An algebraic approach to higher-order theories and rewriting

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Original motivation

- Understood from a talk by Fiore in Lyon:
  - will to explain variable binding by a λ-calculus limited to 2nd order;
  - complex categorical picture.
- Alternative point of view:
  - the λ-calculus perfectly explains variable binding;
  - simple categorical picture: cartesian closed categories.

Introduction

Here:

- A categorical semantics for
  - higher-order rewriting (Klop, Nipkow, Wolfram), and
  - permutation equivalence (Bruggink).
- In cartesian closed 2-categories.
- Corollary: a semantics for variable binding in cartesian closed categories.

Related work

- Cartesian closed sketches (Kinoshita, Power, Takeyama and Wells).
- Bruggink's permutation equivalence, Hilken's 2-λ-calculus, Jacobs' book.
- Capriotti's semantics for flat permutation equivalence in sesqui-categories.
- By extension, Fiore et al., my father and Maggesi.

Plan

1. Introduction.
2. The known case:
   - rewrite systems as fp 2-signatures,
   - their fp 2-categorical semantics.
3. Sketch of the cartesian closed case.

Types

Given a set X, let $\mathcal{L}_0(X)$ be the set of formulas, as generated by the grammar:

$$A, B ::= 1 \mid x \mid A \times B.$$

Proposition

This extends to a monad on Set *with substitution as multiplication.*

Definition

Let $\mathcal{S}_0(X) = \mathcal{L}_0(X)^* \times \mathcal{L}_0(X)$ be the set of sequents over X.

1-Signatures

Definition

A 1-signature $\Sigma = (X_0, X_1)$ consists of:

- a set X_0 of sorts,
- a map $X_1 \to \mathcal{S}_0(X_0)$ of operations.

The latter amounts to a set indexed by sequents in $\mathcal{S}_0(X_0)$.

Example: combinatory logic

- One sort $X_0 = \{t\}$.
- Three operations $X_1 = \{K, S, A\}$:
  - $K, S \colon 1 \to t$,
  - $A \colon t, t \to t$,

  i.e.,
  - $K, S \mapsto ([], t)$,
  - $A \mapsto ([t, t], t)$.

Non-example: the λ-calculus

- Arity of λ: "$t^t \to t$".
- Not expressible by our arities.
- Workarounds do exist.

A category of 1-signatures

Consider 1-signatures $\Sigma = (X_0, X_1)$ and $\Sigma' = (Y_0, Y_1)$.
A morphism $\Sigma \to \Sigma'$ consists of:

- a map $f_0 \colon X_0 \to Y_0$,
- a map $f_1 \colon X_1 \to Y_1$ making

$$\begin{tikzcd}
X_1 \arrow[r, "f_1"] \arrow[d] & Y_1 \arrow[d] \\
\mathcal{S}_0(X_0) \arrow[r, "\mathcal{S}_0(f_0)"'] & \mathcal{S}_0(Y_0)
\end{tikzcd}$$

commute.

A category of 1-signatures

Proposition

This yields a category Sig_1 *of 1-signatures.*

The term monad

Given a 1-signature $\Sigma = (X_0, X_1)$, let $\mathcal{L}_1(\Sigma)$ be the 1-signature with:

- sorts X_0,
- operations $A_1, \ldots, A_n \longrightarrow A$ the terms $x_1 \colon A_1, \ldots, x_n \colon A_n \vdash M \colon A$ in the language generated by Σ.

Proposition

$\mathcal{L}_1$ extends to a monad on Sig_1.

Parallel operations

For a 1-signature $\Sigma = (X_0, X_1)$, let $\Sigma_{||}$ be the pullback:

$$\begin{tikzcd} \Sigma_{||} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X_1 \arrow[d] \\ X_1 \arrow[r] & \mathcal{S}_0(X_0). \end{tikzcd}$$

Example: $\mathcal{L}_1(\Sigma)_{||}$ is the set of parallel terms over Σ.

Proposition

This extends to a functor $(-)_{||} \colon \mathrm{Sig}_1 \to \mathrm{Set}$.

Rewrite systems = 2-signatures

Definition

A rewrite system, or a 2-signature, X consists of:

- a 1-signature $\Sigma = (X_0, X_1)$,
- a map $X_2 \to \mathcal{L}_1(\Sigma)_{||}$ of rewrite rules.

Definition

The rewrite relation generated by X is the smallest congruence generated by the rewrite rules.

A bit more general than the original definition.

