Introduction

- Mainly an introduction to the theory of nerves (Berger, Leinster, Weber, ...).
- Why?
Why nerves?

Some of us design **algebraic** structures modelling computation.
- Categorical semantics (Lambek-Scott, Seely, Lafont, Benton-Hyland-Bierman, ...).
- Polygraphic approach (Burroni, Lafont, ...).

Some design **graphical** structures modelling computation.
- Proof nets, interaction nets (Girard, Lafont, Mazza).
- Polygraphic approach (Burroni, Lafont, ...).

Polygraphs not so graphical: need a rewrite

\[
\begin{array}{c}
\downarrow \\
\end{array}
\rightarrow
\]

Hirschowitz
Algebraic vs. graphical

- Algebraic approaches define categories of models.
- Most graphical approaches define one model.

Graphical seems weaker.

Nerve theory

Graphical $\leadsto$ algebraic (in good cases).

Specifically:
- Given a nice monad $T$,
- compute a sketch $S_T$,
- such that

$T$-algebras $\overset{\simeq}{\longrightarrow}$ Models of $S_T$. 
Why nerves?

If $T$ is nice:

$T$-algebras $\simeq$ Models of $S_T$.

- Here: nice means local right adjoint.
- Has to do with being graphical.

Hope: this can be useful in computer science, and in Geocal in particular.
Contents

- One example: (symmetric) multicategories (Lambek?).
- One counterexample: 2-categories.
Shapes

Consider the category $\mathcal{S}$ looking like:

$$
\begin{array}{c}
\vdots \\
\sigma_0 \\
\vdots \\
\sigma_{n-1} \\
\star \\
\vdots \\
\tau \\
\sigma_0 \\
\vdots \\
\sigma_{p-1} \\
\end{array}
\quad \quad \begin{array}{c}
\cdots \\
\tau \\
\cdots \\
\sigma_0 \\
\vdots \\
\sigma_{p-1} \\
\end{array}
$$
Presheaves over shapes

The category $\hat{S} = [S^{op}, \text{Set}]$ has multigraphs as objects.

**Example**

- $F(\star) = \{0, \ldots, 7\}$,
- $F(0) = \{e_2\}$,
- $F(2) = \{e_0, e_1\}$,
- $F(3) = \{e_3\}$,
- $F(\tau)(e_0) = 0$,
- $F(\tau)(e_1) = 1$,
- $F(\sigma_0)(e_0) = 1, \ldots$
Presheaves over shapes

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- $F(\sigma_0)(e_0) = 1$, \ldots

When every vertex is at most 1-in 1-out: forget the bullets!
Multicategories

There is a “free multicategory” monad $\mathcal{M}$ on $\hat{S}$, which we will reconstruct graphically.

We now:

- define a recipient presheaf $\mathcal{R}$ on $S$;
- and a sequence of subpresheaves $\mathcal{T}_0, \mathcal{T}_1, \ldots \subseteq \mathcal{R}$;
- consider the union $\mathcal{T}_\omega = \bigcup_{i \in \omega} \mathcal{T}_i \subseteq \mathcal{R}$.

Then:

- $\mathcal{T}_\omega(n) \cong$ morphisms $n \to 1$ in the free multicategory over 1;
- And we reconstruct $\mathcal{M}$ from this $\mathcal{T}_\omega$. 
The recipient presheaf

- Recall \( S \):

\[
\begin{array}{ccc}
\ldots & n & \ldots \\
\sigma_0 & \sigma_{n-1} & \tau \\
\tau & \sigma_0 & \sigma_{p-1} & \ldots \\
\ldots & p & \ldots \\
\end{array}
\]

and observe that \( S/\ast \cong 1 \).
The recipient presheaf

- Let $\mathcal{R}(\star)$ have as sole element the representable $y\star$, seen as a functor $S/\star \to \hat{S}$;
- $\mathcal{R}(n)$ is the set of functors $S/n \to \hat{S}$ of the shape:
  \[
  \begin{array}{c}
  \star \\
  t \downarrow \\
  f \\
  s_0 \\
  \star \\
  \end{array}
  \begin{array}{c}
  \star \\
  \ldots \\
  \star \\
  \end{array}
  \\
  \begin{array}{c}
  s_{n-1} \\
  \end{array}
  \\
  \text{with } f \text{ finite (let us call them multi-cospans),}
  \\
  \text{modular isomorphism of multi-cospans } S/n \to \hat{S}.
  \\

Intuition

Multigraphs with arity and handles.
Consider the following presheaf $\mathcal{T}_0 \subseteq \mathcal{R}$:

- $\mathcal{T}_0(\star) = \mathcal{R}(\star) = \{\star\}$,
- $\mathcal{T}_0(n)$ is the singleton

\[
\begin{array}{ccc}
\star & \tau & \star \\
\downarrow & & \downarrow \\
n & \sigma_0 & \sigma_{n-1} \\
\star & \vdots & \star \\
\end{array}
\]

(when $n = 1$).
Pictorially, $\mathcal{T}_0$ has, for each $n$, one element

\[
\begin{array}{c}
\sigma_0 \\
\sigma_n-1 \\
\ldots
\end{array}
\]

plus

(when $n = 1$).
Role of the diagram: distinguish the dangling wires.
Step

Now define $\mathcal{I}_{n+1}$ to be the union of $\mathcal{I}_n$ and $\mathcal{T}_n'$, which has:

- $\mathcal{T}_n'(\star) = \emptyset$
Now define $\mathcal{T}_{n+1}$ to be the union of $\mathcal{T}_n$ and $T'_n$, which has:

- $T'_n(m)$ is the set of diagrams $(f, s, t)$ as above such that:
  - there exist $p + q - 1 = m$, $i \in p$, and
  - diagrams $h \in \mathcal{T}_n(p)$ and $k \in \mathcal{T}_n(q)$, such that $f$ is:
This just glues two multigraphs together along the chosen edge:
Wrap up

Definition

Let $\mathcal{I}_\omega$ be the union of all the $\mathcal{I}_n$s.
Result

Theorem (Coherence at 1)

For all $n$, $\mathcal{T}_\omega(n)$ is isomorphic to the set $M(n)$ of morphisms $n \to 1$ in the free multicategory on 1. Furthermore, composition and identities are given by the operations on the $\mathcal{T}_n$’s, e.g,

\[ \mathcal{T}_\omega(p) \times \mathcal{T}_\omega(q) \cong M(p) \times M(q) \]

\[ \text{glueing at } i \quad \circ_i \quad \text{commutes.} \]
The monad

We now derive the monad from the presheaf $T_\omega$.

Consider the functor:

\[
\hat{S} \longrightarrow \hat{S} \\
F \mapsto T F,
\]

where $T(F)(s) = \bigsqcup_{x \in T_\omega(s)} \hat{S}(x(id_s), F)$.

**WTF?**

Sorry, I have to show you this key formula.
Understanding the key formula

\[ T(F)(s) = \bigoplus_{x \in T_\omega(s)} \hat{S}(x(id_s), F) \]

- Recall that \( T_\omega(s) \) is a set of diagrams \( S/s \to \hat{S} \), e.g.,

\[
\begin{array}{c}
\ast \\
t \downarrow \\
f \\
s_0 \\
s_{n-1} \\
\ast \\
\ast \\
\end{array}
\]

- So that for \( x \in T_\omega(s) \),
  - \( x(id_s) \) is a presheaf on \( S \),
  - here \( f \).
  - And a natural transformation \( f \to F \) is a labelling of \( f \) in \( F \).
Example

Let:

- $F(\star) = \{x, y, z\}$,
- $F(0) = \{a\}$,
- $F(2) = \{b\}$,
- $F(3) = \{c\}$,
- $F(\tau)(a) = y$,
- $F(\tau)(b) = x$,
- $F(\sigma_0)(b) = x, \ldots$

Seen as specifying operations

\[ a: l \to y \quad b: x \otimes y \to x \quad c: x \otimes y \otimes z \to y. \]
Example

Then remember $f$:

A natural transformation $f \to F$ sends each element to $F$, consistently.
A last time

The key formula again:

\[ T(F)(s) = \bigsqcup_{x \in T_\omega(s)} \hat{S}(x(id_s), F), \]

i.e.,

\[ T(F)(s) = \bigsqcup_{(f, \ldots ) \in T_\omega(s)} \hat{S}(f, F). \]
Results

Theorem (Not me)

- $\mathcal{T}$ is a lra monad, and a club [Kelly, Leinster, Weber]:
  - $\mathcal{T}$ preserves pullbacks.
  - Naturality squares for $\mu$ and $\eta$ are pullbacks.
  - Generic factorisations.
  - $\mathcal{T}$ is sketchable, i.e., algebraic.

- $\mathcal{T}$ is isomorphic to the “free multicategory” monad $\mathcal{M}$. 
Symmetric multicategories

In passing: the technique trivially extends to symmetric multicategories.

Key observation

The involved functors $S/n \rightarrow \hat{S}$ (the multi-cospans) have no automorphisms.

(I.e., no non-trivial endo-isomorphisms.)
Let us now illustrate why this is key.
A new base category

We may define a new category \( S_2 \) whose representables look like

\[
\begin{array}{c}
\ldots \\
\downarrow \\
\downarrow \\
\cdots
\end{array}
\]

\[ p \]

\[ n \]
Operations

- Horizontal composition: glueing along backgrounds.
- Vertical composition: glueing along wires.
Argh

- One defines $\mathcal{T}_\omega$ using these operations and identities,
- then $\mathcal{T}$ using the key formula

$$\mathcal{T}(F)(s) = \bigsqcup_{x \in \mathcal{T}_\omega(s)} \hat{S}(x(id_s), F),$$

with:

Hope

$\mathcal{T} \cong \text{“free 2-category”}.$

But this is wrong, because of the 0-ary case.
Very similar to an old error by Carboni and Johnstone.
Observation

Our diagrams do have automorphisms:

\[
\begin{array}{cc}
x & y \\
\end{array} \quad \rotatebox{90}{$\cong$} \quad \begin{array}{cc}
y & x \\
\end{array}
\]

Think of two 2-cells \( id \rightarrow id \).
Consequences of our observation

- Consider the presheaf $F$ on $S_2$ with one object, no morphism, and exactly two 2-cells 0 and 1.
- And the two labellings of the above $\begin{bmatrix} x & y \end{bmatrix}$ in $\{0, 1\}$:
  - $x \mapsto 0$, $y \mapsto 1$, and
  - $x \mapsto 1$, $y \mapsto 0$.
- Since $\begin{bmatrix} x & y \end{bmatrix} \cong \begin{bmatrix} y & x \end{bmatrix}$, they count as two labellings of the same element of $\mathcal{T}(0, 0)$.

Problem

By the Eckmann-Hilton argument they are equal in the free 2-category on $F$. 
Conclusion

- A technique to define algebraic structures from graphical ones.
- Problems when the graphs have automorphisms.

Room for improvement:
- Leinster-Weber provide too few tools to prove that $\mathcal{T}$ is a monad (even with conditions on $\mathcal{T}_\omega$),
- Better handle graphs with automorphisms (perhaps weaken the theory by not quotienting under isomorphism).