Yoneda meets concurrent game semantics

Clovis Eberhart and Tom Hirschowitz

Sheaf models

- Sheaf models: a denotational semantics using ideas from
  - causal models (Nielsen, Plotkin and Winskel, early 80's),
  - presheaf models (Joyal, Nielsen and Winskel, early 90's),
  - game semantics (Abramsky et al., Hyland and Ong, early 90's).
- Basic idea: innocent and concurrent strategies as sheaves on *ad hoc* sites.
- Applied to CCS and π-calculus (with Pous and Seiller, '12 - '15), and to non-deterministic λ-calculus (Tsukada and Ong, '15).

Goal

- In our work on CCS and π: (innocent) strategies automatically derived from algebraic gadget describing the game (a playground).
- Playground theory $\rightsquigarrow$ bisimilar transition systems for terms and strategies.
- Tsukada and Ong do not use it (hence prove adequacy entirely by hand).
- Ongoing work with Clovis: organise their game into a playground.
- Non-trivial: techniques used for CCS and π subtly fail.

Here: less ambitious goal

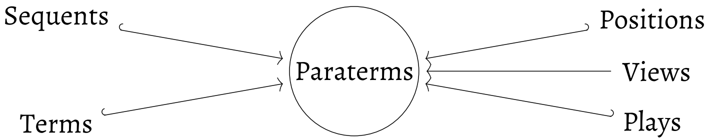
Common generalisation.

Overview

- Start from a natural deduction presentation of your preferred language:

  Sequents + terms as typing derivation trees.

- Step 1 (reap): identify



- Step 2 (sow): derive automatically
  - strategies,
  - innocent strategies as sets of views: innocent behaviours,
  - innocent strategies as sets of plays: innocent strategies,
  - translation: terms $\longrightarrow$ innocent strategies,
  - ... (work in progress),

  from Yoneda theory.

Main idea

- Terms are trees, standardly presented as pointed presheaves.
- Innocent behaviours are also trees, presented as presheaves over branches.

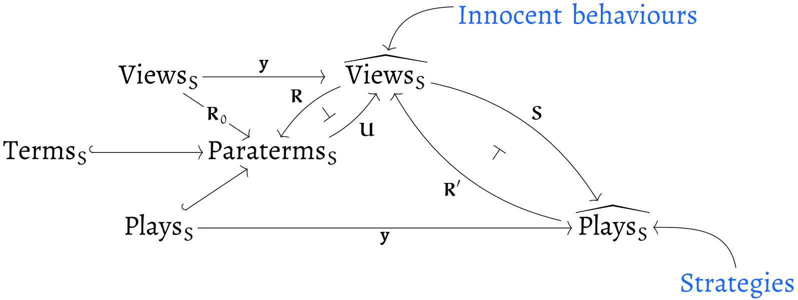
Heart of the matter

Connection between these two presheaf-based presentations of trees.

Main technical tool: Yoneda theory, which is about presheaves.

Overview of the process

- For any sequent S, Terms_S, Views_S, Plays_S, $\mathrm{Paraterms}_S$.
- Yoneda embedding, $\mathbf{y} : \mathscr{C} \to \widehat{\mathscr{C}}$.



- Innocent strategies: image of $\mathbf{S}$.
- Translation: composite $\mathbf{S} \circ \mathbf{U}$.

Running example

Here, stripped down example:

The division calculus

- Terms:

$$T ::= 0 \mid (T_1|\ldots|T_n) \mid (T_1 + \ldots + T_n) + \text{recursion.}$$

- Configurations: finite multisets of terms $[T_1, \ldots, T_n]$.
- Computation:

$$[\ldots, ((T_1|\ldots|T_n) + \ldots), \ldots] \longrightarrow [\ldots, T_1, \ldots, T_n, \ldots].$$

Natural deduction presentation

Sequents $= \{\star\}$.

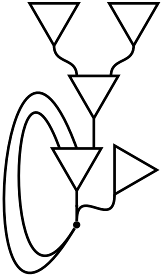
For rules, omitting recursion and non-deterministic branching, one rule for each n:

$$\frac{T_1 \quad \dots \quad T_n}{T_1 | \dots | T_n}\,(i \in \{1, \dots, n\})$$

Terms as finite pointed multigraphs (by example)

Consider $\textbf{let rec}\, X = 0 + (X | X | (0 | 0))\ \textbf{in}\, X$.

Represent it as a derivation tree with loops:



with • the distinguished node.

$\mathrm{Terms}_\star \hookrightarrow \mathrm{MGph}_\star$ (category of pointed multigraphs).

