Contraction-free proofs and games for Linear Logic

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Long term goals

Very unreasonable, hardly confess-able:

Logic: better understand mathematical existence (cf. AC by Coquand-Berardi-Bezem);

Proof theory: general cut elimination;

Programming languages: correctness of compilation and program transformations.

Through graphical games?

Progression

In logic:

1. Multiplicative Additive fragment of Linear Logic (MALL);
2. Today: a hack on exponentials.
3. One day: quantifiers, AC, etc.

Related work

Game models of LL:

- Abramsky-Jagadeesan-Malacaria, Hyland-Ong, Nickau.
- Abramsky-Melliès, Melliès.
- Girard.
- Delande-Miller.
- Melliès-Mimram.

Contraction elimination:

- Dyckhoff.
- Kashima.

Formulae

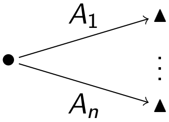
MALL formulae with units, but without atoms.

$$\begin{array}{rcccccccc} A, B, \mathcal{C}, \ldots \in \mathcal{F} & ::= & \mathbf{1} & \mid & A \otimes B & \mid & A \oplus B & \mid & \mathbf{0} \\ & \mid & \perp & \mid & A ⅋ B & \mid & A \,\&\, B & \mid & \top. \end{array}$$

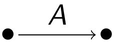
- De Morgan duality (vertically): $A^{\perp\perp} = A$.
- Sidedness: $(\Gamma \vdash A, \Delta) \approx (\Gamma, A^{\perp} \vdash \Delta)$.

Sequents as graphs

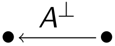
- Understand a sequent $\vdash A_1, \ldots, A_n$ as the neighbourhood of $\bullet$ in the labeled graph:



- Sidedness: identify



with

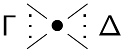


The cut rule

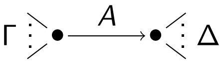
- The rule:

$$\frac{\Gamma \vdash A \qquad A \vdash \Delta}{\Gamma \vdash \Delta}$$

- Graphically:



becomes



This leads to taking as positions of our game ...

Hypersequents

Definition

A *hypersequent* is:

- a directed graph
- labeled in formulae,
- whose underlying undirected graph is acyclic.

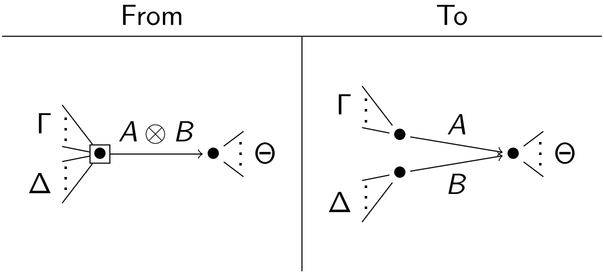
Positions

Definition

A *position* is a hypersequent with a partition of its vertices into Proponents ($\vartriangle$) and Opponents ($\blacktriangle$).

Notation: $\bullet$ means either Proponent or Opponent.

Tensor move

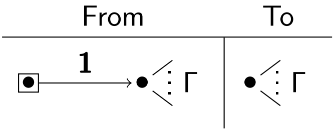


Corresponding rules:

$$\frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B} \qquad\qquad \frac{A, B \vdash \Theta}{A \otimes B \vdash \Theta}$$

- Teams after the move: by inverse image.
- *Active* vertex: $\boxdot$.

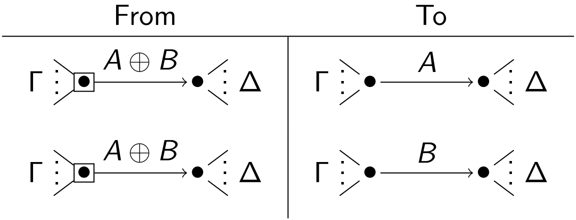
Unit move



Corresponding rules:

$$\frac{}{\vdash \mathbf{1}} \qquad \frac{\vdash \Gamma}{\mathbf{1} \vdash \Gamma}$$

Plus moves



Corresponding rules:

$$\frac{\vdash \Gamma, A}{\vdash \Gamma, A \oplus B} \qquad \frac{\vdash \Gamma, B}{\vdash \Gamma, A \oplus B} \qquad \frac{A \vdash \Delta \qquad B \vdash \Delta}{A \oplus B \vdash \Delta}$$

Remark

Bijections between:

- moves,
- positive rules,
- premises of negative rules.

