# Fully-abstract games for $\pi$

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Galop '14, Grenoble

# Standard games semantics (HO, AJM, Nickau)

- Initial aim: denotational semantics for functional languages.
- Went beyond: references, control, etc.
- Even concurrent languages (Laird, Ghica-Murawski), but for coarse behavioural equivalence, may testing equivalence.

## Asynchronous game semantics (Abramsky, Melliès, Mimram)

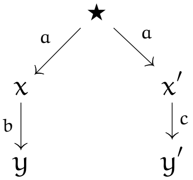
- Aim: denotational semantics for full linear logic.
- Main ingredient: plays should not need to record the exact order of moves, only their causal dependencies.
- Technically: games played on event structures.
- (Though plays remain sequences of moves, asynchrony being implemented as stability of strategies under permutations.)

# Asynchronous game semantics and concurrency

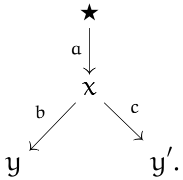
Surprisingly, never applied beyond confluent languages.

Possible reason:

- Standard notion of strategy: set of `accepted' plays.
- Akin to trace equivalence in concurrency.
- Does not account for branching behaviour.
- E.g., does not distinguish between




and




- Hence, could not have been finer than Laird or Ghica-Murawski.

# Concurrent games (Winskel, Castellan, Clairambault, Rideau)

Keep the idea of games on event structures.

**New notion of strategy**

May accept plays in more than one way.

E.g.,

- toss a coin, `privately',
- remember the result,
- modify behaviour accordingly,
- which may show only a few moves later.

To our knowledge, only starting to lead to new models of concurrent languages.

# This work

Follow up on previous work on CCS (with Pous, ICE '11 and SACS, then CALCO '13).

- Same ideas, independently:
  - concurrent notion of play,
  - strategies may accept plays in several ways.
- Implemented very differently!

# How differently?

1. Truly multi-player game.
2. Plays form a proper category.
   - May be several morphisms between plays.
   - Departs from plays as configurations in event structures.
   - Probably close to Levy's notion of morphisms between plays.
3. Borrow notion of strategies following presheaf models (JNW):
   - prefix-closed sets of plays;
   - functors $\mathrm{Plays}^{op} \to 2$, where $2$ is the poset $0 \leqslant 1$;
   - functors $\mathrm{Plays}^{op} \to$ sets.
4. Reincorporate game semantical innocence.

Last point is not obvious, let's explain a bit more.

# Innocent presheaves

- Subcategory Views $\rightarrowtail$ Plays.
- Innocent presheaves: Views$^{op} \to$ sets.

**Recovering global behaviour**

By right extension along Views$^{op} \rightarrowtail$ Plays$^{op}$.

(Works thanks to the more general notion of morphism.)

- Boolean setting: play accepted iff all views are.
- General case: way to accept a play = compatible family of ways to accept views.

# What do we obtain?

- A concurrent game semantics for $\pi$.
- Equality of strategies is very intensional.
- But the model enjoys a lot of structure. Simple categorical tools $\rightsquigarrow$
  - parallel composition of strategies,
  - global behaviour of a strategy, hence
  - semantic fair testing: $S \sim T$.

**Main result**

$P \sim_s Q$ iff $[\![P]\!] \sim [\![Q]\!]$, where $\sim_s$ is standard fair testing equivalence.

# Why fair testing?

- Detects branching behaviour.
- Perceived as one of the most sensible testing equivalences.
- Testing equivalences more intrinsic than LTSs.
- E.g., bisimilarity depends on the chosen LTS.

# This talk

The game for $\pi$ is the crux of the construction: emphasised here.

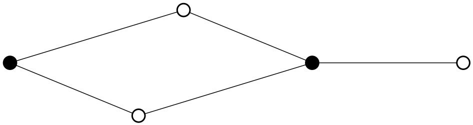
1. The game for $\pi$.
2. Innocent presheaves and global behaviour.
3. Semantic fair testing.
4. Idea of the translation $\pi \longrightarrow$ Strategies.

# What is meant by `a game'

- Set of positions: players and channels (truly multi-player).
- Maybe suprising: moves are not just a binary relation:
  - more than when there is a move $X \to Y$,
  - rather, how one moves from $X$ to $Y$.
- Here, the how is represented as an object $M$ in a category of diagrams.