Example: combinatory logic

- Remember K, S, A.
- Two rewrite rules (with usual shorthand):

$$Kxy \to x \qquad\qquad Sxyz \to xy(xz).$$

A category of 2-signatures

A morphism $X \to Y$ of 2-signatures consists of

- a morphism $(f_0, f_1) \colon (X_0, X_1) \to (Y_0, Y_1)$ of 1-signatures,
- a map $f_2 \colon X_2 \to Y_2$ making

$$
\begin{tikzcd}
X_2 \arrow[r, "f_2"] \arrow[d] & Y_2 \arrow[d] \\
\mathcal{L}_1(X)_{||} \arrow[r, "\mathcal{L}_1(f_1)_{||}"'] & \mathcal{L}_1(Y)_{||}
\end{tikzcd}
$$

commute.

A category of 2-signatures

Proposition

This yields a category Sig.

Reductions and permutation equivalence

- We now define a monad $\mathcal{L}$ on Sig.
- We then use $\mathcal{L}$ to construct adjunctions

$$
\begin{tikzcd}
\mathrm{Sig} \arrow[r, bend left, "\mathcal{L}"] \arrow[r, phantom, "\perp"] & \mathcal{L}\text{-}\mathrm{Alg} \arrow[l, bend left, "\mathcal{U}"] \arrow[r, bend left, "\mathcal{F}"] \arrow[r, phantom, "\perp"] & \text{2FPCat.} \arrow[l, bend left, "\mathcal{V}"]
\end{tikzcd}
$$

- Reminiscent of multigraphs $\dashv$ monoidal categories through multicategories.
- Though missing the language for representable $\mathcal{L}$-algebras.

Reductions and permutation equivalence

- $\mathcal{L}$ preserves underlying 1-signature.
- The rewrite rules of $\mathcal{L}(X)$ are the reductions, modulo permutation equivalence, which we now define.
- Along the way, we sketch:
  - how $\mathcal{L}(X)$ yields an fp 2-category,
  - how any fp 2-category yields an $\mathcal{L}$-algebra.

Reductions

First, for $(r \in X(G \vdash M, N \colon t))$, we have the formation rule:

$$\frac{\ldots \qquad \Gamma \vdash P_i : M_i \to N_i \colon G_i \qquad \ldots}{\Gamma \vdash r(P_1, \ldots, P_n) : M[M_1, \ldots, M_n] \to N[N_1, \ldots, N_n] \colon A} .$$

2-categorically:

$$\begin{tikzcd} \prod \Gamma \arrow[r, bend left, "{(M_1, \ldots, M_n)}"] \arrow[r, bend right, "{(N_1, \ldots, N_n)}"'] \arrow[r, phantom, "\Downarrow P"] & \prod G \arrow[r, bend left, "M"] \arrow[r, bend right, "N"'] \arrow[r, phantom, "\Downarrow r"] & A, \end{tikzcd} \quad \text{with } P = (P_1, \ldots, P_n).$$

Reductions

There is a rule for sequential composition:

$$\frac{\Gamma \vdash P : M_1 \to M_2 : A \qquad \Gamma \vdash Q : M_2 \to M_3 : A}{\Gamma \vdash P \,;_{M_2} Q : M_1 \to M_3 : A}\ .$$

2-categorically:

$$\begin{tikzcd}
\prod G \arrow[rr, bend left=50, "M_1"] \arrow[rr] \arrow[rr, bend right=50, "M_3"'] & {} \arrow[phantom, u, shift right=2ex, "\Downarrow P"] \arrow[phantom, d, shift left=2ex, "\Downarrow Q"] & A.
\end{tikzcd}$$

Reductions

Plus a few other rules:

- for empty reductions, i.e., identities,
- for reductions in context, i.e., making reductions into a congruence.

Permutation equivalence

- We then quotient out reductions by permutation equivalence.
- Also inductively defined.
- Permutation actually happens in the rules equating

$$
\begin{tikzcd}
C \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & B \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow r"] & A,
\end{tikzcd}
\quad \text{with} \quad
\begin{tikzcd}
C \arrow[r] \arrow[r, bend left=60] \arrow[r, phantom, shift left=3, "\Downarrow"] & B \arrow[r] & A \\
C \arrow[r] & B \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift right=3, "\Downarrow r"] & A
\end{tikzcd}
$$

and the other way around. Namely:

$$r(P \,;_{N_2}\, Q) \equiv M_1[P] \,;_{M_1[N_2]}\, r(Q).$$

The monad

Theorem

This defines a functor $\mathcal{L}\colon \mathsf{Sig} \to \mathsf{Sig}$*, which is a monad with substitution as multiplication.*

