An algebraic approach to higher-order theories and rewriting

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Original motivation

- Understood from a talk by Fiore in Lyon:
  - will to explain variable binding by a $\lambda$-calculus limited to 2nd order;
  - complex categorical picture.

- Alternative point of view:
  - the $\lambda$-calculus perfectly explains variable binding;
  - simple categorical picture: cartesian closed categories.
Introduction

Here:

- A categorical semantics for
  - higher-order rewriting (Klop, Nipkow, Wolfram), and
  - permutation equivalence (Bruggink).
- In cartesian closed 2-categories.
- Corollary: a semantics for variable binding in cartesian closed categories.
Related work

- Cartesian closed sketches (Kinoshita, Power, Takeyama and Wells).
- Bruggink’s permutation equivalence, Hilken’s 2-λ-calculus, Jacobs’ book.
- Capriotti’s semantics for flat permutation equivalence in sesqui-categories.
- By extension, Fiore et al., my father and Maggesi.
Plan

1 Introduction.
2 The known case:
   ▶ rewrite systems as fp 2-signatures,
   ▶ their fp 2-categorical semantics.
3 Sketch of the cartesian closed case.
Types

Given a set $X$, let $\mathcal{L}_0(X)$ be the set of formulas, as generated by the grammar:

$$A, B ::= 1 \mid x \mid A \times B.$$ 

**Proposition**

This extends to a monad on Set with substitution as multiplication.

**Definition**

Let $S_0(X) = \mathcal{L}_0(X)^* \times \mathcal{L}_0(X)$ be the set of sequents over $X$. 
1-Signatures

Definition

A 1-signature $\Sigma = (X_0, X_1)$ consists of:

- a set $X_0$ of sorts,
- a map $X_1 \rightarrow S_0(X_0)$ of operations.

The latter amounts to a set indexed by sequents in $S_0(X_0)$. 
Example: combinatory logic

- One sort $X_0 = \{ t \}$.
- Three operations $X_1 = \{ K, S, A \}$:
  - $K, S : 1 \to t$, i.e., $K, S \mapsto ([] \mapsto t)$,
  - $A : t, t \to t$, $A \mapsto ([t, t], t)$. 
Non-example: the $\lambda$-calculus

- Arity of $\lambda$: "$t^t \rightarrow t$".
- Not expressible by our arities.
- Workarounds do exist.
A category of 1-signatures

Consider 1-signatures \( \Sigma = (X_0, X_1) \) and \( \Sigma' = (Y_0, Y_1) \).
A morphism \( \Sigma \to \Sigma' \) consists of:

- a map \( f_0 : X_0 \to Y_0 \),
- a map \( f_1 : X_1 \to Y_1 \) making

\[
\begin{array}{ccc}
X_1 & \xrightarrow{f_1} & Y_1 \\
\downarrow & & \downarrow \\
S_0(X_0) & \xrightarrow{S_0(f_0)} & S_0(Y_0)
\end{array}
\]

commute.
A category of 1-signatures

**Proposition**

This yields a category $\text{Sig}_1$ of 1-signatures.
The term monad

Given a 1-signature $\Sigma = (X_0, X_1)$, let $\mathcal{L}_1(\Sigma)$ be the 1-signature with:

- sorts $X_0$,
- operations $A_1, \ldots, A_n \rightarrow A$ the terms $x_1 : A_1, \ldots, x_n : A_n \vdash M : A$ in the language generated by $\Sigma$.

**Proposition**

$\mathcal{L}_1$ extends to a monad on $\text{Sig}_1$. 
Parallel operations

For a 1-signature $\Sigma = (X_0, X_1)$, let $\Sigma_{\parallel}$ be the pullback:

$$
\begin{array}{c}
\Sigma_{\parallel} \\
\downarrow \\
X_1
\end{array}
\longrightarrow
\begin{array}{c}
X_1 \\
\downarrow \\
S_0(X_0).
\end{array}
$$

Example: $\mathcal{L}_1(\Sigma)_{\parallel}$ is the set of parallel terms over $\Sigma$.

**Proposition**

*This extends to a functor* $(-)_{\parallel}: \mathbf{Sig}_1 \to \mathbf{Set}$. 
Rewrite systems $\equiv$ 2-signatures

**Definition**
A rewrite system, or a 2-signature, $X$ consists of:
- a 1-signature $\Sigma = (X_0, X_1)$,
- a map $X_2 \to \mathcal{L}_1(\Sigma)\|_1$ of rewrite rules.