Moves $X \to Y$ are cospans $X \to M \leftarrow Y$.

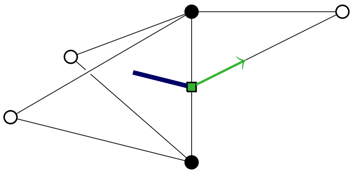
# Positions



- ● ≈ player.
- ○ ≈ channel.
- Edges : `player knows channel'.

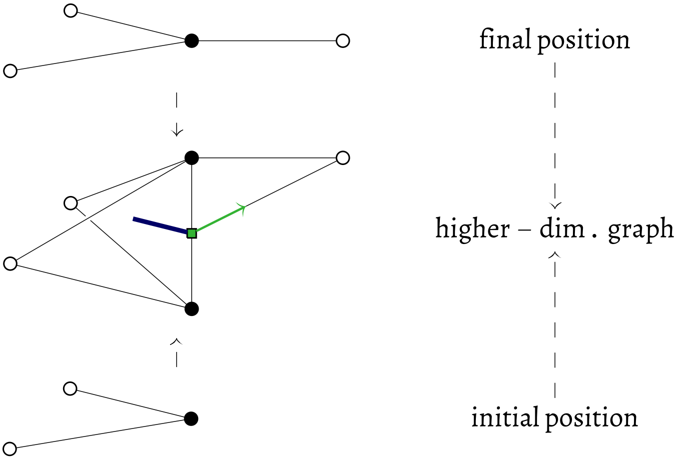
Now, what are diagrams? A kind of higher-dimensional graph.

# Higher-dimensional graph for the input move

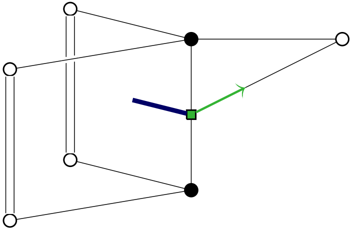


- Thicker, blue line = `pipe' = carrier channel.
- Green arrow = received channel.
- `Late' feel: the input arrow is fresh (for input alone).
- One such graph for each arity (here 3).
- Formal definition: please ask if interested, i find it cute.

# The input move



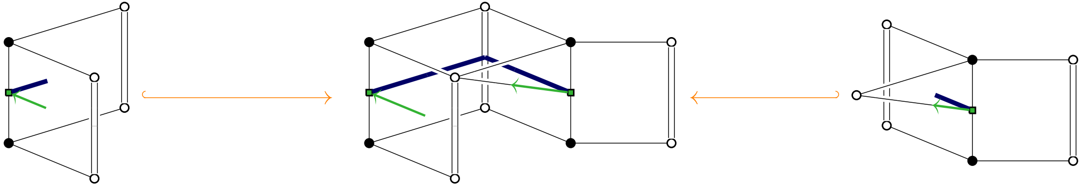

drawn for conciseness as:



And written as $\iota_{2,1} : 3 \twoheadrightarrow 2$.

# Moves: input/output

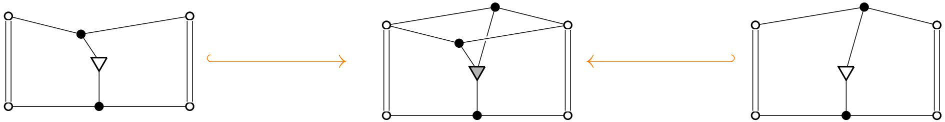
Using the previous convention:



Output Synchronisation Input

Orange arrows: cospan morphisms.

