Exponential decay for solutions to semilinear damped wave equation

Stéphane Gerbi and Belkacem Said-Houari

LAMA, Université de Savoie, Chambéry, France

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Michel Chipot’s 60th Birthday.
1. Introduction
2. Local existence
3. Global existence and decay rate
Consider the damped wave equation:

\[
\begin{cases}
  u_{tt} - \Delta u - \omega \Delta u_t + \mu u_t = u|u|^{p-2} & x \in \Omega, \ t > 0 \\
  u(x, t) = 0, & x \in \partial \Omega, \ t > 0 \\
  u(x, 0) = u_0(x), \ u_t(x, 0) = u_1(x) & x \in \Omega
\end{cases}
\]

(1)

in a bounded regular domain \( \Omega \subset \mathbb{R}^N \), \( p \geq 2 \) and \( \omega, \mu \) are positive constants.

Questions to be asked:

1. Local existence,
2. Global existence,
3. Starting in the stable manifold, what is the decay rate of the solution to \( u = 0 \)?

Definition of a solution

Definition 1

For $T > 0$, we denote

$$
Y_T = \left\{ u \in C^0 ([0, T], H^1_0(\Omega)) \cap C^1 ([0, T], L^2(\Omega)) \cap C^2 ([0, T], H^{-1}(\Omega)) \mid u_t \in L^2 ([0, T], L^2(\Omega)) \right\}
$$

Given $u_0 \in H^1_0(\Omega)$ and $u_1 \in L^2(\Omega)$, a function $u \in Y_T$ is a local solution to (1), if $u(0) = u_0$, $u_t(0) = u_1$ and

$$
\int_{\Omega} u_{tt}\phi dx + \int_{\Omega} \nabla u \nabla \phi dx + \omega \int_{\Omega} \nabla u_t \nabla \phi dx + \mu \int_{\Omega} u_t \phi dx = \int_{\Omega} |u|^{p-2} u \phi dx,
$$

for any function $\phi \in H^1_0(\Omega)$ and a.e. $t \in [0, T]$. 

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Let us first define the Sobolev critical exponent $\bar{p}$ as:

$$\bar{p} = \begin{cases} \frac{2N}{N-2}, & \text{for } \omega > 0 \text{ and } N \geq 3 \\ \frac{2N-2}{N-2}, & \text{for } \omega = 0 \text{ and } N \geq 3 \\ \infty, & \text{if } N = 1, 2 \end{cases}$$

We first state a local existence theorem whose proof is given by Gazzola and Squassina [GS06, Theorem 3.1].

**Theorem 2**

Assume $2 < p \leq \bar{p}$. Let $u_0 \in H_0^1(\Omega)$ and $u_1 \in L^2(\Omega)$. Then there exist $T > 0$ and a unique solution of (1) over $[0, T]$ in the sense of definition 1.
Existence time

Definition 3

Let $2 < p \leq \bar{p}$, $u_0 \in H_0^1(\Omega)$ and $u_1 \in L^2(\Omega)$. We denote $u$ the solution of (1). We define:

$$T_{\text{max}} = \sup \left\{ T > 0 , \ u = u(t) \text{ exists on } [0, T] \right\}$$

Since the solution $u \in Y_T$ (the solution is “enough regular”), let us recall that if $T_{\text{max}} < \infty$, then

$$\lim_{\substack{t \to T_{\text{max}} \\ t < T_{\text{max}}}} \| \nabla u \|_2 + \| u_t \|_2 = +\infty .$$

If $T_{\text{max}} < \infty$, we say that the solution of (1) blows up and that $T_{\text{max}}$ is the blow up time.

If $T_{\text{max}} = \infty$, we say that the solution of (1) is global.
Functional setting

We define the following functions:

\[
I(t) = I(u(t)) = \| \nabla u \|_2^2 - \| u \|_p^p,
\]

(2)

\[
J(t) = J(u(t)) = \frac{1}{2} \| \nabla u \|_2^2 - \frac{1}{p} \| u \|_p^p,
\]

(3)

\[
E(t) = E(u(t)) = J(t) + \frac{1}{2} \| u_t \|_2^2
\]

(4)

The potential well depth is defined as:

\[
d = \inf_{u \in H^1_0(\Omega) \setminus \{0\}} \max_{\lambda \geq 0} J(\lambda u).
\]

“Nehari manifold”:

\[
\mathcal{N} = \left\{ u \in H^1_0(\Omega) \setminus \{0\}; \; I(t) = 0 \right\}.
\]

\[
\mathcal{N}^+ = \left\{ u \in H^1_0(\Omega); \; I(t) > 0 \right\} \cup \{0\} \quad \text{and} \quad \mathcal{N}^- = \left\{ u \in H^1_0(\Omega); \; I(t) < 0 \right\}.
\]

The stable set \( \mathcal{W} \) and unstable set \( \mathcal{U} \) are defined respectively as:

\[
\mathcal{W} = \left\{ u \in H^1_0(\Omega); \; J(t) \leq d \right\} \cap \mathcal{N}^+ \quad \text{and} \quad \mathcal{U} = \left\{ u \in H^1_0(\Omega); \; J(t) \leq d \right\} \cap \mathcal{N}^-.
\]
About the potential depth

We have:

\[ d = \min_{u \in \mathcal{N}} J(u). \]

As it was remarked by Gazzola and Squassina, this alternative characterization of \( d \) shows that

\[ \beta = \text{dist}(0, \mathcal{N}) = \min_{u \in \mathcal{N}} \| \nabla u \|_2 = \sqrt{\frac{2d\rho}{p - 2}} > 0. \]