**Definition**
The rewrite relation generated by $X$ is the smallest congruence generated by the rewrite rules.

A bit more general than the original definition.
Example: combinatory logic

- Remember $K$, $S$, $A$.
- Two rewrite rules (with usual shorthand):

\[ K_{xy} \rightarrow x \quad S_{xyz} \rightarrow xy(xz). \]
A category of 2-signatures

A morphism $X \to Y$ of 2-signatures consists of

- a morphism $(f_0, f_1): (X_0, X_1) \to (Y_0, Y_1)$ of 1-signatures,
- a map $f_2: X_2 \to Y_2$ making

\[
\begin{array}{ccc}
X_2 & \xrightarrow{f_2} & Y_2 \\
\downarrow & & \downarrow \\
\mathcal{L}_1(X) & \xrightarrow{\mathcal{L}_1(f_1)} & \mathcal{L}_1(Y)
\end{array}
\]

commute.
A category of 2-signatures

**Proposition**

(This yields a category $\text{Sig}$.)
Reductions and permutation equivalence

- We now define a monad $\mathcal{L}$ on $\text{Sig}$.
- We then use $\mathcal{L}$ to construct adjunctions

$\text{Sig} \quad \mathcal{L} \quad \bot \quad \mathcal{U} \quad \rightleftharpoons \quad \mathcal{L} \text{-Alg} \quad \mathcal{F} \quad \bot \quad \mathcal{V} \quad \rightleftharpoons \quad \text{2FPCat}$.

- Reminiscent of multigraphs $\dashv$ monoidal categories through multicategories.
- Though missing the language for representable $\mathcal{L}$-algebras.
Reductions and permutation equivalence

- \( \mathcal{L} \) preserves underlying 1-signature.
- The rewrite rules of \( \mathcal{L}(X) \) are the reductions, modulo permutation equivalence, which we now define.
- Along the way, we sketch:
  - how \( \mathcal{L}(X) \) yields an fp 2-category,
  - how any fp 2-category yields an \( \mathcal{L} \)-algebra.
Reductions

First, for \( r \in X(G \vdash M, N : t) \), we have the formation rule:

\[
\begin{align*}
\ldots \quad & \Gamma \vdash P_i : M_i \rightarrow N_i : G_i \quad \ldots \\
\Gamma \vdash r(P_1, \ldots, P_n) : M[M_1, \ldots, M_n] \rightarrow N[N_1, \ldots, N_n] : A
\end{align*}
\]

2-categorically:

\[
\begin{array}{ccc}
(M_1, \ldots, M_n) & \xrightarrow{P} & (N_1, \ldots, N_n) \\
\Pi \Gamma & \Downarrow & \Pi G \\
\end{array}
\quad
\begin{array}{ccc}
M & \Downarrow r & A, \quad \text{with } P = (P_1, \ldots, P_n).
\end{array}
\]

\[
\begin{array}{ccc}
M & \Downarrow r & A, \quad \text{with } P = (P_1, \ldots, P_n).
\end{array}
\]
Reductions

There is a rule for sequential composition:

\[
\frac{\Gamma \vdash P : M_1 \rightarrow M_2 : A \quad \Gamma \vdash Q : M_2 \rightarrow M_3 : A}{\Gamma \vdash P ;_{M_2} Q : M_1 \rightarrow M_3 : A}.
\]

2-categorically:

\[
\begin{array}{c}
\text{\(\prod\)} \hspace{1cm} G \\
\downarrow \hspace{1cm} P \\
M_1 \leftrightarrow \hspace{1cm} A. \\
\downarrow \hspace{1cm} Q \\
M_3
\end{array}
\]
Reductions

Plus a few other rules:

- for empty reductions, i.e., identities,
- for reductions in context, i.e., making reductions into a congruence.
Permutation equivalence

- We then quotient out reductions by permutation equivalence.
- Also inductively defined.
- Permutation actually happens in the rules equating

\[
\begin{align*}
C \xrightarrow{\downarrow} B \xrightarrow{\downarrow r} A, \quad &\text{with} \quad C \xrightarrow{\downarrow} B \xrightarrow{\downarrow} A, \\
&\text{and the other way around. Namely:}
\end{align*}
\]

\[
r(P;_{N_2} Q) \equiv M_1[P] ;_{M_1[N_2]} r(Q).
\]
The monad

Theorem

This defines a functor $\mathcal{L} : \text{Sig} \to \text{Sig}$, which is a monad with substitution as multiplication.
The adjunction