# Moves, continued



Left fork

Fork

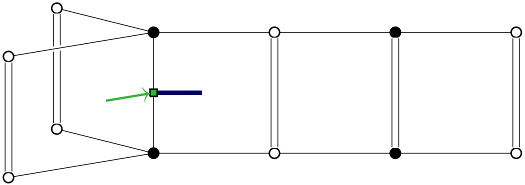
Right fork

Channel creation

Tick

# Local vs. global moves

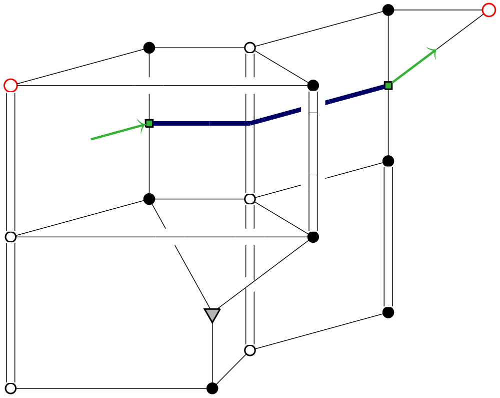
- Until now, moves were local: only involved players were shown.
- Global moves obtained by embedding into larger positions.
- E.g.:



  - First player outputs to the environment.
  - Second player does not notice anything.

# Plays

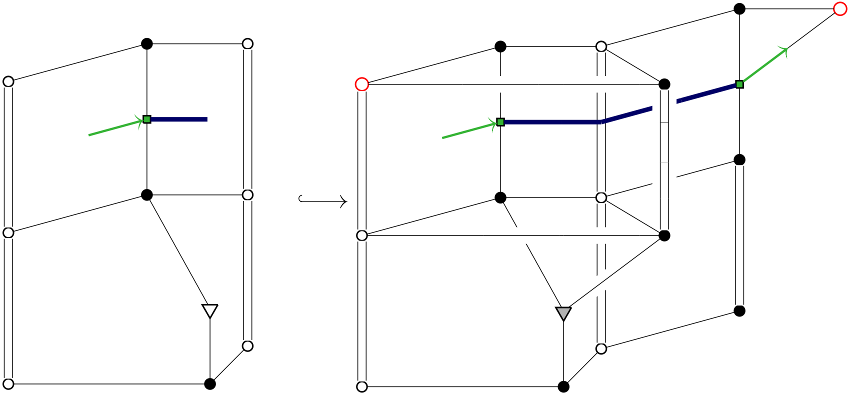
Obtained by piling up global moves:



(identifying the two red nodes)

Feature a certain amount of concurrency.

# Category of plays over position $X$: $P_X$



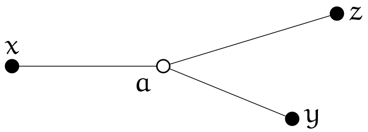
- Plus prefix inclusion.
- Possibly several morphisms between two plays.
- Otherwise, close to configuration posets of event structures.

# Naive, non-deterministic strategies over position $X$

**Definition 1. Strategy over $X$**

Presheaf $\mathbb{P}_X^{op} \to$ sets.

Too general: consider the position




and the naive strategy

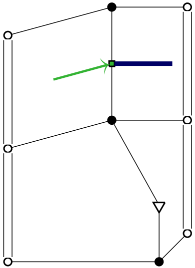
- accepting $x \to y$,
- accepting environment $\to z$,
- but refusing $x \to z$.

Players $x$ and $z$ should not be allowed to choose with whom they synchronise.

# Non-deterministic, innocent strategies

Views: let $V_X \subseteq P_X$ consist of histories of exactly one player.

Example:



**Definition 2. Innocent strategies**

Presheaf $V_X^{op} \longrightarrow$ sets.

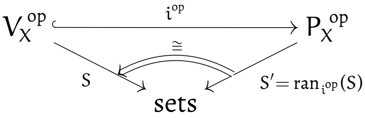
Problem: no obvious inclusion innocent $\subseteq$ naive.

# Fair testing: overview

- Global behaviour: essentially, innocent $\mapsto$ naive.
- Interaction, a.k.a. parallel composition.
- Fair testing.

# Global behaviour

By right Kan extension:




## Explicit formula

- General : $S'(p) = \int_{v \in V_X} S(v)^{P_X(v,p)}$.
- Boolean case; p accepted iff all its views are : $S'(p) = \bigwedge_{\{(v \xrightarrow{\alpha} p) \in P_X\}} S(v)$

## Global behaviour

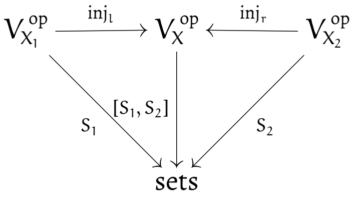
By restricting to closed-world plays: $S \mapsto S' \mapsto \bar{S}$.

