's 1993 (JNW) approach to concurrency theory,

with

- the interactive approach to behavioural equivalences:
  - testing semantics in process algebra (Hennessy and De Nicola, Beffara),
  - Krivine realisability,
  - game semantics (Hyland and Ong, Abramsky et al.), Girard's ludics.

# Review of JNW

- Category $\mathbb{P}$ of (non-empty) paths, i.e.:
  - objects: non-empty words over an alphabet $\mathcal{A}$;
  - morphisms: prefix extensions, e.g., $abc \to abcd$.
- Presheaves $\widehat{\mathbb{P}}$, i.e., functors $\mathbb{P}^{op} \to \mathrm{Set}$.
- Presheaves are like trees. Examples $ab + ac$ and $a(b + c)$.
- Natural transformations are like functional simulations. Example.

# Positions

Let $\mathbb{C}$ be:

$$
\begin{tikzcd}
\cdots & n & \cdots & p & \cdots \\
 & & \star . \arrow[ul, bend left=40, "0"] \arrow[ul, bend right=10, "n-1"'] \arrow[ur, bend left=10, "0"] \arrow[ur, bend right=40, "p-1"'] & &
\end{tikzcd}
$$

**Definition**

Positions are presheaves on $\mathbb{C}$.

Positions form a category $\mathbb{B}$.

# Example

- $F(\star) = \{a, b\}$,
- $F(1) = \{X_1, X_3\}$,
- $F(2) = \{X_2\}$,
- $F(\star \xrightarrow{0} 1)(X_1) = a$,
  Notation: $X_1 \cdot 0 = a$.
- $X_2 \cdot 0 = a$, $X_2 \cdot 1 = b$,
- $X_3 \cdot 0 = b$.

$$\begin{tikzcd}
X_1 & & X_2 & & X_3 \\
& a \arrow[ul, "0"] \arrow[ur, "0"'] & & b \arrow[ul, "1"] \arrow[ur, "0"'] &
\end{tikzcd}$$

The category of elements $\int F$.

# Moves from natural deduction, example: input

$$\frac{a_1, \ldots, a_n \vdash P}{a_1, \ldots, a_n \vdash a_i.P}$$

Add an object $\iota^-_{n,i}$ to $\mathbb{C}$:

$$\begin{tikzcd} \star \arrow[r, bend left, "0"] \arrow[r, phantom, "\vdots"] \arrow[r, bend right, "n-1"'] & n \arrow[r, bend left, "s"] \arrow[r, bend right, "t"'] & \iota^-_{n,i} \end{tikzcd}$$

and quotient by $s \circ j = t \circ j$ for all $j \in n$.

# Motivating the definition

The category of elements of the representable $\iota^{-}_{3,2}$ is the partially ordered set generated by

$$
\begin{tikzcd}
t \cdot 0 \arrow[drr] \arrow[dddd, equal] & & & \\
& & t \arrow[dd] & t \cdot 2 \arrow[l] \arrow[dddd, equal] \\
& t \cdot 1 \arrow[ur] \arrow[dddd, equal] & & \\
& & \iota^{-}_{3,2} & \\
s \cdot 0 \arrow[drr] & & & \\
& & s \arrow[uu] & s \cdot 2 \arrow[l] \\
& s \cdot 1. \arrow[ur] & &
\end{tikzcd}
$$

Output: do the same with $\iota^{+}_{n,i}$.

# Forking

The category of elements of the representable $\pi_3$ is the partially ordered set generated by

$$
\begin{tikzcd}
t_1 0 = t_2 0 \arrow[drr] \arrow[drrrr, bend left] \arrow[dddd, equal] & & & & & \\
& & t_1 \arrow[ddr] & & t_2 \arrow[ddl] & t_1 2 = t_2 0 \arrow[l] \arrow[lll, bend right] \arrow[dddd, equal] \\
& t_1 1 = t_2 1 \arrow[ur] \arrow[urrr, bend right] \arrow[dddd, equal] & & & & \\
& & & \pi_3 & & \\
s0 \arrow[drrr] & & & & & \\
& & & s \arrow[uu] & & s2 \arrow[ll] \\
& s1. \arrow[urr] & & & &
\end{tikzcd}
$$

