Innocent strategies as sheaves, and interactive equivalences for CCS

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Programming languages: a technology

Claim

Research in programming languages is mainly **technological**.

Applies a non-formalised method, e.g.:

- Syntax.
- Quotienting by variable renaming \( (x \mapsto x = y \mapsto y) \).
- Reduction relation to model program execution.
- Reasoning on reduction.
Programming languages: a technology

Claim
Research in programming languages is mainly technological.

Long-term goal
Contribute to finding a general setting for this.

Leads to stupid questions like:
- What is a programming language?
- What is an observational equivalence?
- What is a compilation?
Related work

- Does not account for calculi with a structural congruence.
- Formats and their functorial interpretation by Plotkin and Turi.
- Anything else?
Outline

- A category $\mathbb{E}$ of executions, with a (Grothendieck) topology on $\mathbb{E}$.

Innocent strategies as sheaves.

- The stack of strategies.

Interaction by amalgamation.

- Notions of observation.

Fair testing $=$ must testing.
Positions

- •’s = players,
- ○’s = channels.
- Close to (multi-hole) active contexts in CCS:
  \[ \nu abc.X_1(a, c)|X_2(a, b, c)|X_3(b, c). \]
Moves from natural deduction: in/out

\[
\frac{a_0, \ldots, a_{n-1} \vdash P}{a_0, \ldots, a_{n-1} \vdash a_i \cdot P}
\]

Output: same with a \( \iota_{n,i}^+ \).
Moves from natural deduction: parallel composition

\[
\frac{\Gamma \vdash P \quad \Gamma \vdash Q}{\Gamma \vdash P \mid Q}
\]

\[
\pi_2
\]

\[
Y \downarrow M \downarrow X
\]
Moves from natural deduction: name creation

\[ \Gamma, a \vdash P \quad \Rightarrow \quad \Gamma \vdash \nu a. P \]
Moves from natural deduction: tick

\[ \Gamma \vdash P \]
\[ \Gamma \vdash \lozenge.P \]

A cheap daimon.
Synchronisation

\[ a.P \mid \bar{a}.Q \rightarrow P \mid Q \]
Executions

Glueings of diagrams of the above kind together:

- horizontally,
- vertically (possibly denumerable).

Keeping track of the base position: $X \hookrightarrow U$. 
A word on representing executions

- These diagrams: formalised as certain presheaves on a category $\mathcal{C}$.
- Basic diagrams: representables.
- Glueing $=$ taking colimits.

More in the end if time permits.
The category $\mathcal{E}$ of executions

- Objects: $X \hookrightarrow U$ well-formed.
- Morphisms: all commuting squares

\[
\begin{array}{ccc}
U & \rightarrow & V \\
\downarrow & & \downarrow \\
X & \rightarrow & Y.
\end{array}
\]

- Obvious functor $\pi: \mathcal{E} \rightarrow \mathcal{B}$:

$$(X \hookrightarrow U) \mapsto X,$$

where $\mathcal{B}$ is the category of positions.
A Grothendieck topology

- We will now introduce a Grothendieck topology on $\mathbb{E}$.
- Whose canonical neighbourhoods will be views, in a sense very close to game semantics.

First we recall the definition of a Grothendieck topology.
Sieves

Definition

A **sieve** on an object $U$ is

- a class of morphisms to $U$
- stable under precomposition by arbitrary morphisms.

Equivalently:

- A subpresheaf of the representable $\mathbb{E}(-, U)$.
- A subfibration of the domain fibration $\mathbb{E}/U \to \mathbb{E}$.
Grothendieck topologies

**Definition**

A **Grothendieck topology** \( J \) on \( \mathcal{E} \) assigns to each object \( U \) a class \( J(U) \) of sieves satisfying

1. the total sieve \( \mathcal{E}(-, U) \) is in \( J(U) \);
2. if \( S \in J(U) \) and \( f : V \to U \), then \( f^*(S) \in J(V) \);
   (A covering sieve restricts to covering sieves on all opens.)
3. if \( S \in J(U) \) and \( R \) is another sieve on \( U \), then if for all \( f : V \to U \) in \( S \) we have \( f^*(R) \in J(V) \), then \( R \in J(U) \).
   (If a sieve covers all the opens of a covering sieve, then it is covering.)