In particular:

- no positive rule (or move) for $\mathbf{0}$;
- no premise for the $\top$ rule.

Summary of the rest for MALL (1)

- Plays and strategies, roughly as usual (asynchronous).
- To win, keep a negative formula. Example:

$$\blacktriangle \xrightarrow{\quad \mathbf{0} \quad} \vartriangle \xleftarrow{\quad \mathbf{0} \quad} \vartriangle$$

- Results:
  - Cut-free plays are finite.
  - Consistency (at least one team loses).
  - Soundness.
  - Incompleteness ($\bot \otimes \bot$).

Summary of the rest for MALL (2)

And more:

- Restrict strategies to be *local* to recover completeness:

Local (winning) strategies form a sheaf.

- Parallel composition and hiding (i.e., cut elimination), using a *factorisation system* on the category of positions.

The problem

- MALL introduction rules are symmetric and decreasing.
- Exponential rules are dissymmetric and increasing (contraction).

Challenge

Reveal the hidden symmetry and control expansion.

Our solution applies to sequent calculus.

A new sequent calculus: NLL

For exponentials:

$$\frac{\Gamma, \bot}{\Gamma, ?A} \qquad \frac{\Gamma, A}{\Gamma, ?A} \qquad \ldots \qquad \frac{\Gamma, \parr^n A}{\Gamma, ?A}\ \textsc{NewD}_n \qquad \ldots$$

$$\frac{\Delta, \mathbf{1} \qquad \Delta, A \qquad \ldots \qquad \Delta, \otimes^n A \qquad \ldots}{\Delta, !A}\ \textsc{NewBang}.$$

- One positive rule per negative premise per move.
- Guess how many moves?
- NewBang has a *reversible* flavor.
- No more contraction, but weakening and dereliction are still there.
- Proof = finite depth tree of rule applications.

NLL

And for tensor:

$$\frac{\Gamma, ?\Theta, A \qquad \Delta, ?\Theta, B}{\vdash \Gamma, \Delta, ?\Theta, A \otimes B} \text{ NewTens.}$$

Thanks to this rule:

- Prove $!A \multimap (!A \otimes !A)$.
- Consequence: LL-provable implies NLL-provable (not cut-free).
- Better, the NLL proof may be chosen *bounded*.

Contraction through cut in NLL

Duplicator: a formula δ_A starting with ? and such that both rules

$$\frac{\Gamma, ?A, ?A}{\Gamma, ?A, \delta_A}\ \textsc{Dup} \qquad\qquad \overline{\delta_A^{\perp}}$$

are derivable in NLL without cuts.

Contracting A consumes a δ_A.

For instance,

$$\delta_A = ?(!A^{\perp} \otimes (?A \parr ?A)).$$

You don't want to see

$$\dfrac{\dfrac{\dfrac{\Gamma, ?A, ?A}{\Gamma, ?A ⅋ ?A} \quad \dfrac{}{?A, !A^{\perp}}}{\Gamma, ?A, !A^{\perp} \otimes (?A ⅋ ?A)}}{\Gamma, ?A, ?(!A^{\perp} \otimes (?A ⅋ ?A))} \qquad \dfrac{\dfrac{\dfrac{?A, !A^{\perp} \quad ?A, !A^{\perp}}{?A, !A^{\perp} \otimes !A^{\perp}}}{!A^{\perp} \multimap (!A^{\perp} \otimes !A^{\perp})}}{!(!A^{\perp} \multimap (!A^{\perp} \otimes !A^{\perp}))},$$

the right-hand one working thanks to rule NewTens.