Game interpretation: positions and plays

- Position: finite set n of players.
- Move: player $i \in n$ divides itself into p new players.

$$(\text{Initial position: } n) \quad \rightsquigarrow \quad (\text{Final position: } n-1+p)$$

  Yields a directed graph with edges $n \xrightarrow{(i,p)} n-1+p$.
- Keeping track of which player has played and which players are created:

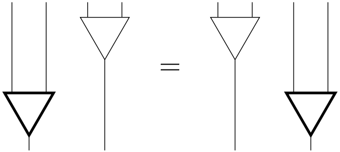
$$n \cong (i-1)+1+(n-i) \to (i-1)+p+(n-i) \cong n-1+p\,.$$

Game interpretation: plays

- Play: path in the graph of moves, up to permutation of independent steps. Example: two players forking in parallel.

$$\begin{array}{ccc} 1+1 & \xrightarrow{(1,2)} & 2+1 \\ {\scriptstyle (2,2)}\downarrow & & \downarrow{\scriptstyle (3,2)} \\ 1+2 & \xrightarrow[(1,2)]{} & 2+2 \end{array}$$

or graphically



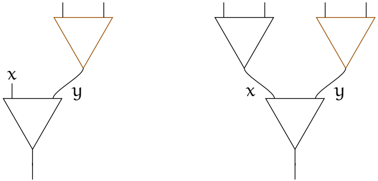
Again!

Plays on one player are just finite trees:

$$\mathrm{Plays}_{\star} \hookrightarrow \mathrm{MGph}_{\star}.$$

Game interpretation: strategies, views and innocence

- Deterministic strategy: prefix-closed set of plays.
- Non-deterministic strategies on $\star$ (one player): $\widehat{\mathrm{Plays}_\star}$.
- Innocent strategy: each player decides on its own.
- Example:

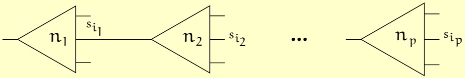


  - If both black parts are accepted then whole plays should be accepted at the same time.
  - Intuition: y cannot 'see' whether x has played or not.
  - The 'view' of y is the left-hand black part.

Views and innocence 2

Definition 1.

Actual view: linear tree



- Actual views $\hookrightarrow$ $\mathbf{MGph}_\star$.
- Remark: Pointed morphism $v \to v' =$ prefix relation $v \leqslant v'$.

Definition 2.

- Formal view: sequence of pairs (arity, input port) $(n_1, i_1) \dots (n_p, i_p)$ with $i_j \in n_j$ for all $j \in p$.
- Form a category by the prefix ordering: morphism $v \to v'$ iff $v \leqslant v'$.

Formal views $\xrightarrow{R_0}$ actual views $\hookrightarrow$ $\mathbf{MGph}_\star$.

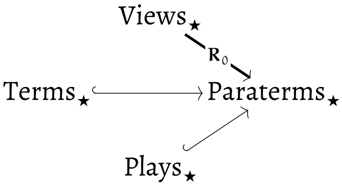
The setting

No surprise:

Definition 3.

$\mathrm{Paraterms}_\star = \mathrm{MGph}_\star$.

We can apply our construction to:

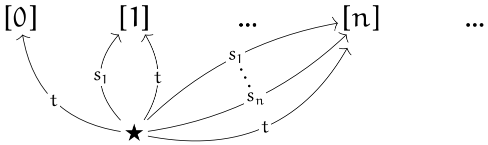


Now the fun begins.

Multigraphs

Let us start by giving our precise representation for multigraphs.

Base category $\mathbb{M}$:



Abuse of notation: morphisms should be indexed by n.

Definition 4.

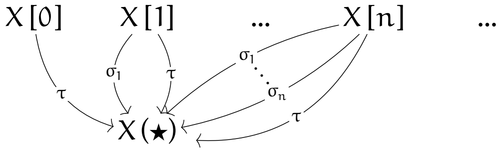
$\mathrm{MGph} = \widehat{\mathbb{M}}$.

- Category of presheaves, i.e., functors $\mathbb{M}^{op} \to \mathrm{Set}$.
- Morphisms = natural transformations.

  Intuition: map vertices/edges to vertices/edges preserving sources and target.

Multigraphs

A multigraph X is thus a diagram in Set:

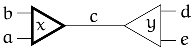


- $X[n]$ is the set of n-ary multiedges, and so on.
- $\sigma_i : X[n] \to X(\star)$ gives the ith source of each such multiedge.
- $\tau : X[n] \to X(\star)$ gives the target of each such multiedge.
- $\sigma_i(e) = e \cdot s_i$, action of s_i.
- Similarly, $e \cdot t$.