# Name creation

The category of elements of the representable $\nu_2$ is the partially ordered set generated by

$$
\begin{tikzcd}
t \cdot 0 \arrow[drr] \arrow[dddd, equal] & & & \\
& & t \arrow[dd] & t \cdot 2 \arrow[l] \\
& t \cdot 1 \arrow[ur] \arrow[dddd, equal] & & \\
& & \nu_2 & \\
s \cdot 0 \arrow[drr] & & & \\
& & s \arrow[uu] & \\
& s \cdot 1. \arrow[ur] & &
\end{tikzcd}
$$

# Tick

$$
\begin{tikzcd}
t0 \arrow[drr] \arrow[dddd, equal] & & & \\
 & & t \arrow[dd] & t2 \arrow[l] \arrow[dddd, equal] \\
 & t1 \arrow[ur] \arrow[dddd, equal] & & \\
 & & \heartsuit_3 & \\
s0 \arrow[drr] & & & \\
 & & s \arrow[uu] & s2 \arrow[l] \\
 & s1. \arrow[ur] & &
\end{tikzcd}
$$

# Synchronisation: the 4th dimension

$$
\begin{tikzcd}
\rho t0 \arrow[drr] \arrow[dddd, equals] & & & & & & \\
& & \rho t \arrow[dd] & \rho t2 = \epsilon t1 \arrow[l] \arrow[rr] \arrow[dddd, equals] & & \epsilon t \arrow[dd] & \epsilon t0 \arrow[l] \arrow[dddd, equals] \\
& \rho t1 \arrow[ur] \arrow[dddd, equals] & & & & & \\
& & \rho \arrow[rr] & & \tau_{3,2,2,1} & \epsilon \arrow[l] & \\
\rho s0 \arrow[drr] & & & & & & \\
& & \rho s \arrow[uu] & \rho s2 = \epsilon s1 \arrow[l] \arrow[rr] & & \epsilon s \arrow[uu] & \epsilon s0 \arrow[l] \\
& \rho s1. \arrow[ur] & & & & &
\end{tikzcd}
$$

# Examples

- The two maximal executions of $\overline{a} \mid b$, which are actually equal.
- If $a = b$, one more execution.
- An execution of $\overline{a} \mid \mu X.(X \mid X)$.
- Some "wrong" examples.

# Restrictions and moves

- A restriction from $X$ to $Y$ is a cospan $Y \hookrightarrow X \xhookleftarrow{id} X$.
- A move from $X$ to $Y$ is a cospan $Y \hookrightarrow M \hookleftarrow X$ of presheaves obtained
  - from a cospan $Y_0 \xhookrightarrow{t\text{'s}} M_0 \xhookleftarrow{s} X_0$,
  - with $M_0$ a representable of dim 2 or 3,
  - by identifying some names.

# Observations

## Definition

*An observation is a presheaf $U \in \widehat{\mathbb{C}}$ isomorphic to a possibly denumerable "composition" of moves and restrictions in* $\mathrm{Cospan}(\widehat{\mathbb{C}})$*:*

$$
\begin{tikzcd}
X_0 \arrow[dr, hook] & & X_1 \arrow[dl, hook'] & \ldots & X_n \arrow[dr, hook] & & X_{n+1} \arrow[dl, hook'] \arrow[dr, hook] & & X_{n+2} \arrow[dl, hook'] & \ldots \\
& M_0 \arrow[drrr, hook] & & \ldots & & M_n \arrow[dl, hook'] & & M_{n+1} \arrow[dllll, hook'] & \ldots & \\
& & & & U & & & & &
\end{tikzcd}
$$

*The base is the morphism $X_0 \hookrightarrow U$.*

# The category $\mathbb{E}$ of observations

- Objects: $X \hookrightarrow U$ in $\widehat{\mathbb{C}}$ with
  - $U$ an observation,
  - $X$ its base;
- Morphisms: all commuting squares

$$\begin{tikzcd} U \arrow[r] & V \\ X \arrow[u, hook] \arrow[r] & Y \arrow[u, hook] \end{tikzcd}.$$

- Obvious functor to positions $\pi \colon \mathbb{E} \to \mathbb{B}$:

$$(X \hookrightarrow U) \mapsto X.$$

# A Grothendieck topology

Let $\star$ have dimension 0, $n$ have dimension 1, and so on up to 3.

**Definition**

Let a sieve $S$ on $X \hookrightarrow U$ in $\mathbb{E}$ be view-covering when it is jointly surjective in dimension 1.

