Categories 1

Premières définitions et premiers exemples

A bit of maths

Objects	Morphisms
Sets	Functions
Monoids	Monoid homomorphisms
Groups	Group homomorphisms
...	...
Algebraic structures	Structure-preserving maps
Topological spaces	Continuous functions
Vector spaces over $\mathbb{R}$	Linear maps
Elements of a preorder	Pairs (x, y) such that $x \leq y$

- For each line, identities are neutral and composition is well-defined and associative. (For the last line, identities = reflexivity and composition = transitivity.)
- Hence, each line of this table is a category.
- General definition ?

Definition of a category

Very formal. Don't try to read everything in detail. Just to convince you that it may be formalized.

Definition

A *category* is given by

- a set of *objects* Obj,
- a set of *morphisms* Mor,
- two functions $dom, cod : Mor \to Obj$,
- a function $id : Obj \to Mor$, and
- a function $\circ : \{(g, f) \in Mor^2 \mid dom(g) = cod(f)\} \to Mor$,

such that ...

Definition of a category (cont)

Definition

...

- For all objects A, B, and morphisms $f : A \to B$ and $g : B \to A$,

$$dom(id_A) = cod(id_A) = A$$
$$f \circ id_A = f$$
$$id_A \circ g = g.$$

- For all morphisms f, g, and h such that $cod(f) = dom(g)$ and $cod(g) = dom(h)$,

$$dom(g \circ f) = dom(f)$$
$$cod(g \circ f) = cod(g)$$
$$h \circ (g \circ f) = (h \circ g) \circ f.$$

(Notation : $f : A \to B$ for $dom(f) = A$ and $cod(f) = B$.)

Definition of a category (cont)

Is everyone happy with this definition ?

Definition of a category (cont)

Is everyone happy with this definition ?

- The ambient meta-theory is unclear.
- Example : what did I mean by “set” and “function” ?
- Several possible answers :

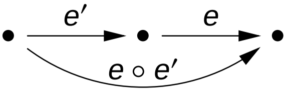
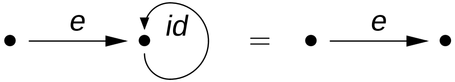
First-order logic	sorts and function (or relation) symbols,
Set theory	sets and functions,
Type theory, CiC	types and functions (or relations), . . .

- The properties fitting the formulation in first-order logic will be called *elementary*.
- Example : construction of functions in topoi.
- Don’t necessarily try to write that down formally, we won’t care too much in the following.

Yet another example of a category

Fact

A monoid is a category with one object.

Monoid	Category with one object
Elements	Morphisms
Multiplication $*$	Composition $\circ$
$e * e'$	
Neutral element 0	Identity *id*
$0 * e = e$	

Very intuitive when the multiplication has a sequential flavor, e.g., the monoid of words over a given alphabet.

Functors

Fact

Monoid homomorphisms are in 1-1 correspondence with functors between categories with one object.

General definition of a functor ?

Functors

Definition

Given two categories $\mathcal{C}$ and $\mathcal{C}'$, a *functor* $F : \mathcal{C} \to \mathcal{C}'$ is a pair of functions

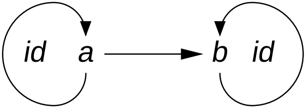
- $F_{Obj} : \mathcal{C}_{Obj} \to \mathcal{C}'_{Obj}$ and
- $F_{Mor} : \mathcal{C}_{Mor} \to \mathcal{C}'_{Mor}$,

(which we both write F, letting the context disambiguate,) such that

- $F(f) : F(dom(f)) \to F(cod(f))$
- $F(id_A) = id_{F(A)}$ and
- $F(g \circ f) = F(g) \circ F(f)$.

A sometimes useful intuition about functors

A functor $F : \mathcal{C} \to \mathcal{C}'$ gives a picture of $\mathcal{C}$ in $\mathcal{C}'$.
Example : let **2** be the dumb category



A functor from it to any category $\mathcal{C}$ is a mere arrow in $\mathcal{C}$.

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That's it. Now you master all of the technicalities of category theory. Almost. Namely, we will need to explain

- natural transformations
- and adjunctions,

which are more complicated and powerful than categories and functors.

Examples of adjunctions

Left adjoint	Right adjoint	Comments
Δ	$\times$	Def of internal product in terms of the external one.
$+$	Δ	Def of internal coproduct in terms of the external one.
Free group generated by a monoid	Forgetful functor	
$\times$	$\rightarrow$	Cf. currying $(A \times B) \rightarrow C \cong A \rightarrow (B \rightarrow C)$.
$\otimes$	$\multimap$	The same, in a linear set-ting.
Forgetful functor	$A \mapsto \begin{array}{c} !A \\ \delta \downarrow \\ !!A \end{array}$	Benton explaining linear logic through an adjunc-tion between the linear and non-linear worlds.

Examples of adjunctions (cont)

Left adjoint	Right adjoint	Comments
$\exists$	Weakening	Lawvere's "quantifiers
Weakening	$\forall$	as adjoints",
$=$	Contraction	and other insights.

Organization of the course

- We want to cover
  - the categorical explanation of programming languages,
  - and basic considerations of categorical logic.
- Logic : deduction rules

$$\text{deduction rules} \xrightarrow{} \text{over}\downarrow \; \text{formulas} \; \text{over}\downarrow \; \text{terms.}$$

- Term languages may be seen as minimal programming languages.
- We will thus start with terms, sometimes digressing to consider features that are unusual in logic, e.g., side effects.