We use $\mathcal{L}$ to construct an adjunction

$$
\begin{array}{ccc}
\text{Sig} & \overset{\mathcal{H}}{\longrightarrow} & 2\text{FPCat} \\
\downarrow & & \downarrow \\
\mathcal{W} & \overset{\mathcal{W}}{\longleftarrow} & \text{Sig}
\end{array}
$$

as the composite

$$
\begin{array}{ccc}
\text{Sig} & \overset{\mathcal{L}}{\longrightarrow} & \mathcal{L}\text{-Alg} \\
\downarrow & & \downarrow \\
\mathcal{U} & \overset{\mathcal{U}}{\longleftarrow} & \text{Sig}
\end{array}
\quad \quad \quad
\begin{array}{ccc}
\mathcal{L}\text{-Alg} & \overset{\mathcal{F}}{\longrightarrow} & 2\text{FPCat}. \\
\downarrow & & \downarrow \\
\mathcal{V} & \overset{\mathcal{V}}{\longleftarrow} & \mathcal{L}\text{-Alg}
\end{array}
$$
Soundness and completeness

**Proposition**

Given a 2-signature $X$, form the fp 2-category $X^*$ with:

- **objects**: tuples of sorts,
- **morphisms**: tuples of terms,
- **preordered hom-categories**:
  - $X^*(M, N) = 1$ iff $M \rightarrow^* N$,
  - $X^*(M, N) = \emptyset$ otherwise.
Soundness and completeness

**Theorem**

For all sorts $t_1, \ldots, t_n, t$, there is a functor

$$\mathcal{H}(X)(\prod t_i, t) \to X^*(\prod t_i, t),$$

which is full and identity-on-objects (to a preorder).

Hence:

- If $P: M \to N$ in $\mathcal{H}(X)$ then $M \to^* N$ in the standard sense.
- Conversely.
Cartesian closedness

- Up to now: first-order rewriting.
- 2-categorical semantics known, or at least folklore.
- New presentation as far as I know.
- Now, sketch for higher-order rewriting.
2-signatures

- Types: $A, B ::= 1 \mid x \mid A \times B \mid B^A$.
- $L_1(\Sigma)$: simply-typed $\lambda$-calculus on $\Sigma$.
- 2-signature: set indexed by pairs of parallel $\lambda$-terms in $\beta$-normal, $\eta$-long form.
Example

- Signature $\Sigma$: $\ell: t^t \to t$, $a: t \times t \to t$.
- Pure $\lambda$-terms with variables in $n$ in bijection with

$$\mathcal{L}_1(\Sigma)(t, \ldots, t \to t).$$

$n$ times

- One rule $\beta: a(\ell(M^t), N^t) \to MN$, i.e.,

\[
\begin{array}{ccc}
\ell \times t & \xrightarrow{\beta} & t \times t \\
\downarrow & & \downarrow a \\
t^t \times t & \xrightarrow{ev} & t
\end{array}
\]
Reductions and permutations

As before, plus formation rules

\[ \Gamma, x : A \vdash P : M \rightarrow N : B \]
\[ \Gamma \vdash \lambda x : A.P : \lambda x : A.M \rightarrow \lambda x : A.N : B^A \]
\[ \Gamma \vdash P : M \rightarrow M' : B^A \quad \Gamma \vdash Q : N \rightarrow N' : A \]
\[ \Gamma \vdash PQ : MN \rightarrow M'N' : B \]

with equations for $\beta\eta$. 
Soundness and completeness

From any 2-signature $X$, higher-order rewriting yields a reduction relation between terms on the underlying 1-signature.

Proposition

$M \rightarrow^* N$ iff there exists a reduction $P : M \rightarrow N$. 
Corollary

- Cartesian closed categories are locally discrete cartesian closed 2-categories: $\downarrow \begin{align*} \mathbf{2CCCat} & \rightarrow \mathbf{CCCat} \end{align*} \downarrow$

- Composing with the above: $\downarrow \begin{align*} \mathbf{K} \rightarrow \mathbf{CCCat} \end{align*} \downarrow$

- Fiore and Hur’s syntactic category on $X$ is a full subcategory of $\mathbf{K}(X)$.

- More objects in $\mathbf{K}(X)$, e.g., $t^t$. 
Uncaught

- The $\pi$-calculus (and all calculi with a structural congruence).
- Would need to incorporate equations between reductions.
- Even with that, no efficient presentation of $\pi$ yet.