An example

- $X(\star) = \{a, b, c, d, e\}$,
- $X[2] = \{x, y\}$,
- $X[n] = \emptyset$ otherwise,
- $x \cdot s_1 = a$, $x \cdot s_2 = b$, $x \cdot t = c$, $y \cdot t = c$, $y \cdot s_1 = d$, $y \cdot s_2 = e$.

Graphically:

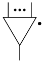


First ingredient of Yoneda theory: the Yoneda embedding

Yoneda embedding

$\mathbf{y} : \mathbb{M} \to \widehat{\mathbb{M}}$.

In our case:

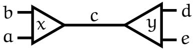
- $\mathbf{y}_\star$ = single vertex.
- $\mathbf{y}_{[n]}$ = the 'typical' n-ary multiedge .
- $\mathbf{y}_{s_i} : \mathbf{y}_\star \to \mathbf{y}_{[n]}$ embeds single vertex as ith source of n-ary multiedge.

Example application of the Yoneda embedding

Previous example = pushout:

$$\begin{array}{ccc} \mathbf{y}_{\star} & \xrightarrow{\mathbf{y}_{t}} & \mathbf{y}_{[2]} \\ {\scriptstyle \mathbf{y}_{t}}\downarrow & \ulcorner & \downarrow \\ \mathbf{y}_{[2]} & \longrightarrow & X. \end{array}$$

Intuition: glue two binary multiedges along their targets.



Second ingredient of Yoneda theory: co-Yoneda lemma

This generalises to:

Any presheaf is canonically a colimit of its elements.

- Morally: a multigraph is a gluing of its multiedges and vertices.
- Elément de langage:

Coends

$$X \cong \int^{c} X(c) \cdot \mathbf{y}_c$$

where

- c ranges over all objects,
- $A \cdot Y := Y + \ldots + Y$ (A times),
- $\sum_c X(c) \cdot \mathbf{y}_c$ would put elements side by side,
- $\int^c X(c) \cdot \mathbf{y}_c$ is a quotient — that's where the gluing occurs.

The mother of all Yoneda situations

Set up

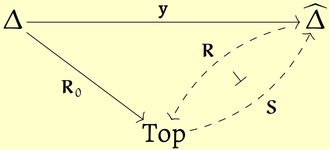
- Δ = base category for simplicial sets.

  Objects = natural numbers a.k.a. finite ordinals.

  Morphisms = monotone maps.
- Top = topological spaces + continuous maps.

The mother of all Yoneda situations

The magic diagram

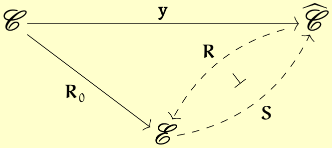


- $\mathbf{R}_0[n]$ = the n-simplex.
- $\mathbf{R}(X) = \mathbf{R}\left(\underline{\int^c X(c) \cdot \mathbf{y}_c}\right) = \int^c X(c) \cdot \mathbf{R}_0(c)$ (using co-Yoneda)

  Glue realisations of X's elements — $x \in X(c)$ being realised as $\mathbf{R}_0(c)$.
- $\mathbf{S}(U)[n] = \mathrm{Top}(\mathbf{R}_0[n], U)$
- $\mathbf{R}$ and $\mathbf{S}$ form an adjunction (in this case even a Quillen equivalence) though we won't need this here.

General Yoneda situation

For $\mathscr{C}$ small and $\mathscr{E}$ cocomplete:

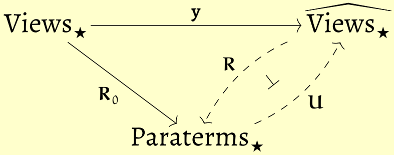


- $\mathbf{R}(X) = \mathbf{R}\left(\underline{\int^{c} X(c)\cdot \mathbf{y}_c}\right) = \int^{c} X(c)\cdot \mathbf{R}_0(c)$.
- $\mathbf{S}(E)(c) = \mathscr{E}(\mathbf{R}_0(c), E)$
- $\mathbf{R}$ and $\mathbf{S}$ form an adjunction.

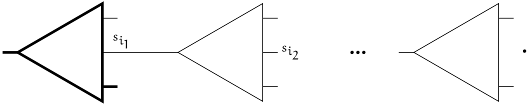
Yoneda situation for pointed multigraphs (= paraterms)

Recall that $\mathrm{Paraterms}_\star = \mathrm{MGph}_\star$ ($= \star/\widehat{\mathbb{M}}$).