# Elementary views

An elementary view from $X$ to $Y$ is a composite of

- a move from a representable,
- followed by a restriction to a representable:

$$n' \hookrightarrow X \xleftarrow{id} X \hookrightarrow M \hookleftarrow n.$$

Keeps track of one trajectory.

# Views

**Definition**

*A* view *is an observation $X \hookrightarrow U$ isomorphic to a possibly denumerable "composition" of elementary views.*

# Views are a canonical covering

**Proposition**

*For any observation $X \hookrightarrow U$, the sieve generated by morphisms from finite views into $U$ is covering.*

**Proposition**

*Any covering sieve contains all morphisms from finite views.*

# The categories $\mathbb{E}_X$

Here we want to relativise to a base position $X$.

**Definition**

Let $\mathbb{E}_X$ have as objects $U \hookleftarrow Y \to X$, and morphisms transformations between such with $X$ fixed.

$$\begin{tikzcd}
U \arrow[rr] & & U' \\
Y \arrow[u, hook] \arrow[rr] \arrow[dr] & & Y' \arrow[u, hook] \arrow[dl] \\
& X &
\end{tikzcd}$$

$\mathbb{E}_X$ inherits a Grothendieck topology from $\mathbb{E}$.

# Strategies as sheaves

**Definition**

Let the category $\mathsf{S}_X$ of strategies on $X$ be $\mathrm{Sh}(\mathbb{E}_X)$.

# The stack of strategies

**Proposition**

*This $X \mapsto \mathsf{S}_X$ extends to a functor $\mathsf{S} \colon \mathbb{B}^{op} \to \mathsf{CAT}$, which is a stack for the restriction of the view-covering topology to $\mathbb{B}$.*

Why stacks?

- Strategies are only sensible up to iso.
- Intuitively, only the number of possible states should matter, not the precise set of states.

# Canonical spatial decomposition

Let $\mathrm{Sq}(X) = \coprod_{n} X(n)$.

## Proposition

$$\mathsf{S}_X \simeq \prod_{(n,x) \in \mathrm{Sq}(X)} \mathsf{S}_n.$$

# Temporal decomposition

- Let $\mathcal{M}_X$ be the set of moves from $X$ (explain the size).
- For each $i \in \mathcal{M}_X$, let $X_i$ be the domain of $i$.

**Theorem**

$$\mathsf{S}_n \simeq \mathrm{Fam}\left(\prod_{i \in \mathcal{M}_n} \mathsf{S}_{X_i}\right).$$

A strategy is determined by

- its initial states, and
- what remains of them after each possible move.

Almost a sketch: would be $\mathsf{S}_n \cong \prod_{i \in \mathcal{M}_n} \mathsf{S}_{X_i}$.

# Scenarioses

In concurrency,

- Physical, or fair scenario: players are really independent;
- Interpreted, or potentially unfair scenario: a scheduler is responsible for parallelism.

# Must testing

Supposing a fixed move $\heartsuit$:

**Definition**

A process $P$ is must orthogonal to a context $C$, when all maximal traces of $C[P]$ play $\heartsuit$ at some point.
Notation: $P \perp^m C$, $P^{\perp^m}$.

**Definition**

$P$ and $Q$ are must equivalent, notation $P \sim_m Q$, when $P^{\perp^m} = Q^{\perp^m}$.

# Must testing in an unfair setting

Usually, only the unfair scenario is formalised:

$$P = (\Omega \mid \overline{a}) \qquad \text{and} \qquad Q = \Omega$$

are must equivalent.

The obvious test $C = a.\heartsuit \mid \Box$ is not orthogonal to $P$.

Indeed, there is an infinite looping trace, maximal.

# Fair testing in an unfair setting

- The example

$$(\Omega \mid \overline{a}) \sim_m \Omega$$

  takes potential unfairness of the scheduler into account.
- Usually people do not want to, and resort to:

**Definition**

A process $P$ is fair orthogonal to a context $C$, when all finite traces of $C[P]$ extend to traces that play $\heartsuit$ at some point.
Notation: $P \perp^f C$, $P^{\perp^f}$.

**Definition**

$P$ and $Q$ are fair equivalent, notation $P \sim_f Q$, when $P^{\perp^f} = Q^{\perp^f}$.

Solves the issue.

# Closed-world observations

## Definition

An observation $X \hookrightarrow U$ is closed-world when both

$$\coprod_{n,i} U(\iota^+_{n,i}) \xleftarrow{\epsilon} \coprod_{n,i,m,j} U(\tau_{n,i,m,j}) \xrightarrow{\rho} \coprod_{n,i} U(\iota^-_{n,i}) \tag{1}$$

are surjective.