The adjunction

We use $\mathcal{L}$ to construct an adjunction

$$\begin{tikzcd} \mathsf{Sig} \arrow[r, bend left, "\mathcal{H}"] \arrow[r, phantom, "\perp"] & \mathsf{2FPCat} \arrow[l, bend left, "\mathcal{W}"] \end{tikzcd}$$

as the composite

$$\begin{tikzcd} \mathsf{Sig} \arrow[r, bend left, "\mathcal{L}"] \arrow[r, phantom, "\perp"] & \mathcal{L}\text{-}\mathrm{Alg} \arrow[l, bend left, "\mathcal{U}"] \arrow[r, bend left, "\mathcal{F}"] \arrow[r, phantom, "\perp"] & \mathsf{2FPCat}. \arrow[l, bend left, "\mathcal{V}"] \end{tikzcd}$$

Soundness and completeness

Proposition

Given a 2-signature X, form the fp 2-category X^* with:

- *objects: tuples of sorts,*
- *morphisms: tuples of terms,*
- *preordered hom-categories:*
  - $X^*(M, N) = 1$ *iff* $M \to^* N$,
  - $X^*(M, N) = \emptyset$ *otherwise.*

Soundness and completeness

Theorem

For all sorts $t_1, \ldots, t_n, t$, there is a functor

$$\mathcal{H}(X)(\Pi t_i, t) \to X^*(\Pi t_i, t),$$

which is full and identity-on-objects (to a preorder).

Hence:

- If $P \colon M \to N$ in $\mathcal{H}(X)$ then $M \to^* N$ in the standard sense.
- Conversely.

Cartesian closedness

- Up to now: first-order rewriting.
- 2-categorical semantics known, or at least folklore.
- New presentation as far as I know.
- Now, sketch for higher-order rewriting.

2-signatures

- Types: $A, B ::= 1 \mid x \mid A \times B \mid B^A$.
- $\mathcal{L}_1(\Sigma)$: simply-typed λ-calculus on Σ.
- 2-signature: set indexed by pairs of parallel λ-terms in β-normal, η-long form.

Example

- Signature Σ: $\quad \ell \colon t^t \to t \qquad a \colon t \times t \to t.$
- Pure λ-terms with variables in n in bijection with

$$\mathcal{L}_1(\Sigma)(\underbrace{t, \ldots, t}_{n \text{ times}} \to t).$$

- One rule $\beta \colon a(\ell(M^{t^t}), N^t) \to MN$, i.e.,

$$\begin{tikzcd}
 & t \times t \arrow[dr, "a"] \arrow[d, phantom, "\Downarrow\beta"] & \\
t^t \times t \arrow[ur, "\ell \times t"] \arrow[rr, bend right, "ev"'] & {} & t.
\end{tikzcd}$$

Reductions and permutations

As before, plus formation rules

$$\frac{\Gamma, x\colon A \vdash P : M \to N\colon B}{\Gamma \vdash \lambda x\colon A.P : \lambda x\colon A.M \to \lambda x\colon A.N\colon B^A}$$

$$\frac{\Gamma \vdash P : M \to M'\colon B^A \qquad \Gamma \vdash Q : N \to N'\colon A}{\Gamma \vdash PQ : MN \to M'N'\colon B}$$

with equations for $\beta\eta$.

Soundness and completeness

From any 2-signature X, higher-order rewriting yields a reduction relation between terms on the underlying 1-signature.

Proposition

$M \longrightarrow^* N$ iff there exists a reduction $P \colon M \longrightarrow N$.

Corollary

- Cartesian closed categories are locally discrete cartesian closed 2-categories: $\begin{tikzcd} \mathbf{2CCCat} \arrow[r, bend left] \arrow[r, phantom, "\perp"] & \mathbf{CCCat}. \arrow[l, bend left] \end{tikzcd}$

- Composing with the above: $\begin{tikzcd} \mathbf{Sig} \arrow[r, bend left, "\mathcal{K}"] \arrow[r, phantom, "\perp"] & \mathbf{CCCat}. \arrow[l, bend left] \end{tikzcd}$

- Fiore and Hur's syntactic category on X is a full subcategory of $\mathcal{K}(X)$.
- More objects in $\mathcal{K}(X)$, e.g., t^{t^t}.

Uncaught

- The π-calculus (and all calculi with a structural congruence).
- Would need to incorporate equations between reductions.
- Even with that, no efficient presentation of π yet.