Here \( f^*(S) = \{ g : W \to V \mid fg \in S \} \).
Our Grothendieck topology

Let $\star$ have dimension 0, $n$ have dimension 1, and so on up to 3.

**Definition**

Let a sieve $S$ on $X \hookrightarrow U$ in $E$ be **view-covering** when it is jointly surjective in dimensions 1 and 2.

- Apart from unused channel names, this also implies surjectivity in dim 0.
- Let’s get to views.
Representable sequents

A **representable sequent**, denoted by $n$ is a position with

- one player,
- knowing $n$ names:

```
  .
```

```
  .  .  .
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Innocent strategies as sheaves...
Elementary views

Definition

An elementary view $V$ from $n$ to $n'$ is a cospan $n \hookrightarrow V \leftarrow n'$ isomorphic to a composite of

- a move $M$ from a representable sequent $n$,
- followed by a restriction to a representable sequent $n'$,

i.e., a cospan of the shape

$$n \hookrightarrow M \leftarrow X \leftarrow n'.$$
Views

Definition

A view is a possibly denumerable (vertical) composition of elementary views in $\text{Cospan}(\hat{C})$.

Examples.
Views form a canonical covering

**Proposition**

*For any execution $X \hookrightarrow U$, the sieve generated by morphisms from views into $U$ is covering.*

**Proposition**

*Any covering sieve contains all morphisms from views.*
Sheaves on a site

Let \( \mathbb{E} \) be equipped with a Grothendieck topology \( J \).

**Definition**

A presheaf \( F \) is a sheaf when for any sieve \( S \) covering \( U \), precomposition by \( S \hookrightarrow U \) yields a bijection

\[
\hat{\mathbb{E}}(U, F) \cong \hat{\mathbb{E}}(S, F).
\]

Let \( \text{Sh}(\mathbb{E}) \) the full subcategory of such.

\( U \rightarrow F \)

\( S \)

(Notation: \( U = \mathbb{E}(\cdot, U) \).)

(Being defined on the opens in \( S \) is enough.)
Relativising to a base position $X$

**Definition**

Let $(\mathbb{E})_X$ have

- as objects $U \leftarrow Y \rightarrow X$, with $Y \leftarrow U$ well-formed, and
- as morphisms commuting diagrams

\[
\begin{array}{ccc}
U & \rightarrow & U' \\
\uparrow & & \uparrow \\
Y & \rightarrow & Y' \\
X. & & \\
\end{array}
\]

$(\mathbb{E})_X$ inherits a Grothendieck topology from $\mathbb{E}$. 
Strategies as sheaves

**Definition**

Let the category $S_X$ of strategies on $X$ be $\text{Sh}((E)_X)$.

Intuition: a strategy $S$ specifies for each execution $E \in (E)_X$ a number of ways for it to accept $E$. 
Restriction to a subposition

- Consider \( Y \to X \), and a strategy \( S \in S_X \).
- Let \( S|_Y \) send \( U \leftrightarrow Z \to Y \) to
  \[ S(U \leftrightarrow Z \to Y \to X). \]
- This extends to morphisms.

**Proposition**

This forms a functor \( S : \mathcal{B}^{op} \to \text{CAT} \).
The stack of strategies

**Proposition**

This functor \( S : \mathcal{B}^{\text{op}} \to \text{CAT} \) is a stack for the “surjective in dim 1” topology on \( \mathcal{B} \).

- Stacks are like sheaves but one dimension up.
- For sheaves: a bijection \( \hat{E}(U, F) \cong \hat{E}(S, F) \).
- For stacks: an equivalence of categories.
- Known fact: sheaves on slices form a stack.
- Here: mild generalisation.
- Why stacks? Intuitively, only the number of possible states should matter, not the precise set of states.
Canonical covering in $\mathbb{B}$