# Global behaviours

- Let $\mathbb{W} \hookrightarrow \mathbb{E}$ be the full subcategory of closed-world observations.
- Let $\mathbb{W}(X)$ be the fibre over $X$ for the projection functor $\mathbb{W} \to \mathbb{B}$.

**Definition**

Let the category of global behaviours on $X$ be simply $\mathsf{G}_X = \widehat{\mathbb{W}(X)}$.

- The inclusion $\mathbb{W}(X) \hookrightarrow \mathbb{W}_X \hookrightarrow \mathbb{E}_X$ induces a functor $\mathsf{Gl} \colon \mathsf{S}_X \to \mathsf{G}_X$.

# Observable criterion

**Definition**

An observable criterion consists for all positions $X$, of a subcategory $\perp\!\!\!\perp_X \hookrightarrow \mathrm{G}_X$.

# Interactive equivalence

## Definition

For any strategy $S$ on $X$ and any pushout $P$

$$\begin{tikzcd} I \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & Z \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} \tag{2}$$

of positions with $I$ of dimension 0, let $S^{\perp\!\!\!\perp_P}$ be the class of all strategies $T$ on $Y$ such that $\mathrm{GI}(S \parallel T) \in \perp\!\!\!\perp_Z$.

- Here $\parallel$ denotes amalgamation in the stack $\mathsf{S}$.
- Let us make this concrete.

# Fair testing

**Definition**

*A closed-world story is successful when it contains a* $\heartsuit_n$.

**Definition**

Given a global behaviour $G \in \mathrm{G}_X$, an extension of a state $s \in G(U)$ to $U'$ is an $s' \in G(U')$ with $i \colon U \to U'$ and $s' \cdot i = s$.

**Definition**

The fair criterion $\perp\!\!\!\perp^f_X$ contains all global behaviours $G$ such that any state $s \in G(U)$ for finite $U$ admits a successful extension.

# Must testing

**Definition**

An extension of $s \in G(U)$ is strict when $U \to U'$ is not surjective.

**Definition**

For any global behaviour $G \in \mathrm{G}_X$, a state $s \in G(U)$ is $G$-maximal when it has no strict extension.

**Definition**

Let the must criterion $\perp\!\!\!\perp_X^m$ consist of all global behaviours $G$ such that for all closed-world $U$, and $G$-maximal $s \in G(U)$, $U$ is successful.

# The key result

**Theorem**

*For any strategy $S$, any state $s \in \mathsf{Gl}(S)(U)$ admits a $\mathsf{Gl}(S)$-maximal extension.*

# Fair vs. must

Thanks to the theorem, we have:

## Lemma

*For all* $S \in \mathsf{S}_X$, $\mathrm{Gl}(S) \in \perp^m_X$ *iff* $\mathrm{Gl}(S) \in \perp^f_X$.

## Proof.

Let $G = \mathrm{Gl}(S)$.
*(⇒)* By the theorem, any state $s \in G(U)$ has a $G$-maximal extension $s' \in G(U')$, for which $U'$ is successful by hypothesis, hence $s$ has a successful extension.
*(⇐)* Any $G$-maximal $s \in G(U)$ admits by hypothesis a successful extension which may only be on $U$ by $G$-maximality, and hence $U$ is successful. □

# Fair equals must

**Theorem**

*For all $S, S' \in \mathsf{S}_X$, $S \sim_m S'$ iff $S \sim_f S'$.*

**Proof.**

*(⇒)* Consider two strategies $S$ and $S'$ on $X$, and a strategy $T$ on $Y$ (as in the pushout $P$). We have:

$$\begin{aligned} \mathrm{Gl}(S \parallel T) \in \perp\!\!\!\perp^f \quad & \text{iff } \mathrm{Gl}(S \parallel T) \in \perp\!\!\!\perp^m \\ & \text{iff } \mathrm{Gl}(S' \parallel T) \in \perp\!\!\!\perp^m \\ & \text{iff } \mathrm{Gl}(S' \parallel T) \in \perp\!\!\!\perp^f. \end{aligned}$$

*(⇐)* Symmetric. □

# Perspectives

Short term:

- Link with CCS.
- Kleene theorem.

Longer term:

- Treat $\pi, \lambda, \ldots$
- Understand the abstract structure.
- What is a compilation?