We denote by \( C_* \) the best constant in the Poincaré-Sobolev embedding \( H^1_0(\Omega) \hookrightarrow L^p(\Omega) \) defined by:

\[ C_*^{-1} = \inf \left\{ \| \nabla u \|_2 : u \in H^1_0(\Omega), \| u \|_p = 1 \right\}. \]

For \( p < \bar{p} \), the embedding is compact and \( C_* \) is related to the potential depth by:

\[ d = \frac{p - 2}{2p} C_*^{-2p/(p-2)}. \]
The main result

Theorem 4

Assume $2 < p \leq \bar{p}$. Let $u_0 \in \mathcal{N}^+$ and $u_1 \in L^2(\Omega)$. Moreover, assume that $E(0) < d$. Then:

- $T_{\text{max}} = \infty$
- there exist two positive constants $\hat{C}$ and $\xi$ independent of $t$ such that:
  \begin{align*}
  0 < \| \nabla u(t) \|_2^2 + \| u_t \|_2^2 &\leq \hat{C}e^{-\xi t}, \ \forall \ t \geq 0.
  \end{align*}

Under the same conditions, Gazzola and Squassina obtained:

- $T_{\text{max}} = \infty$
- there exists $C > 0$ such that:
  \begin{align*}
  0 < \| \nabla u(t) \|_2^2 + \| u_t \|_2^2 &\leq \frac{C}{t}, \ \forall \ t > 0.
  \end{align*}
Remark 1

- **Multiplying (1) by \( u_t \), integrating over \( \Omega \) and using integration by parts we obtain:**

\[
\frac{dE(t)}{dt} = -\omega \| \nabla u_t \|_2^2 - \mu \| u_t \|_2^2 , \forall t \geq 0.
\]

Thus the function \( E \) is decreasing along the trajectories.

- **If there exists** \( t_0 \in [0, T) \) **such that**

\[
E(t_0) < d
\]

the same result stays true. It is the reason why we choose \( t_0 = 0 \).

- **The condition** \( E(0) < d \) **is equivalent to the inequality:**

\[
C^p_\star \left( \frac{2p}{p-2} E(0) \right)^{\frac{p-2}{2}} < 1
\]
Sketch of the proof: two preliminaries lemmas

**Lemma 5**

Assume $2 \leq p \leq \bar{p}$. Let $u_0 \in \mathcal{N}^+$ and $u_1 \in L^2(\Omega)$. Moreover, assume that $E(0) < d$. Then $u(t, .) \in \mathcal{N}^+$ for each $t \in [0, T)$.

**Proof:** After some calculations, and using the remarks, we easily show that under the condition $E(0) < d$, we obtain:

$$\|u(t)\|_p^p < \|\nabla u(t)\|_2^2 \quad \forall t \in [0, T).$$

Hence $\|\nabla u\|_2^2 - \|u\|_p^p > 0$, $\forall t \in [0, T)$. This shows that

$$u(t, .) \in \mathcal{N}^+, \forall t \in [0, T).$$

**Lemma 6**

Assume $2 < p \leq \bar{p}$. Let $u_0 \in \mathcal{N}^+$ and $u_1 \in L^2(\Omega)$. Moreover, assume that $E(0) < d$. Then the solution of the problem (1) is global in time.
Define a Lyapunov function: a small perturbation of the energy

For $\varepsilon > 0$, to be chosen later, we define

$$L(t) = E(t) + \varepsilon \int_\Omega u_t u dx + \frac{\varepsilon \omega}{2} \| \nabla u \|^2_2 .$$

(5)

It is straightforward to see that $L(t)$ and $E(t)$ are equivalent in the sense that there exist two positive constants $\beta_1$ and $\beta_2 > 0$ depending on $\varepsilon$ such that for $t \geq 0$

$$\beta_1 E(t) \leq L(t) \leq \beta_2 E(t).$$

(6)

By taking the time derivative of the function $L$ defined above in equation (5), using problem (1), and performing several integration by parts, we get:

$$\frac{dL(t)}{dt} = -\omega \| \nabla u_t \|^2_2 - \mu \| u_t \|^2_2 + \varepsilon \| u_t \|^2_2 - \varepsilon \| \nabla u \|^2_2$$

$$+ \varepsilon \| u \|_p^p - \varepsilon \mu \int_\Omega u_t u dx .$$

(7)
Using Young and Poincaré’s inequality, the characterization of $d$ and $C_*$, we obtain:

**Differential inequality**

It exists $\xi > 0$ such that:

$$\frac{dL(t)}{dt} \leq -\xi L(t), \quad \forall t \geq 0.$$  

Integrating the previous differential inequality between 0 and $t$, it exists $C > 0$, such that:

$$L(t) \leq Ce^{-\xi t}, \quad \forall t \geq 0.$$  

Consequently, using the equivalence between $L$ and $E$, it exists $\hat{C} > 0$, such that:

$$E(t) \leq \hat{C}e^{-\xi t}, \quad \forall t \geq 0.$$
Remark 2

- Note that we can obtain the same results as in Theorem 4 in the case of the absence of the strong damping i.e. $\omega = 0$, by taking the following Lyapunov function:
  \[ L(t) = E(t) + \varepsilon \int_{\Omega} u_t u dx \]

- It is clear that the following problem:
  \[
  \begin{cases}
  u_{tt} - \text{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) - \omega \Delta u_t + \mu u_t = u|u|^{p-2} & x \in \Omega, \ t > 0 \\
  u(x, t) = 0, & x \in \partial \Omega, \ t > 0 \\
  u(x, 0) = u_0(x), \ u_t(x, 0) = u_1(x) & x \in \Omega
  \end{cases}
  \]

  could be treated with the same method and we obtained also an exponential decay of the solution if the initial condition is in the positive Nehari manifold and its energy is lower than the potential well depth.
Thank you for your attention