Cut anticipation

Definition

A proof in NLL is *bounded* when it is either cut-free, or of the form

$$\dfrac{\dfrac{\pi_1}{A} \qquad \dfrac{\pi_2}{A^{\perp}, \Gamma}}{\Gamma}$$

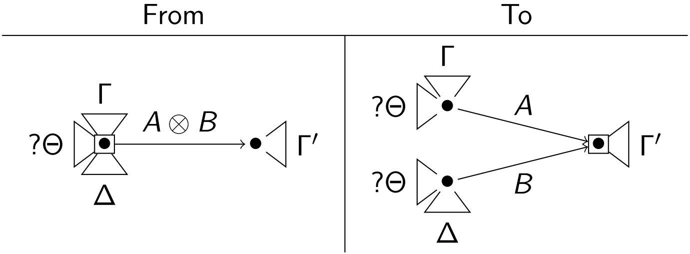
with π_1 and π_2 cut-free.

Theorem

Each provable sequent in LL admits a bounded proof in NLL.

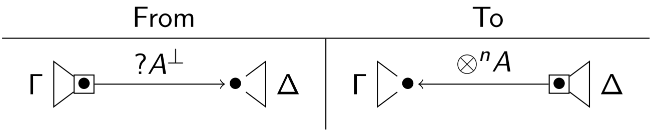
The converse does not hold.

The new move for tensor



Duplicates a lot!

Moves for exponentials



An infinite play

Start from any position of the shape

$$\bullet \xleftarrow{1 \otimes 1} \bullet \xleftarrow{!1} \boxed{U} \xrightarrow{!1} \bullet \xrightarrow{1 \otimes 1} \bullet$$

for some position U.
Break a tensor:

$$\bullet \xleftarrow{1 \otimes 1} \bullet \xleftarrow{!1} \boxed{U} \xrightarrow{!1} \bullet \xrightarrow{1} \bullet \xleftarrow{1} \bullet \xleftarrow{!1} \boxed{U} \xrightarrow{!1} \bullet \xrightarrow{1 \otimes 1} \bullet,$$

i.e.,

$$\bullet \xleftarrow{1 \otimes 1} \bullet \xleftarrow{!1} \boxed{U'} \xrightarrow{!1} \bullet \xrightarrow{1 \otimes 1} \bullet.$$

Boom.

Restricting the game

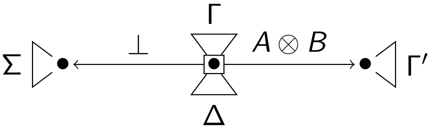
Restrict the game as follows:

- Mark one vertex with a *token*.
- The only vertex to play is the one holding the token.
- The token may be passed along a negative edge (see example below).

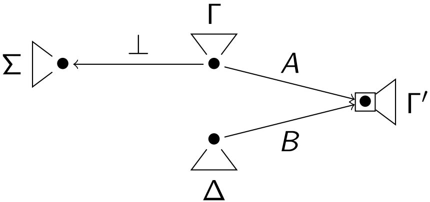
Who holds the token first?

For any formula, it does not matter!

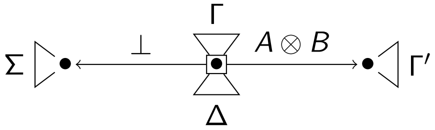
Passing the token along a negative edge



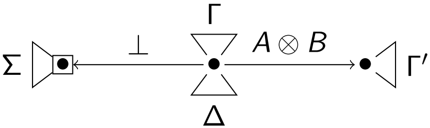
Passing the token along a negative edge



Passing the token along a negative edge



Passing the token along a negative edge



Last but not least: finiteness

Theorem

Plays are finite in the game with a token.

Chambéry-style proof

Thanks to René David and Karim Nour.

The new symmetry

To validate or refute A:

- Proponents want to find P and, $\forall O$, win:

$$\triangle \xrightarrow{\quad P \quad} \triangle \xrightarrow{\quad A \quad} \blacktriangle \xrightarrow{\quad O \quad} \blacktriangle .$$

- Proponents want to find O and, $\forall P$, win:

$$\triangle \xrightarrow{\quad P \quad} \triangle \xrightarrow{\quad A \quad} \blacktriangle \xrightarrow{\quad O \quad} \blacktriangle .$$

Summary

- Validity in our game is consistent, and sound for MAELL provability.
- But incomplete.
- Cut anticipation in proofs $\rightsquigarrow$ finite plays.

Will this hack make it to a full-fledged, game-based logic?

Perspectives

- Proof theory, i.e., full completeness (through understanding *innocent* strategies in our setting).
- Quantifiers.
- Programming languages, calculi (λ, π ...?).