**Proposition**

For a given position $X$, the collection of morphisms $n \to X$ (for all $n$) is covering in $\mathbb{B}$.

**Proposition**

Any covering contains it.
Canonical covering continued

For any square

\[
\begin{array}{ccc}
Y & \xrightarrow{n} & n \\
\downarrow & & \downarrow x \\
m & \xrightarrow{x'} & X
\end{array}
\]

with \( x \neq x' \), \( Y \) has dimension 0.
Canonical covering continued

Proposition

If $Y$ has dim 0, then $S_Y \simeq 1$.

Indeed:

- any execution $U$ on $Y$ is covered by the empty family,
- which has a unique $\emptyset \rightarrow F$ for any sheaf $F$,
- so $F(U) \simeq 1$, which determines $F$ up to iso.
Canonical spatial decomposition

Let $\text{Sq}(X) = \coprod_n X(n)$.

**Proposition**

$S_X \simeq \prod_{(n,x) \in \text{Sq}(X)} S_n$. 
Temporal decomposition

- Let $\mathcal{M}_X$ be the set of possible moves from $X$ (explain the size).
- For each $i \in \mathcal{M}_X$, let $X_i$ be the domain of the corresponding move.
- For any $\mathcal{C}$, let $\text{Fam}(\mathcal{C})$ denote the category with
  - objects families $f : X \to \text{ob} \mathcal{C}$,
  - morphisms $f \to g$ the pairs of
    - $u : X \to Y$ such that $gu = f$, and
    - $v : X \to \mathcal{C}_1$ with
      \[ \text{dom} v(x) = f(x) \quad \text{cod}(v(x)) = g(u(x)). \]

Examples.
Temporal decomposition

\textbf{Theorem}

\textit{Equivalence of categories:} \( S_n \cong \text{Fam} \left( \prod_{i \in \mathcal{M}_n} S_{X_i} \right) \).

A strategy is determined by
- its initial states, and
- what remains of them after each possible move.

Almost a sketch: would be a bijection of sets

\[ S_n \cong \prod_{i \in \mathcal{M}_n} S_{X_i}. \]
Scenarios

In concurrency,

- Physical, or **fair** scenario: players are really independent;
- Interpreted, or **potentially unfair** scenario: a scheduler is responsible for parallelism.
**Must testing**

Supposing a fixed move $\heartsuit$:

**Definition**

A process $P$ is **must orthogonal** to a context $C$, when all maximal traces of $C[P]$ play $\heartsuit$ at some point.

Notation: $P \perp^m C$, $P \perp^m$.

**Definition**

$P$ and $Q$ are **must equivalent**, notation $P \sim^m Q$, when $P \perp^m = Q \perp^m$. 
Must testing in an unfair setting

Usually, only the unfair scenario is formalised:

\[ P = (\Omega \mid \bar{a}) \quad \text{and} \quad Q = \Omega \]

are must equivalent.

The obvious test \( C = a.\heartsuit \mid \square \) is not orthogonal to \( P \).

Indeed, there is an infinite looping trace, maximal.
Fair testing in an unfair setting

- The example

\[(\Omega \mid \overline{a}) \sim_m \Omega\]

takes potential unfairness of the scheduler into account.

- Usually people do not want to, and resort to:

**Definition**

A process \(P\) is **fair orthogonal** to a context \(C\), when all finite traces of \(C[P]\) extend to traces that play ♥ at some point.

Notation: \(P \perp_f C\), \(P \perp_f\).

**Definition**

\(P\) and \(Q\) are **fair equivalent**, notation \(P \sim_f Q\), when \(P \perp_f = Q \perp_f\).

Solves the issue.
Closed-world observations

**Definition**

An observation \( X \leftrightarrow U \) is **closed-world** when both

\[
\prod_{n,i} U(\nu^+_{n,i}) \leftrightarrow \prod_{n,i,m,j} U(\tau_{n,i,m,j}) \overset{\rho}{\longrightarrow} \prod_{n,i} U(\nu^-_{n,i})