# Interaction, a.k.a. parallel composition

- Split the players of position $X$ into two teams.
- Obtain two subpositions $X_1 \hookrightarrow X \hookleftarrow X_2$ sharing no player.
- We have

$$V_X \simeq V_{X_1} + V_{X_2}.$$

- Let $S_1$ play against $S_2$ by *copairing* :



# Fair testing

- Successful play: one with at least one ✓.
- $S \perp T$: all unsuccessful executions of $\overline{[S, T]}$ extend to successful ones.

**Definition 3. Semantic fair testing equivalence**

$S \sim S'$ iff $\forall T, S \perp T \Leftrightarrow S' \perp T$.

# A syntax for strategies

- Idea:

  A strategy ≈ what remains of it after each atomic view b.

- Generic syntax:

$$\frac{\ldots \quad n_b \vdash S_b \quad \ldots}{n \vdash_D \left\langle (S_b)_b \right\rangle} \qquad \frac{\ldots \quad n \vdash_D D_i \quad \ldots}{n \vdash \bigoplus_{i \in p} D_i}$$

where $b : n_b \twoheadrightarrow n$ ranges over

- $\iota_{n,a} : n+1 \twoheadrightarrow n$,
- $o_{n,a,b} : n \twoheadrightarrow n$,
- $\pi^l_n : n \twoheadrightarrow n$,
- $\pi^r_n : n \twoheadrightarrow n$,
- $\nu_n : n+1 \twoheadrightarrow n$,
- $\checkmark_n : n \twoheadrightarrow n$.

# The translation (examples)

$$P|Q \mapsto \boxed{\left\langle \begin{array}{rcl} \pi_n^l & \mapsto & [\![P]\!] \\ \pi_n^r & \mapsto & [\![Q]\!] \\ - & \mapsto & \emptyset \end{array} \right\rangle}$$

$$\nu a\,.\,P \mapsto \boxed{\left\langle \begin{array}{rcl} \nu_n & \mapsto & [\![P]\!] \\ - & \mapsto & \emptyset \end{array} \right\rangle}$$

$$a\,.\,P + a\,.\,Q \mapsto \boxed{\left\langle \begin{array}{rcl} \iota_{n,a} & \mapsto & [\![P]\!] \oplus [\![Q]\!] \\ - & \mapsto & \emptyset \end{array} \right\rangle}$$

...

## Main result

Recall standard fair testing equivalence:

- $P \in \perp\!\!\!\perp$ iff $\forall\ P \Rightarrow Q, \exists\, Q \stackrel{\checkmark}{\Longrightarrow} R$,
- $P \sim_s Q$ iff $\forall R, ((P|R) \in \perp\!\!\!\perp) \Leftrightarrow ((Q|R) \in \perp\!\!\!\perp)$.

**Theorem.**

$P \sim_s Q$ iff $[\![P]\!] \sim [\![Q]\!]$.

# Summary

- Concurrent game semantics through innocent presheaves.
- Full abstraction for a semantic analogue of fair testing equivalence.
- Techniques from categorical combinatorics:
  - various kinds of presheaves,
  - right Kan extension.

# Left out from this talk

Playgrounds: an algebraic structure $\approx$ rule of the game.

**Crucial point in playgrounds**

Plays fibred over positions.

I.e., operation to restrict a play on $X$ along $Y \to X$.

Much more satisfactory treatment than in previous work (using a factorisation system).

(Please ask if interested...)

# FAQ

Aren't strategies too syntactic (they contain terms)?

First answer: can't expect otherwise.

- HO strategies $\approx$ $\beta$-normal $\eta$-long terms.
- In concurrency: no notion of normal form.

Second answer, maybe better viewpoint. Not really a game semantics, rather:

- a categorical combinatorics approach
- to a slightly generalised syntax / semantics duality.

Promising, thanks to high-level tools from categorical combinatorics.

# Future work

- Scale the approach to Join, HO$\pi$P (passivation), $\lambda$,...
- Tools for generating playgrounds (with Clovis Eberhart).
- Investigate morphisms between playgrounds (c.f. compilation).
- Link with exotic settings like cellular automata.
- `Double category of elements' $\rightsquigarrow$ new notion of abstract rewriting system.

Thanks for your attention!