Yoneda situation for pointed multigraphs

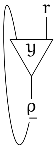


$\mathbf{R}_0((n_1, i_1), \ldots, (n_p, i_p))$ is



Example

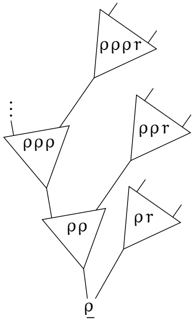
$\mathbf{R} \circ \mathbf{U}$ ‘unfolds’ its argument.

- Let G denote 

 (root underlined).

- $\mathbf{U}(G) : \mathrm{Views}_{\star}^{op} \longrightarrow \mathrm{Set}$ with $\mathbf{R}(\mathbf{U}(G))$:

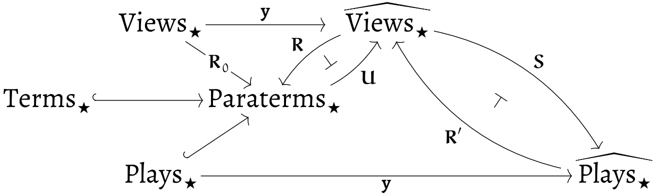
$$
\begin{aligned}
\varepsilon &\mapsto \{\rho\} \\
((2,1)) &\mapsto \{\rho\,\rho\} \\
((2,2)) &\mapsto \{\rho\,r\} \\
(2,1)^{+} &\mapsto \{\rho \ldots \rho\} \\
((2,1)^{\star}, (2,2)) &\mapsto \{\rho \ldots \rho\, r\} \\
- &\mapsto \emptyset
\end{aligned}
$$



Two combined Yoneda situations

Innocent behaviours

$\mathbf{U}(G)(v) = \mathrm{Paraterms}_\star(\mathbf{R}_0(v), G)$



Strategies

$\mathbf{S}(B)(P) = \widehat{\mathrm{Views}_\star}(\mathbf{U}(P), B)$

Definition 5.

Innocent strategy: one in the image of $\mathbf{S}$.

- $\mathbf{S} \circ \mathbf{U}$ maps terms to innocent strategies (translation).
- $\mathbf{S} \circ \mathbf{R}'$ maps naive strategies to innocent strategies (innocentisation).

When does the innocent strategy associated to term T accept play P?

$$\mathbf{S}(\mathbf{U}(T))(P)$$

$$\cong \widehat{\mathrm{Views}_\star}(\mathbf{U}(P), \mathbf{U}(T)) \qquad \text{(defn. of } \mathbf{S}\text{)}$$

$$\cong \int_{v \in \mathrm{Views}_\star} [\mathbf{U}(P)(v), \mathbf{U}(T)(v)] \qquad \text{(nat. transfos = ends ; } [A, B] \text{ means hom-set)}$$

$$\cong \int_{v \in \mathrm{Views}_\star} [[\mathbf{R}_0(v), P], [\mathbf{R}_0(v), T]] \qquad \text{(defn. of } \mathbf{U}\text{)}$$

$$\cong \int_{v \in \mathrm{Views}_\star} [[\mathbf{R}_0(v), P] \cdot \mathbf{R}_0(v), T] \qquad ([A, [B, C]] = (C^B)^A \cong C^{A \cdot B} = [A \cdot B, C])$$

$$\cong \left[\int^{v \in \mathrm{Views}_\star} [\mathbf{R}_0(v), P] \cdot \mathbf{R}_0(v), T\right] \qquad (\textstyle\int_v [A_v, B] \approx \prod_v B^{A_v} \cong B^{\sum_v A_v} \approx [\int^w A_v, B])$$

$$\cong [\mathbf{R}(\mathbf{U}(P)), T] \qquad \text{(defn. of } \mathbf{R}\text{)}$$

$$= [\mathrm{unfolding}(P), T].$$

Ways of accepting P ≅ ways of mapping each branch of P into T.

Summary

On a simplistic example:

- Step 1: notion of paraterm $\supseteq$ terms, plays, views.
- Step 2: two combined Yoneda situations.

$\rightsquigarrow$

- Automatic definition of innocent behaviours and (innocent) strategies
- and various links between the involved notions:
  - translation terms $\longrightarrow$ strategies,
  - adjunction innocent behaviours $\leftrightarrow$ strategies.

Next steps

- Extend to CCS, π, and λ and recover desired constructions.
- Does playground theory generalise?
  - Not shown: transition system on strategies.
  - Construct one for paraterms ($\approx$ GoI).
  - When is the translation a bisimulation?

Thanks for your attention!