\]

are surjective.
Global behaviours

- Let $\mathcal{W} \hookrightarrow \mathcal{E}$ be the full subcategory of closed-world observations.
- Let $\mathcal{W}(X)$ be the fibre over $X$ for the projection functor $\mathcal{W} \rightarrow \mathcal{B}$.

**Definition**

Let the category of global behaviours on $X$ be simply $G_X = \overline{\mathcal{W}(X)}$.

- Cf. Joyal, Nielsen, and Winskel.
- The inclusion $\mathcal{W}(X) \hookrightarrow (\mathcal{E})_X$ induces a functor $G_I : S_X \rightarrow G_X$. 
Observable criterion

**Definition**

An **observable criterion** consists for all positions $X$, of a subcategory $\bot_X \hookrightarrow G_X$. 
Interactive equivalence

Definition

For any strategy $S$ on $X$ and any pushout $P$

$$
\begin{array}{c}
I \\
\downarrow \\
X
\end{array} 
\quad 
\begin{array}{c}
\longrightarrow \\

Y \\
\downarrow \\
\longrightarrow \\
Z
\end{array}
$$

(2)

of positions with $I$ of dimension 0, let $S \perp_P$ be the class of all strategies $T$ on $Y$ such that $Gl(S \parallel T) \in \perp Z$.

- Here $\parallel$ denotes amalgamation in the stack $S$.
- Let us make this concrete.
Fair testing

**Definition**

A closed-world execution is **successful** when it contains a $\heartsuit_n$.

**Definition**

Given a global behaviour $G \in G_X$, an **extension** of a state $s \in G(U)$ to $U'$ is an $s' \in G(U')$ with $i: U \to U'$ and $s' \cdot i = s$.

**Definition**

The **fair** criterion $\perp^f_X$ contains all global behaviours $G$ such that any state $s \in G(U)$ for finite $U$ admits a successful extension.
# Must testing

**Definition**
An extension of $s \in G(U)$ is **strict** when $U \rightarrow U'$ is not surjective.

**Definition**
For any global behaviour $G \in G_X$, a state $s \in G(U)$ is $G$-**maximal** when it has no strict extension.

**Definition**
Let the **must** criterion $\perp^m_X$ consist of all global behaviours $G$ such that for all closed-world $U$, and $G$-maximal $s \in G(U)$, $U$ is successful.
The key result

**Theorem**

*For any strategy \( S \), any state \( s \in \text{Gl}(S)(U) \) admits a \( \text{Gl}(S) \)-maximal extension.*
Fair vs. must

Thanks to the theorem, we have:

Lemma

For all $S \in S_X$, $Gl(S) \in \bot^m_X$ iff $Gl(S) \in \bot^f_X$.

Proof.

Let $G = Gl(S)$.

$(\Rightarrow)$ By the theorem, any state $s \in G(U)$ has a $G$-maximal extension $s' \in G(U')$, for which $U'$ is successful by hypothesis, hence $s$ has a successful extension.

$(\Leftarrow)$ Any $G$-maximal $s \in G(U)$ admits by hypothesis a successful extension which may only be on $U$ by $G$-maximality, and hence $U$ is successful.
Fair equals must

**Theorem**

For all $S, S' \in S_X$, $S \sim_m S'$ iff $S \sim_f S'$.

**Proof.**

$(\Rightarrow)$ Consider two strategies $S$ and $S'$ on $X$, and a strategy $T$ on $Y$ (as in the pushout $P$). We have:

$$Gl(S \parallel T) \in \perp^f \iff Gl(S \parallel T) \in \perp^m$$

$$\text{iff } Gl(S' \parallel T) \in \perp^m$$

$$\text{iff } Gl(S' \parallel T) \in \perp^f.$$ 

$(\Leftarrow)$ Symmetric.
Perspectives

Short term:
- We have a translation of CCS processes into this model.
- Identify the equivalence induced by this translation.

Longer term:
- Treat $\pi, \lambda, \ldots$
- Understand the abstract structure.
- What is a compilation?